Quantum deformations of $N = 1$, $D = 3$ Lorentz supersymmetry

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Contents

The search for quantum gravity is linked with studies of noncommutative space-times and quantum deformations of space-time symmetries. The considerations of simple dynamical models in quantized gravitational background indicate that the presence of quantum gravity effects generates noncommutativity of $D = 4$ space-time coordinates, and as well the Lie-algebraic space-time symmetries (e.g. Euclidean, Lorentz, Kleinian, quaternionic and their ingomogeneous versions and superextensions) are modified into respective quantum symmetries, described by noncocommutative Hopf algebras, named quantum deformations. Therefore, studing all aspects of the quantum deformations in details is an important issue in the search of quantum gravity models and their superextensions (SUGRA).

For classifications, constructions and applications of quantum Hopf deformations of an universal enveloping algebra $U(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ ($\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$, as a linear space, with a linear $\mathbb{Z}_2$-grading function $\deg(\cdot)$: $\deg(\mathfrak{g}_a)=a\in\{\bar{0},\bar{1}\}$), Lie bisuperalgebras $(\mathfrak{g},\delta)$ play an essential role, where the *cobracket* δ is a linear superskew-symmetric map:

$$\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}=\mathfrak{g}_{\bar{0}}\wedge\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}\wedge\mathfrak{g}_{\bar{1}}\oplus\mathfrak{g}_{\bar{0}}\wedge\mathfrak{g}_{\bar{1}}, \tag{1}$$

which conserves the grading function $\deg(\cdot)$:

$$\delta(\mathfrak{g}_{\bar{0}})\ \in\ \mathfrak{g}_{\bar{0}}\wedge\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}\wedge\mathfrak{g}_{\bar{1}},\qquad \delta(\mathfrak{g}_{\bar{1}})\ \in\ \mathfrak{g}_{\bar{0}}\wedge\mathfrak{g}_{\bar{1}}. \tag{2}$$

provided that

$$\deg(\mathfrak{g}_a\wedge\mathfrak{g}_b)=\deg(\mathfrak{g}_a)+\deg(\mathfrak{g}_b)=a+b=c\mod 2,\quad a,b,c\in\{\bar{0},\bar{1}\}. \tag{3}$$

For general element $x=x_{\bar{0}}+x_{\bar{1}}$ we use the linearity of $\delta(x)$ in the argument x. It should be noted also that the superskew-symmetric bilinear space $\mathfrak{g}\wedge\mathfrak{g}$ is defined as follows:

$$x_a\wedge y_b:=x_a\otimes y_b-(-1)^{ab}y_b\otimes x_a,\quad \forall x_a\in g_a,\ \forall y_b\in g_b,\quad a,b\in\{\bar{0},\bar{1}\} \tag{4}$$

and for general elements $x=x_{\bar{0}}+x_{\bar{1}}$ and $y=y_{\bar{0}}+y_{\bar{1}}$ we use the bilinearity of $x\wedge y$.

Moreover the cobracket δ satisfies the relations consisted with the superbracket $[\![\cdot,\cdot]\!]$ in the Lie superalgebra $\mathfrak{g}$:

$$\begin{aligned} &\delta([\![x,y]\!]) = [\![\delta(x),\Delta_0(y)]\!] + [\![\Delta_0(x),\delta(y)]\!], \quad x,y\in\mathfrak{g}, \\ &(\delta\otimes\mathrm{id})\delta(x_a) + (-1)^{\phi}\mathrm{cycle} = 0, \quad \forall x_a\in g_a, \;\; a\in\{\bar{0},\bar{1}\}, \end{aligned} \tag{5}$$

where $\Delta_0(\cdot)$ is a trivial (non-deformed) coproduct

$$\Delta_0(x) = x\otimes 1 + 1\otimes x, \tag{6}$$

The first relation in (5) is a condition of the 1-cocycle and the second one is the co-Jacobi identity. The Lie bisuperalgebra $(\mathfrak{g},\delta)$ is a correct infinitesimalization of the quantum Hopf deformation of $U(\mathfrak{g})$ and the operation δ is an infinitesimal part of difference between a coproduct Δ and an oposite coproduct $\tilde{\Delta}$ in the Hopf algebra, $\delta(x) = h^{-1}(\Delta-\tilde{\Delta}) \mod h$, where h is a deformation parameter. Any two Lie bialgebras $(\mathfrak{g},\delta)$ and $(\mathfrak{g},\delta')$ are isomorphic (equivalent) if they are connected by a $\mathfrak{g}$-automorphism φ satisfying the condition

$$\delta(x) = (\varphi\otimes\varphi)\delta'(\varphi^{-1}(x)) \tag{7}$$

for any $x\in\mathfrak{g}$.

