The moduli space of Special Bohr-Sommerfeld submanifolds

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Mirror Symmetry in the broadest context can be characterized (or even defined?) as a **duality** between

complex geometry | *symplectic geometry*

of Kahler manifolds: from $\mathbb{R}$ - geometry point of view Kahler manifold $= (M, I, \omega)$ where I is a *complex structure* and ω is a Kahler form = *symplectic form*.
Thus every Kahler manifold carries two geometries — complex and symplectic — therefore must be studied from these two different viewpoints.
Main interest: compact algebraic variety, by the very definition admits Kahler form of the Hodge type ($[\omega] \in H^2(M, \mathbb{Z})$), which is not unique of course.
Duality means that for mirror partners M, W certain derivation from complex geometry of M is equivalent to the corresponding derivation from symplectic geometry of W and *vice versa*.
Example: Homological Mirror Symmetry by M. Kontsevich says that derived category of coheret sheaves $D^b(CohM)$ and Fukaya -Floer catogery $FF(W)$ are equivalent.

But if we would like to study geometrical objects:

complex geometry	symplectic geometry
$\Downarrow$	$\Downarrow$
complex submanifolds	symplectic submanifolds
holomorphic vector bundles	lagrangian submanifolds
$\Downarrow$	$\Downarrow$
linear systems	?
(semi) stable vector bundles	?

main difference

finitness	infinitness

Example: Special Lagrangian Geometry for Calabi - Yau manifolds, proposed by N. Hitchin and J. McLean: finite dimemsional moduli space formed by lagrangian submanifolds which satisfy some *speciality* condition.

Main Question: *Is it possible to construct a finite dimensional moduli space of certain "special" lagrangian submanifolds for arbitrary Kahler (or algebraic) variety?*

Possible Solution: *Attach to a very ample divisor* D_α *the space of smooth compact homologically non trivial exact lagrangian submanifolds on the complement* $X\backslash D_\alpha$ *modulo Hamiltonian isotopies on* $X\backslash D_\alpha$*, and then globalizing the attachment over the projective space* $|L_D|$ *get modified moduli space* $\tilde{\mathcal{M}}_{SBS}$*.*

Recall the definition of exact lagrangian submanifolds in the situation we are interested in:
Let (X, L_D) be a smooth compact simply connected algebraic variety, L_D - very ample line bundle. By the very defintion there exists hermitian structure h on L_D s.t. for any $\alpha \in H^0(X, L_D)$ the form $\omega_h = \mathrm{d}(\mathrm{I}(\mathrm{d}\Psi_\alpha))$ is non degenerated on $X\backslash D_\alpha$. Then ω_h is globally defined on X and doesn't depend on α.
Lagrangian $S \subset X\backslash D_\alpha$ is **exact** iff $\mathrm{I}(\mathrm{d}\Psi_\alpha)|_{\mathrm{S}} \equiv 0$.
We present the following alternative

Def. *Lagrangian* $S \subset X\backslash D_\alpha$*is called* D - *exact w.r.t.* $D_\alpha \in |L_D|$ *iff for any loop* $\gamma \subset S$ *for generic disc* $B_\gamma \subset X, \partial B_\gamma = \gamma$*, one has* $\mathrm{ind}(\mathrm{B}_\gamma \cap \mathrm{D}_\alpha) = \int_{\mathrm{B}_\gamma} \omega_{\mathrm{h}}$.

Theorem. *Exactness w.r.t.* $\mathrm{I}(\mathrm{d}\Psi_\alpha) \cong$ D - *exactness w.r.t.* D_α.

Corollary. *For any Hamiltonian deformation* S_t *of a a given exact* $S_0 \subset X \backslash D_\alpha$ *such that* $S_t \cap D_\alpha = \emptyset$, *each* S_t *is exact.*

On the other hand it was proved that any exact lagrangian $K \subset T^*S$ is smoothly homotopic to $S \Rightarrow$

Theorem. *For any exact* $S \subset X \backslash D_\alpha$ *there exists Darboux - Weinstein neiborhood* $\mathcal{O}_{DW}(S) \subset X \backslash D_\alpha$ *which contains no exact lagrangian submanifolds, Hamiltonian non isotopic to* S.

