Kerr-Schild Geometry Indicates Compton Size of Electron

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September 2007, Dubna, Spin-07

Kerr-Schild Formalism. Debney, Kerr and Schild, J. Math. Phys. **10** (1969) 1842

The Kerr-Schild ansatz for metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h e^3_\mu e^3_\nu, \quad \sqrt{-g} = 1. \qquad (1)$$

$\eta_{\mu\nu}$ - auxiliary Minkowski space-time.

Principal null direction

$$e^3 = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv \qquad (2)$$

is expressed via complex function $Y(x)$ in the null Cartesian coordinates $\sqrt{2}\zeta = x + iy$, $\sqrt{2}\bar{\zeta} = x - iy$, $\quad \sqrt{2}u = z - t$, $\sqrt{2}v = z + t$.

Null tetrad $e^a, a = 1, 2, 3, 4$. Real directions $e^3 = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv$ and $e^4 = dv + he^3$, and two complex conjugate directions

$$e^1 = d\zeta - Y dv, \quad e^2 = d\bar{\zeta} - \bar{Y} dv.$$

The Kerr theorem.

The geodesic and shearfree (GSF) null congruences $e^3(Y)$ satisfy the conditions

$Y,_2 = Y,_4 = 0.$

GSF congruences are determined by $Y(x)$ which is a solution of the equation

$$F(Y, l_1, l_2) = 0 \ , \tag{3}$$

where F is an arbitrary analytic function of the projective twistor coordinates

$$Y, \quad l_1 = \zeta - Y v, \quad l_2 = u + Y\bar{\zeta} \ . \tag{4}$$

Integration of the Einstein-Maxwell field equations for the geodesic and shear-free congruences was fulfilled in DKS, leading to the following form of the function

$$h = \frac{1}{2} M (Z + \bar{Z}) - \frac{1}{2} A \bar{A} Z \bar{Z} \ , \qquad (5)$$

of the Kerr-Schild ansatz:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2 h e^3_\mu e^3_\nu. \qquad (6)$$

Necessary functions $Z = P/\tilde{r}$, $Y(x)$ and parameters are determined by the generating function F.

$$P Z^{-1} = \tilde{r} = - \quad dF/dY \qquad (7)$$

is complex radial distance, factor P is connected with Killing vector or the boost of the source.

There was obtained a system of differential equations for functions A, and M.

Electromagnetic sector:

$$A_{,2} - 2Z^{-1}\bar{Z}Y_{,3}\,A = 0, \quad A_{,4} = 0, \tag{8}$$

$$\mathcal{D}A + \bar{Z}^{-1}\gamma_{,2} - Z^{-1}Y_{,3}\,\gamma = 0, \quad \gamma_{,4} = 0, \tag{9}$$

where $\mathcal{D} = \partial_3 - Z^{-1}Y_{,3}\,\partial_1 - \bar{Z}^{-1}\bar{Y}_{,3}\,\partial_2$.

The strength tensor of self-dual electromagnetic field is given by the tetrad components

$$\mathcal{F}_{12} = AZ^2, \quad \mathcal{F}_{31} = \gamma Z - (AZ)_{,1}\ . \tag{10}$$

Gravitational sector:

Real function M.

$$M_{,2} - 3Z^{-1}\bar{Z}Y_{,3}\,M = A\bar{\gamma}\bar{Z}, \tag{11}$$

$$\mathcal{D}M = \frac{1}{2}\gamma\bar{\gamma}, \quad M_{,4} = 0. \tag{12}$$

For any holomorphic $F(Y) \Rightarrow$ GSF congruence $\Rightarrow$ *algebraically special solution.*

Final integration of eqs. I and II was given only for $\gamma = 0$ and quadratic in Y function $F(Y)$, (Debney, Kerr and Schild 1969) $\Rightarrow$ a broad class of exact solutions containing **Kerr-Newman solution** as a very important particular case:

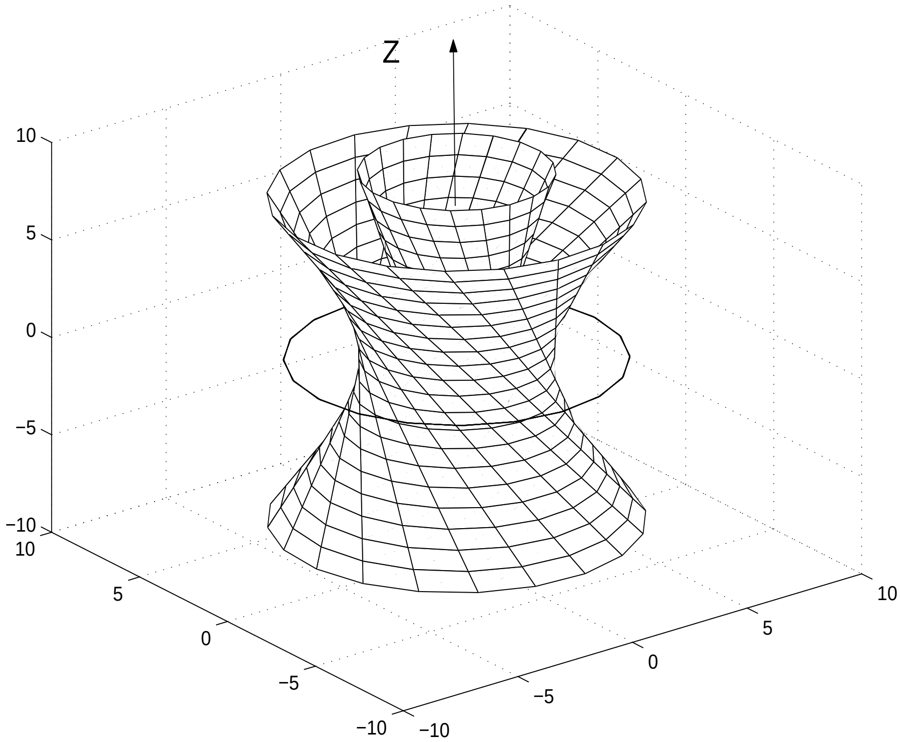
Metric $g_{\mu\nu} = \eta_{\mu\nu} + 2hk_\mu k_\nu$, where $h = \frac{mr - e^2/2}{r^2 + a^2\cos^2\theta}$.