Of our special interest here are the quasitriangle Lie bisuperalgebras $(\mathfrak{g},\delta_{(r)}){:=}(\mathfrak{g},\delta,r)$, where the cobracket $\delta_{(r)}$ is given by the classical r-matrix $r\in\mathfrak{g}\wedge\mathfrak{g}$ as follows:

$$\delta_{(r)}(x)=[\![r,\Delta_0(x)]\!]. \tag{8}$$

Because the co-bracket $\delta_{(r)}$ conserves the grading then from (8) we see that the r-matrix r is even, $\deg(r)=\bar{0}$, i.e.:

$$r\ \in\ \mathfrak{g}_{\bar{0}}\wedge\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}\wedge\mathfrak{g}_{\bar{1}}. \tag{9}$$

Moreover it is easy to see also from (7) and (8) that *two quasitriangular Lie bisuperalgebras $(\mathfrak{g},\delta_{(r)})$ and $(\mathfrak{g},\delta_{(r')})$ are isomorphic iff the classical r-matrices r and r' are isomorphic, i.e.:*

$$(\varphi\otimes\varphi)r'=r. \tag{10}$$

Therefore for a classification of all nonequivalent quasitriangular Lie bisuperalgebras $(\mathfrak{g},\delta_{(r)})$ of the given Lie superalgebra $\mathfrak{g}$ we need to find all nonequivalent (nonisomorphic) classical r-matrices. Because nonequivalent quasitriangular Lie bisuperalgebras uniquely determine non-equivalent quasitriangular quantum deformations (Hopf algebras) of $U(\mathfrak{g})$ therefore the classification of all nonequivalent quasitriangular Hopf superalgebras is reduced to the classification of all nonequivalent classical r-matrices.

Let $\mathfrak{g}^* := (\mathfrak{g}, *)$ be a real form of a classical complex Lie superalgebra $\mathfrak{g}$, where $*$ is an antilinear involutive (semiinvolutive) antiautomorphism of $\mathfrak{g}$, then *the bisuperalgebra* $(\mathfrak{g}^*, \delta_{(r)})$ *is real iff the classical* r*-matrix* r *is* $*$*-anti-real* *(*$*$*-anti-Hermitian).*[1] Indeed, the condition of $*$-reality for the bisuperalgebra $(\mathfrak{g}^*, \delta)$ means that

$$\delta(x)^{*\otimes *} = \delta(x^*). \tag{11}$$

Applying this condition to the relations (8) we abtain that

$$r^{*\otimes *} = -r, \tag{12}$$

i.e. the r-matrix r is $*$-anti-Hermitian.

[1] All bialgebras over the simple complex and real Lie algebras are quasitriangular, due to Whitehead lemma (see e.g. N. Jacobson, *"Lie algebras", Dover Publications, Inc., New York* (1979)). It is more likely that the Whitehead lemma is valid also for all classical Lie superalgebras.

Quite recetly in the paper: J. Lukierski, V.N. Tolstoy, Quantizations of D=3 Lorentz symmetry, Eur. Phys. J. **C77** (2017) 226 we investigate the quantum deformations of the complex Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ and its real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2;\mathbb{R})$. Namely, firstly we obtain the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices (bialgebras) for all these Lie algebras and then Hopf deformations corresponding to these bialgebras were presented in explicite form. In particular, it was shown that $D=3$ Lorentz symmetry $\mathfrak{o}(2,1)\,(\simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2;\mathbb{R}))$ has two standard q-deformations and one Jordanian. In this talk I would like to present some super-analog of these results, namely I first give the the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices for complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ (which is a minimal supersymmetric extension of the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$) and its real forms $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$. In particular, we show that $N=1$, $D=4$ Lorentz supersymmetry,

$$\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R})), \qquad (13)$$

has two nonequivalents tandard q-deformations and two nonequivalent Jordanian and super-Jordanian deformations. The isomorphic Lie superalgebras (13) and their quantum deformations play very important role in physics as well as in mathematical considerations, so the structure of these deformations should be understood with full clarity. It should be noted also that the importance of $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ and its deformations follows also from the unique role of the $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ algebra as the lowest-dimensional rank one noncompact simple classical Lie superalgebra.