For a given D_α take the space $\mathcal{H}(D_\alpha)$ which consists of all **smooth compact homologically non trivial D - exact lagrangian submanifolds** of $X \backslash D_\alpha$. Define the quotient space

$$\tilde{\mathcal{M}}_{SBS}(D_\alpha) = \mathcal{H}(D_\alpha)/\mathrm{HamIso}(\mathrm{X}\backslash\mathrm{D}_\alpha).$$ Then

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Corollary. *The quotient space is discrete for any* $D_\alpha \in |L_D|$:

$$\tilde{\mathcal{M}}_{SBS}(D_\alpha) = \{< S_1 >, ..., < S_k >, ...\}.$$

Globalize the construction over projective space $|L_D|$:
$p_2 : \tilde{\mathcal{M}}_{SBS}(L_D, h, \mathrm{topS}) \to |\mathrm{L_D}|, \quad \mathrm{p}_2^{-1}(\mathrm{D}_\alpha) = \tilde{\mathcal{M}}_{\mathrm{SBS}}(\mathrm{D}_\alpha).$
We call $\tilde{\mathcal{M}}_{SBS}(L_D, h, \mathrm{topS})$ **the modified moduli space of special Bohr - Sommerfeld cycles**:

Theorem. *The modified moduli space* $\tilde{\mathcal{M}}_{SBS}(L_D, h, \mathrm{topS})$ *is smooth open Kahler manifold of dimension* $h^0(X, L_D) - 1$.

Example: Consider the case $X = \mathbb{CP}^1, L_D = \mathcal{O}(3)$:

$\cdot$ if $D_\alpha = \{p_1, p_2, p_3\} \Rightarrow$ each p_i defines unique class $< S_i >$: take small γ_i around p_i and blow it to have symplectic area 1;

$\cdot$ if $D_\alpha = \{p_1, p_2 = p_3\} \Rightarrow$ it remains class $< S_1 >$ only;

$\cdot$ if $D_\alpha = \{p_1 = p_2 = p_3\} \Rightarrow$ no D - exact loops.

Summing up, $\tilde{\mathcal{M}}_{SBS}(\mathcal{O}(3), S^1) = V \backslash R$, where
$\mathbb{CP}^1 \times \mathbb{CP}^2 \supset V = \{a_0 z_0^3 + a_1 z_0^2 z_1 + a_2 z_0 z_1^2 + a_3 z_1^3 = 0\}$ and
$R \subset V$ is the ramification divisor for $\pi : V \to \mathbb{CP}^3$.
Again the answer = **"algebraic variety** $\setminus$ **ample divisor"**.

Dependence on hermitian structure. Space of **appropriate** hermitian structures on L_D is isomorphic to infinite dimensional open ball: for each pair h_0, h_1 it exists $f \in C^\infty(X, \mathbb{R})$ s.t. $|\alpha|_{h_1} = e^f |\alpha|_{h_0}$ for every $\alpha \in H^0(X, L_D)$ and therefore $\omega_{h_1} = \phi^{t=1}_{\mathrm{grad} f} \omega_{h_0}$ - gradient flow transforms $\omega_{h_0} \mapsto \omega_{h_1}$ for small f. Then $\phi^{t=1}_{\mathrm{grad} f} S_0 = S_1$ - lagrangian w.r.t. ω_{h_1} and still D - exact if $S_t = \phi^{t}_{\mathrm{grad} f} S_0 \cap D_\alpha = \emptyset$ for each $t \in [0:1]$ (possible to manage). Therefore **geometry** of $\tilde{\mathcal{M}}_{SBS}(L_D, h, \mathrm{topS})$ doesn't depend on h.

As a byproduct of the above construction one gets certain **lagrangian invariants** for our algebraic variety X in terms of L_D:

Def. *For a generic appropriate hermitian connection h on very ample line bundle L_D and generic divisor $D_\alpha \in |L_D|$ let $\kappa_1 = \mathrm{Card}(\mathcal{H}(\mathrm{D}_\alpha))$. At the same time it can be done for any positive tensor products L_D^k which gives $\kappa_k = \kappa_1(L_D^k)$.*

These numbers (possible infinite) do not depend on the choice of h $\Rightarrow \mathcal{K}(L_D) = \sum_{k=1}^\infty \kappa_k q^k$ is a **lagrangian** invariant of X.