Electromagnetic field $A_\mu = \frac{er}{r^2 + a^2\cos^2\theta} k_\mu$.

It is remarkable simple form. All the complications are included in the form of the the **Kerr congruence** $k_\mu(x) = e^3_\mu(Y)\sqrt{2}/(1 + Y\bar{Y})$ which is determined by function $Y(x)$ (solution of the eq. $F(Y, x) = 0$.

Singularity is by $r + ia\cos\theta \equiv \partial_Y F = 0 \Rightarrow r = \cos\theta = 0$.

The Kerr congruence is vortex of null lines (twistors)



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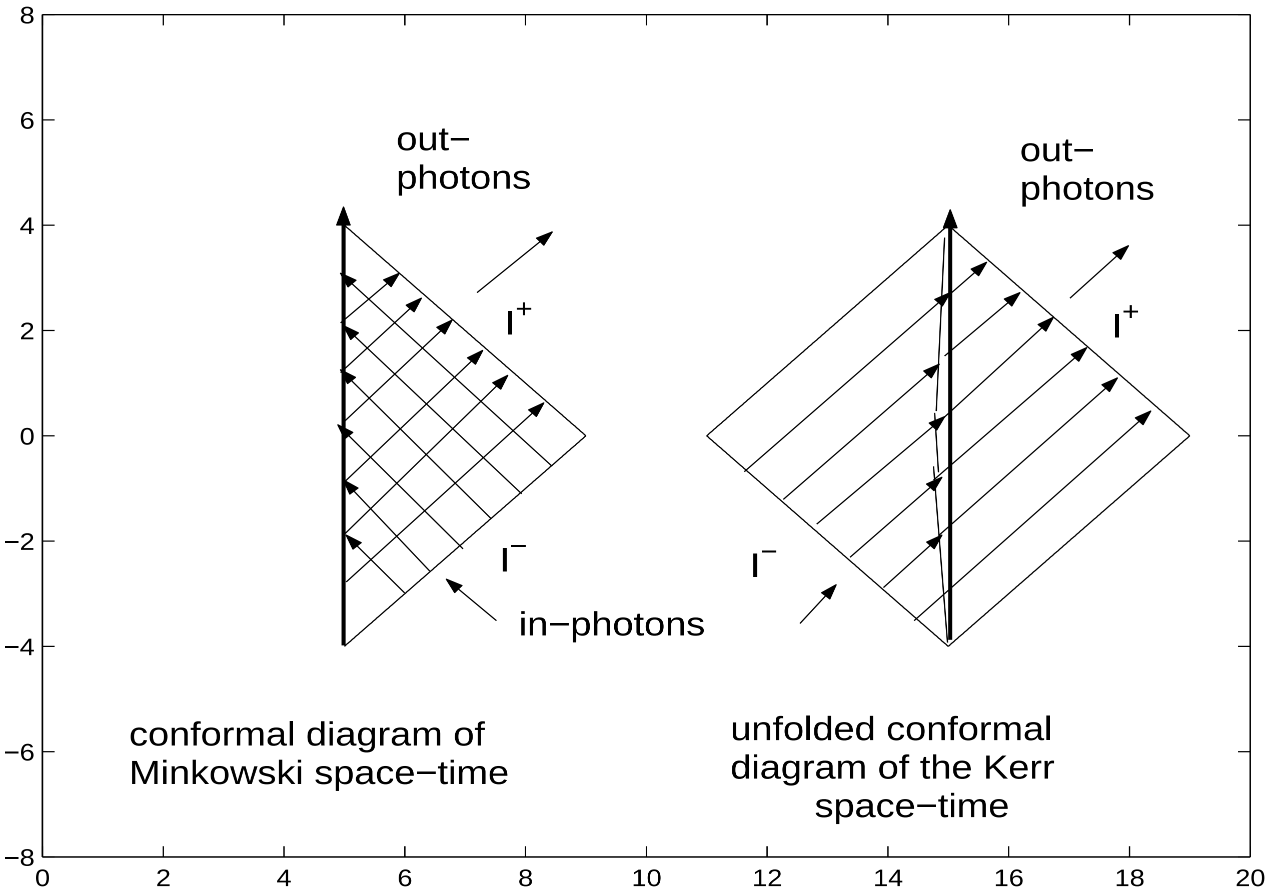
The Kerr singular ring and the Kerr congruence.

The Kerr singular ring is a branch line of space on two sheets: "negative" and "positive" where the fields change their signs and directions. Congruence covers the space-time twice.



Twosheetedness of the Kerr space-time. The 'in' and 'out' electromagnetic fields are positioned on different sheets, they are **aligned to Kerr congruence** and don't interact with each other.

Classical extended electron.
Gyromagnetic ratio $g = 2$ as that of the Dirac electron (Carter 1968).
Spin $J = ma = 1/2$ is very high with respect to mass $m = 10^{-22}$, $\mid a \mid >> m$ and BH horizons are absent, singular ring is naked.
Kerr's ring as a string (A.B. 1974, 1978, 1993, 1995, 2003 ... 2006). Compton size $a \sim \hbar/m$.

Problem of the Kerr source. W.Israel(1970), A.B.(1974,1988,2000), M.Gürses and F.Gürsey(197
V.Hamity(1976), C.López(1984)...
Alternatives:
a/ Alice string + twosheetedness,
b/ disklike source.
Compromise:
rotating **superconducting disk** – a "mirror gates" in the "Alice mirror world". Similar twovaluedness appears in the models of the cosmic "Alice" strings which are connected with superconducting properties of the source. The "negative" sheet looks as a mirror image of the "positive" one.

Smooth and regularized Kerr source. A.B., E. Elizalde, S.Hildebrandt and G. Magli, PRD (2002)

Ansatz $g_{\mu\nu} = \eta_{\mu\nu} + 2hk_\mu k_\nu$, where

$h = f(r)/(r^2 + a^2 \cos^2\theta)$ (Gürses and Gürsey 1974).

Regularized solutions have tree regions:

i) the Kerr-Newman exterior, $r > r_0$, where $f(r) = mr - e^2/2$,

ii) interior $r < r_0 - \delta$, where $f(r) = f_{int}$ and function $f_{int} = \alpha r^n$, and $n \geq 4$ to suppress the singularity at $r = 0$, and provide the smoothness of the metric up to the second derivatives.

iii) a narrow intermediate region providing a smooth interpolation between i) and ii).

Non-rotating source: by $n = 4$ and $\alpha = 8\pi\Lambda/6$.

Interior: Constant curvature $R = -24\alpha$.

Energy density $\rho = \frac{1}{4\pi}(f'r - f)/\Sigma^2$. Tangential and radial pressures $p_{rad} = -\rho, \quad p_{tan} = \rho - \frac{1}{8\pi}f''/\Sigma$, where $\Sigma = r^2$.

De Sitter - for $\alpha > 0$, AdS for $\alpha < 0$, and flat interior for $\alpha = 0$.

The resulting sources may be considered as the bags filled by a special matter with positive ($\alpha > 0$) or negative ($\alpha < 0$) energy density.

The transfer from the external Kerr-Newman solution to the internal region (source) may be considered as a phase transition from 'true' to 'false' vacuum.

Assuming that transition region is very thin, one can consider the following useful graphical representation.

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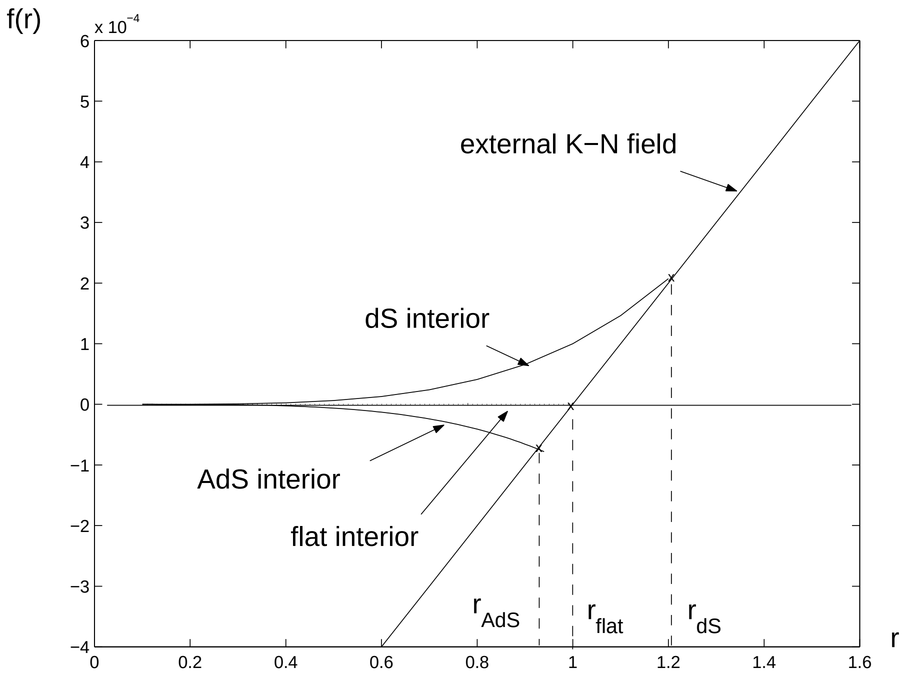


Regularization of the Kerr spinning particle by matching the external field with dS, flat or AdS interior.

The point of phase transition r_0 is determined by the equation $f_{int}(r_0) = f_{KN}(r_0)$.

It yields

$$m = \frac{e^2}{2r_0} + \frac{4}{3}\pi r_0^3 \rho. \qquad (13)$$

The first term on the right side is electromagnetic mass of a charged sphere with radius r_0, $M_{em}(r_0) = \frac{e^2}{2r_0}$, while the second term is the mass of this sphere filled by a material with a homogenous density ρ, $M_m = \frac{4}{3}\pi r_0^3 \rho$.

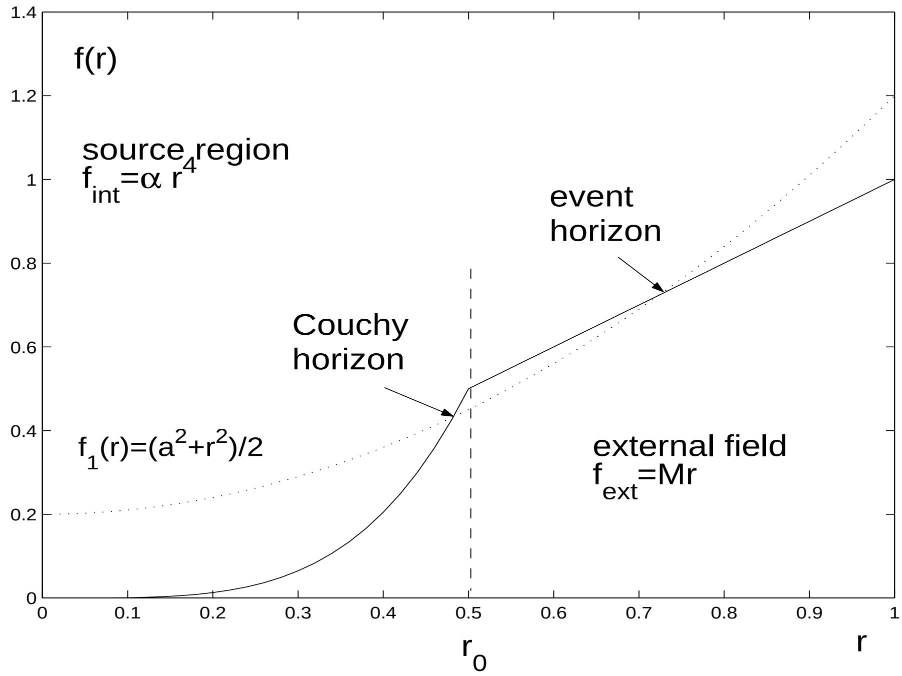
The point of intersection r_0 acquires a deep physical meaning, providing the energy balance by the mass formation.