Example. Again $X = \mathbb{CP}^1, L_D = \mathcal{O}(1)$. As we have seen $\kappa_1 = 0, \quad \kappa_2 = 1, \quad \kappa_3 = 3, \ldots$ and not hard to compute:

$$\mathcal{K}(\mathcal{O}(1)) = \sum_{k=1}^{\infty}(2^{k-1} - 1)q^k$$

(not the same as just *topological* $\mathrm{rkH_1(X \backslash D_\alpha, \mathbb{Z}) = k - 1}$).

Of course, here we preassume the finitness of the numbers $\approx$ the modified moduli space is **algebraic**, as it is in all our examples!

But if it is not the case?

Example. Riemann surface $\Sigma + I \mapsto$ *algebraic curve*, $I \mapsto (G, \Omega), \int_\Sigma \Omega = 2g - 2 \Rightarrow$ hermitian structure on $T^*\Sigma$. Thus $\alpha \in H^0(T^*\Sigma) \mapsto \Psi_\alpha = -\ln|\alpha|$ on $\Sigma \backslash \{p_1, \ldots, p_l\}$ and the graph $\Gamma_\alpha = W(\Sigma \backslash \{p_1, \ldots. p_l\})$ is finite.

Proposition. *Every primitive class from $H_1(\Sigma \backslash \{p_i\}, \mathbb{Z})$ is realized by smooth exact loop $\gamma \subset \Sigma \backslash \{p_i\}$ such that $\int_\gamma I(\mathrm{d}\Psi_\alpha) = 0$. The realization is unique up to Hamiltonian isotopy.*

In this case $\kappa = \infty$, but it is possible to cut off certain *finite component* from the modified moduli space:

To make the story **finite** we can **stabilize** the construction, taking the smooth exact loops which **approach compact closed components** of the **Weinstein skeleton**:
in the situation above for a holomorphic section α the function

$$\Psi_\alpha = -\ln|\alpha|_h$$

is a Kahler potential on $X \backslash D_\alpha$.
The Morse properties of Ψ_α are well known:
the Morse index of any critical point is **less or equal** to n;
the **Weinstein skeleton** of $X \backslash D_\alpha$, defined as the union of **all finite trajectories** of the gradient flow of Ψ_α, is *homotopic* to $X \backslash D_\alpha$ and is *isotropic* at smooth points.

In particular any smooth part of compact n - dimensional component of the Weinstein skeleton $W(X \backslash D_\alpha)$ must be lagrangian! $\Rightarrow$ and we have finite number of such components!

SYNTHESIS. *Cut from the modified moduli space* $\tilde{\mathcal{M}}_{SBS}$ *a stable component* $\tilde{\mathcal{M}}^{st}_{SBS}$ *consists of the classes* $< S_i > \in \mathcal{H}(D_\alpha)$ *which present Hamiltonian desingularizations of the compact closed components of the Weinstein skeleta* $W(X \backslash D_\alpha)$.

Def. *A Hamiltonian desingularization of a cycle* $\Delta \subset W(X \backslash D_\alpha)$ *is a homotopy* $S_t \subset X \backslash D_\alpha, t \in [0:1]$ *such that* $S_0 = \Delta$, *and for other* $t \in (0;1]$ *family* $\{S_t\}$ *is Hamiltonian isotopy of smooth B-S lagrangian submanifolds.*

Note if Δ is smooth itself $\Rightarrow$ one takes $S_t = \Delta$ for every $t \in [0;1]$.

Main Conjecture. *For arbitrary compact smooth simply connected algebraic variety* X *and a very ample line bundle* $L_D \to X$ *the stable component of the modified moduli space is algebraic:* $\tilde{\mathcal{M}}^{st}_{SBS} \cong Y \backslash D$ *where* D *is a compact algebraic variety and* $D \subset Y$ *is an ample divisor.*

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work in progress...