Transfer to rotating case is trivial: the replacement

$$r \rightarrow \Sigma = r^2 + a^2 \cos^2\theta, \qquad (14)$$

where new r and θ are the oblate spheroidal coordinates.

The Kerr source represents a rigidly rotating disk with the boundary $r = r_0$. In the corotating with disk coordinate system, the matter of the disk looks homogenous distributed, however, because of the relativistic effects the energy-momentum tensor increases strongly near the boundary of the disk.



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Matching the (rotating) internal "de Sitter" source with the external Kerr-Schild field. The dotted line $f_1(r) = (r^2 + a^2)/2$ determines graphically the position of horizons as the roots of the equation $f(r) = f_1(r)$.

Supersymmetric Superconducting Bag as Source of the Kerr Spinning Particle

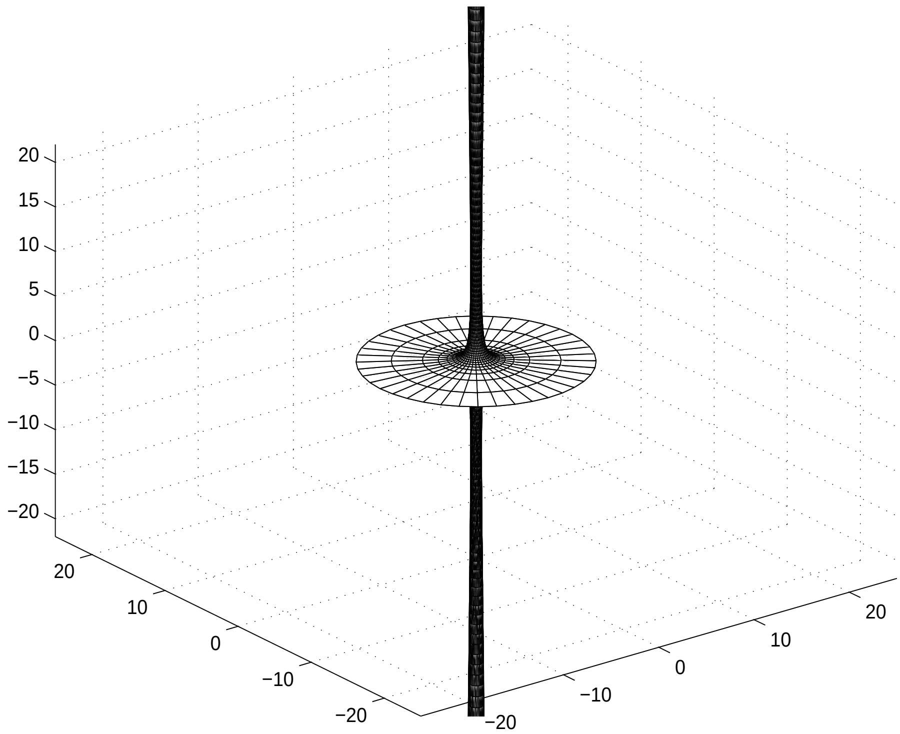
$U(I)\times\tilde{U}(I)$ field model for cosmic strings (Witten 1985)

$$L = -(D^\mu\phi_A)(\overline{D_\mu\phi_A}) - \frac{1}{4}F_A^{\mu\nu}F_{A\mu\nu}$$
$$(\tilde{D}^\mu\phi_B)(\overline{\tilde{D}_\mu\phi_B}) - \frac{1}{4}F_B^{\mu\nu}F_{B\mu\nu} - V.$$

Two sorts of superconductivity, A and B. Two Higgs fields ϕ_A and ϕ_B, and two gauge fields A_μ and B_μ describe the 'true' and 'false' vacua.

Supersymmetric version of this model (J.Morris, Phys. Rev.**D56**, 2378 (1997)) contains fife chiral fields, $V = V(\Phi_A, \bar{\Phi}_A, \Phi_B, \bar{\Phi}_B, Z)$. Field Z synchronizes transfer $true \leftrightarrow false$ vacua for spherical bag (A.B., Grav.Cosm. 2000).

In the limit of a very thin disk a stringy singularity develops on the border of disk. This case corresponds to the Israel-Hamity source 1970-1976.



The Kerr disk-like source and two axial null beams k^μ_{L} and k^μ_{R}.

Dirac-Kerr electron. (A.B.PRD(2004),hep-th/0507109)
The axial null half-strings play peculiar role controlling the Kerr twistorial structure.
Dirac equation in the Weyl basis, $\Psi^\dagger = (\phi_\alpha, \chi^{\dot\alpha})$,

$$\sigma^\mu_{\alpha\dot\alpha}(i\partial_\mu + eA_\mu)\chi^{\dot\alpha} = m\phi_\alpha, \ \bar\sigma^{\mu\dot\alpha\alpha}(i\partial_\mu + eA_\mu)\phi_\alpha = m\chi^{\dot\alpha}.$$

Null vectors $k^\mu_{\mathrm{L}} = \bar\chi\sigma^\mu\chi, \ k^\mu_{\mathrm{R}} = \bar\phi\bar\sigma^\mu\phi$ determine
current $J_\mu = e(\bar\Psi\gamma_\mu\Psi) = e(\bar\chi\sigma_\mu\chi + \bar\phi\bar\sigma_\mu\phi)$,
momentum $p^\mu = \frac{m}{2}(k^\mu_{\mathrm{L}} + k^\mu_{\mathrm{R}})$ and **spin** (polarization) $n^\mu = \frac{1}{2}(k^\mu_{\mathrm{L}} - k^\mu_{\mathrm{R}})$ of electron.
Kerr's geometry is also fixed by two null vectors k^μ_{L} and k^μ_{R} controlling *spin* (orientation) and *Lorentz deformation* of congruence.
Dirac wave manages the Kerr congruence.
Null tetrad is completed by vectors $m^\mu = \phi\sigma^\mu\chi$, and $\bar m^\mu = (\phi\sigma^\mu\chi)^+$ determined by phase of the wave function. Phase of Dirac wave synchronizes null tetrad in space-time, playing the role of an 'order parameter'.

The simplest class of the exact stationary Kerr-Schild solutions.

Electromagnetic field is **aligned with the Kerr congruence:**

$$F^{\mu\nu} k_\mu = 0. \qquad (15)$$

Kerr -Schild metric $g^{\mu\nu} = \eta^{\mu\nu} - 2hk^\mu k^\nu$, where

$$h = m(Z + \bar{Z})/(2P^3) - A\bar{A}Z\bar{Z}/2. \qquad (16)$$

Stationary case, $P = 2^{-1/2}(1 + Y\bar{Y})$ and A has the general form

$$A = \psi(Y)/P^2, \qquad (17)$$

and ψ is an arbitrary holomorphic function of Y.

Kerr-Newman solution is very particular case: $\psi(Y) = e = const.$

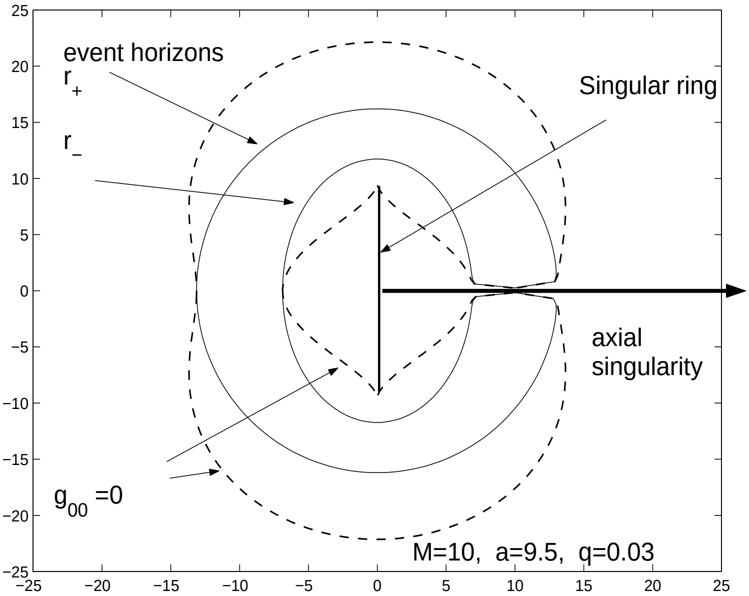
In general case function $Y(x) = e^{i\phi}\tan\frac{\theta}{2} \in CP^1$ is coordinate on projective sphere, and there is an infinite set of the *exact solutions*, in which function $\psi(Y)$ is holomorphic on the punctured sphere at the set of points $\{Y_i, i = 1, 2, ...\}$, $\psi(Y) = \sum_i \frac{q_i}{Y(x) - Y_i}$.

In these solutions $\psi(Y)$ is singular at a set of angular directions $\phi_i,\ \theta_i$, and there appear semi-infinite ılightlike beams, (singular pp-strings) along some of the null rays of the Kerr congruence. **How act such beams on the BH horizon?**

Singular beams lead to formation of the holes in the black hole horizon, which opens up the interior of the “black hole” to external space.

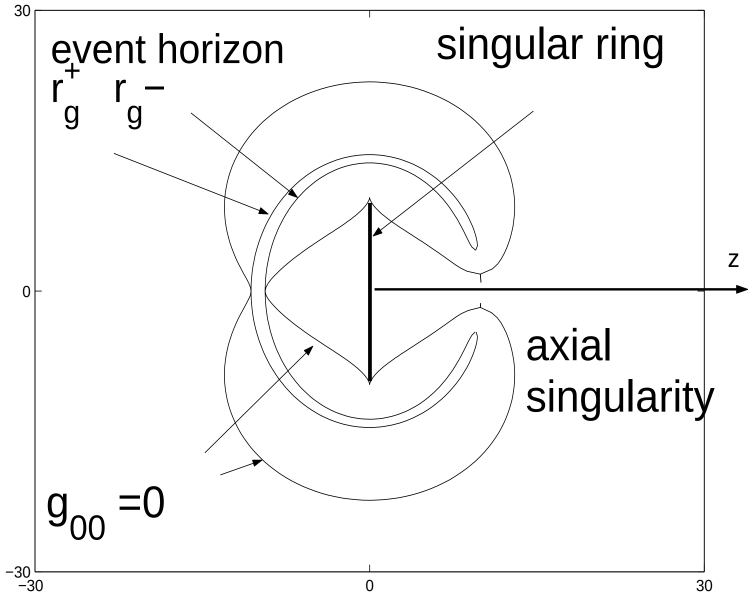
Black holes with holes in the horizon

A.B., E.Elizalde, S.R.Hildebrandt and G.Magli, Phys. Rev. **D74** (2006) 021502(R)



Singular beam forms a small hole in the horizon.

The boundaries of **ergosphere** (punctured) are determined by $g_{00} = 0$. The **event horizons**,r_+ and r_-, are null surfaces obeying
$(\partial_r S)^2\{r^2+a^2+(q/\tan\frac{\theta}{2})^2-2Mr\}-(\partial_\theta S)^2 = 0.$
They are joined by a tunnel, by forming a simply connected surface.



Near extremal black hole with a hole in the horizon, caused by a lightlike singular beam. The r^+ and r^- surfaces are joined, by forming a simply connected surface.

Wave and nonstationary electromagnetic excitations.

Exact *self-consistent* solutions with $\gamma \neq 0$ are absent, however, EM eqs. may be solved. (Eq.I) $A = \psi/P^2$, where $\psi,_2 = \psi,_4 = 0$, but now $\psi = \psi(Y, \tau)$, where τ is complex retarded time obeying $\tau,_2 = \tau,_4 = 0$.

(Eq.II) $DA + \bar{Z}^{-1}\gamma,_2 - Z^{-1}Y,_3\,\gamma = 0$,
where $D = \partial_3 - Z^{-1}Y,_3\,\partial_1 - \bar{Z}^{-1}\bar{Y},_3\,\partial_2$.

Integration yields

$$\gamma = \frac{2^{1/2}\dot{\psi}}{P^2 Y} + \phi(Y, \tau)/P, \qquad (18)$$

By any nonstationarity, $\dot{\psi} \neq 0$, **there appears a pole in** ψ **or** γ, **and there appears inevitably a singular beam.**

It was shown that for the slowly varying EM field $\gamma \to 0$, and the approximate KS solutions containing singular beams tend to the corresponding exact stationary Kerr-Schild solutions.

The analytical twistorial structure of the Kerr spinning particle leads to the appearance of an extra axial stringy system. As a result, the Kerr spinning particle acquires a simple stringy skeleton which is formed by a topological coupling of the Kerr circular string and the axial stringy system. The projective spinor coordinate Y is a projection of sphere on complex plane. It is singular at $\theta = \pi$, and such a singularity will be present in any holomorphic function $\psi(Y)$. Therefore, *all the aligned e.m. solutions turn out to be singular at some angular direction* θ. The simplest modes

$$\psi_n = qY^n \exp i\omega_n\tau \equiv q(\tan\frac{\theta}{2})^n \exp i(n\phi + \omega_n\tau) \tag{19}$$

can be numbered by index $n = \pm 1, \pm 2, ...$

The leading wave terms are

$$\mathcal{F}|_{wave} = f_R \; d\zeta \wedge du + f_L \; d\bar{\zeta} \wedge dv, \tag{20}$$

where

$$f_R = (AZ)_{,1}\,; \qquad f_L = 2Y\psi(Z/P)^2 + Y^2(AZ)_{,1} \tag{21}$$

are the factors describing the “left” and “right” waves propagating along the z^- and z^+ semi-axis correspondingly.

The parameter $\tau = t - r - ia\cos\theta$ takes near the z-axis the values $\tau_+ = \tau|_{z^+} = t - z - ia, \quad \tau_- = \tau|_{z^-} = t + z + ia.$

The leading wave for $n = 1$,

$$\mathcal{F}_1^- = \frac{4qe^{i2\phi + i\omega_1\tau_-}}{\rho^2}\, d\bar\zeta \wedge dv,$$

propagates to $z = -\infty$ along the z^- semi-axis.

The leading wave for $n = -1$,

$$\mathcal{F}_{-1}^+ = -\frac{4qe^{-i2\phi + i\omega_{-1}\tau_+}}{\rho^2}\, d\zeta \wedge du,$$

is singular at z^+ semi-axis and propagates to $z = +\infty$.

The $n = \pm 1$ partial solutions represent asymptotically the singular plane-fronted e.m. waves propagating without damping.

There are corresponding self-consistent solution of the Einstein-Maxwell field equations which are singular plane-fronted waves (pp-waves). The Maxwell equations take the form $\partial^\mu \partial_\mu \mathcal{A} = J = \delta(string)$, and can easily be integrated leading to the solutions

$$\mathcal{A}^+ = [\Phi^+(\zeta) + \Phi^-(\bar{\zeta})] f^+(u) du, \qquad (22)$$

$$\mathcal{A}^- = [\Phi^+(\zeta) + \Phi^-(\bar{\zeta})] f^-(v) dv, \qquad (23)$$

where $\Phi^\pm$ are arbitrary analytic functions, and functions $f^\pm$ describe the arbitrary retarded and advanced waves. Therefore, the wave excitations of the Kerr ring lead to the appearance of singular lightlike beams (pp-waves) which propagate outward along the z^+ and/or z^- semi-axis.

Asymptotically exact Kerr-Schild solutions with aligned wave excitations.

Corresponding solutions with wave electromagnetic excitations are asymptotically exact in the low-frequency limit. In the far zone the wave beams tend to the exact $pp-$wave solutions. Similar 'axial' strings where also obtained by fermionic wave excitations.

Wave excitations propagating in the direction Y_i are described by means of the function

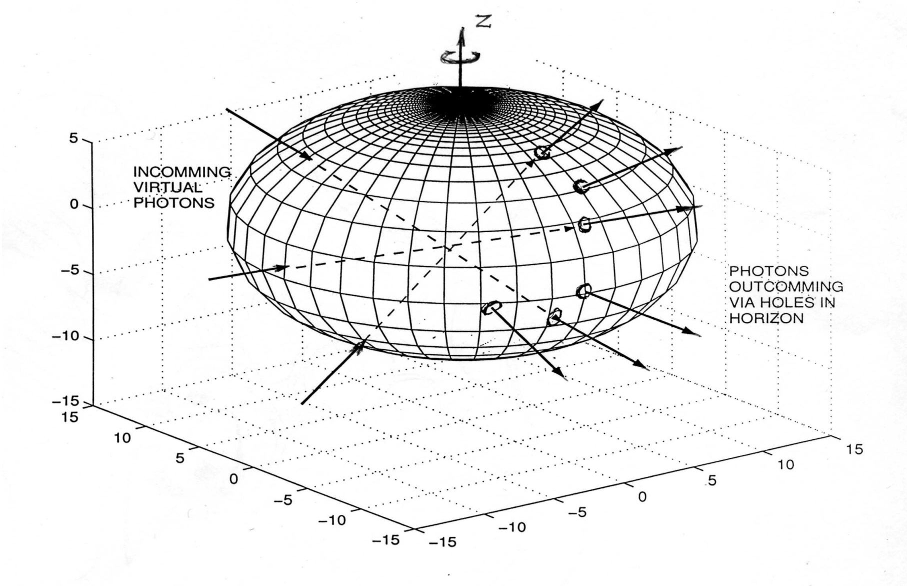
$$\psi_i(Y,\tau) = q_i(\tau)\,\text{exp}\{i\omega\tau\}\frac{1}{Y - Y_i}, \tag{24}$$

where the extra dependence is on the retarded time τ. The right hand sides of the gravitational KS equations are proportional to γ and $\gamma\bar{\gamma}$ and are quite small for low-frequency aligned wave excitations, since the function γ will be of the order: $\gamma \sim \dot{\psi} \sim i\omega\psi$.

Note that the smallness of function $\psi(Y,\tau)$ *not* required. The electromagnetic field $A=\psi/P^2$ and the metric distortion via the function $H=\{mr-\psi(Y)^2/2\}/(r^2+a^2\cos^2\theta)$ can be in fact very strong. At the same time, the right hand sides of the gravitational equations are determined by $\gamma\sim\dot{\psi}$, and in the low-frequency limit $\omega\rightarrow 0$, the corresponding eqs. tend towards the resolved stationary eqs., and solutions tend to exact stationary Kerr-Schild solutions with singular beams.

Quantum deformation of the BH horizon.

Since solutions tend to exact in the limit $\gamma \to 0$, one can consider a black hole immersed into the zero-point field of the virtual photons. One has a sum over small excitations in the diverse directions $\psi(Y,\tau) = \sum_i \frac{q_i(\tau)}{Y - Y_i}$, which lead to a flow and migration of many singular micro-beams and also to an instantaneous appearance and disappearance of the micro-holes at the horizon.

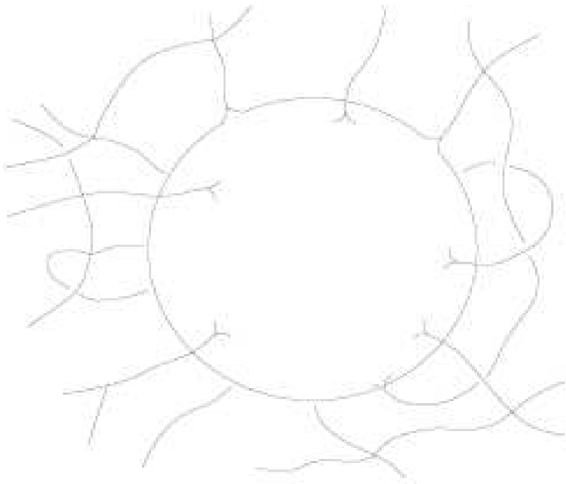


It suggests a new mechanism of black hole emission, which is both a semiclassical alternative and a complement to the quantum tunnelling conjectures on BH evaporation.

The old idea - dynamical origin of Hawking radiance (J.York Jr. 1983, C. Rovelli 1996). Microstructure of gravitational field related to zero point fluctuations.

Micro-structure of the emission from BH may be related with specific, aligned excitations of BH, singular beams (or pp-wave strings).

Loop quantum gravity: appearance of strong topological deformations of the horizon (taken from paper by A. Ashtekar, J. Baez and K. Krasnov, Adv. Theor. Math. Phys. **4**, 1 (2000)). The quantum singular hairs of the black hole pierce the horizon, forming strings outgoing from the horizon with different topological loop numbers.



Quantum horizon and polymeric structure of a BH as obtained from loop quantum gravity. (A. Ashtekar, J. Baez and K. Krasnov, Adv.Theor.Math.Phys.**4**,1(2000))

Aligned excitations lead to formation of the micro-holes at the horizon by singular strings which are null *twistor* lines aligned with the Kerr principal null congruence. The polymer structure of the vacua turns out actually to be a twistor structure of the black hole, which looks very entangled for multi-black-hole (that is, multi-particle) solutions.

The case of quadratic Kerr's generating functions F(Y) is well studied. A.B. and G. Magli, Phys.Rev.**D 61**044017 (2000).

The considered in DKS function F is quadratic in Y,

$$F \equiv a_0 + a_1 Y + a_2 Y^2 + (qY + c)\lambda_1 - (pY + \bar{q})\lambda_2, \quad (25)$$

where the coefficients c and p are real constants and $a_0, a_1, a_2, q, \bar{q}$, are complex constants. The Killing vector of the solution is determined as

$$\widehat{K} = c\partial_u + \bar{q}\partial_\zeta + q\partial_{\bar{\zeta}} - p\partial_v. \quad (26)$$

Writing the function F in the form

$$F = AY^2 + BY + C, \quad (27)$$

one can find two solutions of the equation $F = 0$ for the function $Y(x)$

$$Y_{1,2} = (-B \pm \Delta)/2A, \quad (28)$$

where $\Delta = (B^2 - 4AC)^{1/2}$.

Complex radial distance

$$\tilde{r} = -\partial F/\partial Y = -2AY - B, \qquad (29)$$

and consequently

$$\tilde{r} = PZ^{-1} = \mp\Delta. \qquad (30)$$

These two roots reflect the known twofoldedness of the Kerr geometry. They correspond to two different directions of congruence on positive and negative sheets of the Kerr spacetime. In the stationary case

$$P = pY\bar{Y} + \bar{q}\bar{Y} + qY + c\ . \qquad (31)$$

Link to the complex world line of the source. The stationary and boosted Kerr geometries are described by a straight complex world line with a real 3-velocity $\vec{v}$ in CM^4:

$$x_0^\mu(\tau) = x_0^\mu(0) + \xi^\mu \tau; \qquad \xi^\mu = (1, \vec{v})\ . \qquad (32)$$

The gauge of the complex parameter τ is chosen in such a way that $Re\ \tau$ corresponds to the real time t.

$\hat{K}$ is a Killing vector of the solution

$$\hat{K} = \partial_\tau x_0^\mu(\tau)\partial_\mu = \xi^\mu \partial_\mu \ . \tag{33}$$

$$P = \hat{K}\rho = \partial_\tau x_0^\mu(\tau) e_\mu^3 \ , \tag{34}$$

where

$$\rho = \lambda_2 + \bar{Y}\lambda_1 = x^\mu e_\mu^3. \tag{35}$$

It allows one to set the relation between the parameters $p, c, q, \bar{q}$, and ξ^μ, showing that these parameters are connected with the boost of the source.

The complex initial position of the complex world line $x_0^\mu(0)$ in Eq. (32) gives six parameters for the solution, which are connected to the coefficients a_0, a_1 a_2 . It can be decomposed as $\vec{x}_0(0) = \vec{c} + i\vec{d}$, where $\vec{c}$ and $\vec{d}$ are real 3-vectors with respect to the space O(3)-rotation. The real part $\vec{c}$ defines the initial position of the source, and the imaginary part $\vec{d}$ defines the value and direction of the angular momentum (or the size and orientation of a singular ring).

It can be easily shown that in the rest frame, when $\vec{v} = 0$, $\vec{d} = \vec{d}_0$, the singular ring lies in the orthogonal to $\vec{d}$ plane and has a radius $a = |\vec{d}_0|$. The corresponding angular momentum is $\vec{J} = m\vec{d}_0$.

Complex Kerr source, complex shift. Appel 1887!

A point-like charge e, placed on **complex z-axis** $(x_0, y_0, z_0) = (0, 0, ia)$, gives the real Appel potential

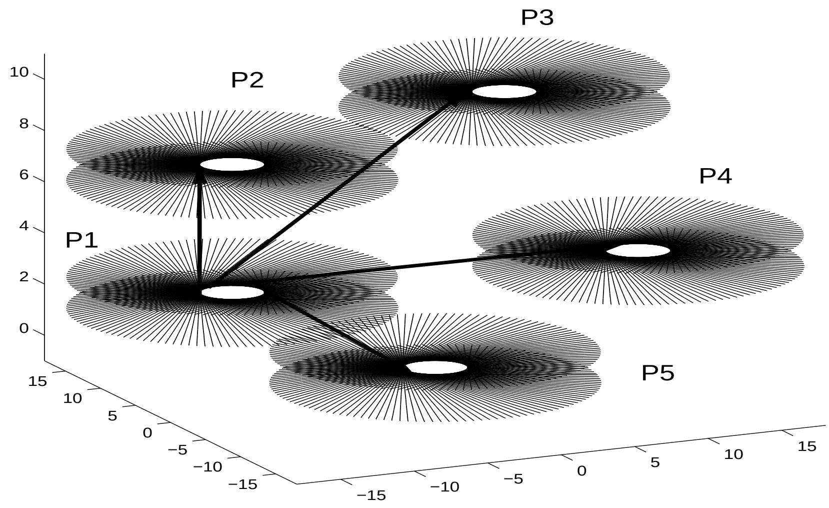
$$\phi_a = Re\ e/\tilde{r}. \tag{36}$$

$\tilde{r} = r + ia \cos\theta$ is the Kerr complex radial coordinate, where r and θ are the oblate spheroidal coordinates. It may be expressed in the usual rectangular Cartesian coordinates x, y, z, t as

$$\tilde{r} = [x^2 + y^2 + (z - ia)^2]^{1/2}. \tag{37}$$

The singular line of the solution corresponds to $r = \cos\theta = 0$.

Appel potential describes *exactly* the e.m. field of the Kerr-Newman solution.



Multisheetedness of the multiparticle KS solutions. The functions $F(Y)$ of higher degrees are formed as a product of quadratic blocks corresponding to n different particles $F(Y) = \prod_i^n F_i(Y)$. The particles i and j are positioned on different Riemannian sheets of the function $F(Y)$ and interact with each other only via a *common* twistor line of the i-th and j-th congruences, by forming a singular null string connecting these particles.

Complex world line and complex Kerr string.

The Appel source can be considered as a mysterious "particle" propagating along a *complex world-line* $x_0^\mu(\tau)$ in CM^4 and parametrized by a complex time $\tau = t + i\sigma$.

Complex source of the Kerr-Newman solution leads to a complex retarded-time construction for the Kerr geometry.

Complex world-lines occupy an intermediate position between particles and strings. Like a string they form two-dimensional surfaces or world-sheets in space-time. In many respects this source is similar to the "mysterious" $N = 2$ complex string of superstring theory.

The Kerr congruence is a track of the complex light cones (left null planes) emanating from the complex world line $x_0^\mu(\tau)$.

Complex retarded-time parameter.

Parameter τ may be defined for each point x of the Kerr space-time and plays the role of a complex retarded-time parameter. Its value for a given point x may be defined by L-projection, using the solution $Y(x)$ and forming the twistor parameters $\lambda_1, \quad \lambda_2$ which fix a left null plane. The points x^μ and x_0^μ are connected by the left null plane spanned by the null vectors e^1 and e^3.

The point of intersection of this plane with the complex world-line $x_0(\tau)$ gives the value of the "left" retarded time τ_L, which is in fact a complex scalar function on the (complex) space-time $\tau_L(x)$.

By using the null plane equation, one can get a retarded-advanced time equation

$$\tau = t \mp \tilde{r} + \vec{v}\vec{R}. \tag{38}$$

For the stationary Kerr solution $\tilde{r} = r + ia\cos\theta$, and one can see that the second root $Y_2(x)$ corresponds to a transfer to the negative sheet of the metric: $r \to -r$; $\quad \vec{R} \to -\vec{R}$, with a simultaneous complex conjugation $ia \to -ia$.