The complex orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C}) = \mathfrak{osp}_0(1|2;\mathbb{C}) \oplus \mathfrak{osp}_1(1|2;\mathbb{C})$ is generated by three even Cartan-Weyl (CW) generators: $H, E_\pm \in \mathfrak{osp}_0(1|2;\mathbb{C}) \simeq \mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$, and two odd CW generators: $v_\pm \in \mathfrak{osp}_1(1|2;\mathbb{C})$ with the defining relations:

$$\begin{gathered}
[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = 2H, \\
[H, v_\pm] = \pm\frac{1}{2}v_\pm, \quad [E_\pm, v_\mp] = v_\pm, \quad [E_\pm, v_\pm] = 0, \\
\{v_\pm, v_\pm\} = \pm\frac{1}{2}E_\pm, \quad \{v_+, v_-\} = -\frac{1}{2}H.
\end{gathered} \tag{14}$$

The the CW generators $H, E_\pm$ of $\mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$ is related with the *Cartesian* basis I_i $(i = 1, 2, 3)$ as follows:

$$H = \imath I_3, \quad E_\pm = \imath I_1 \mp I_2. \tag{15}$$

For convenience we set also

$$v_1 := v_+, \quad v_2 := v_-. \tag{16}$$

In therms of the generators $\{I_i, v_\alpha | i = 1, 2, 3; \alpha = 1, 2\}$ the defining relations (14) take the form:

$$\begin{gathered}
[I_i, I_j] = \varepsilon_{ijk} I_k, \quad [I_i, v_\alpha] = -\frac{\imath}{2}(\sigma_i)_{\beta\alpha} v_\beta, \\
\{v_1, v_1\} = \frac{1}{2}(\imath I_1 - I_2), \quad \{v_2, v_2\} = -\frac{1}{2}(\imath I_1 + I_2), \quad \{v_1, v_2\} = -\frac{\imath}{2} I_3,
\end{gathered} \tag{17}$$

where σ_i, $(i = 1, 2, 3)$ are the 2×2 Pauli matrices, and $(\alpha, \beta = 1, 2)$.

It is well known that the Lie algebra $\mathfrak{o}(3;\mathbb{C})\simeq\mathfrak{sl}(2;\mathbb{C})$, which is a subalgebra of the superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$, has two real forms: compact $\mathfrak{o}(3)\simeq\mathfrak{su}(2)$, and noncompact $\mathfrak{o}(2,1)\simeq\mathfrak{sl}(2;\mathbb{R})\simeq\mathfrak{su}(1,1)$. These real form of the subalgebra $\mathfrak{o}(3;\mathbb{C})\simeq\mathfrak{sl}(2;\mathbb{C})$ are raised up to the odd part $\mathfrak{osp}_{\mathbf{1}}(1|2;\mathbb{C})$, that is to the whole superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$.

I. *The compact pseudoreal form* $\mathfrak{osp}^*(1|\mathfrak{o}(3))\simeq\mathfrak{osp}^*(1|\mathfrak{su}(2))$.
In therms of the generators $\{I_i, v_\alpha | i=1,2,3;\alpha=1,2\}$ this form is defined by the following conjugation:

$$I_i^* \;=\; -I_i,\qquad v_1^* \;=\; k_* v_2,\quad v_2^* \;=\; -k_* v_1, \tag{18}$$

where $k_*=1$ if the conjugation $(^*)$ respect the grading of the Lie superbracket, i.e.

$$[\![x_a,x_b]\!]^* \;=\; (-1)^{ab}[\![x_b^*,x_a^*]\!] \quad (\text{graded}) \tag{19}$$

and $k_*=\imath$ if

$$[\![x_a,x_b]\!]^* = [\![x_b^*,x_a^*]\!] \quad (\text{ungraded}) \tag{20}$$

for all homogeneous elements $x_a\in\mathfrak{g}_a$, $x_b\in\mathfrak{g}_b$ ($\mathfrak{g}:=\mathfrak{osp}(1|2;\mathbb{C})$). We see that the conjugation $(^*)$ prolonged to the odd generators v_α is an antilinear antiautomorphism of four order provided that $(v_\alpha^*)^*=-v_\alpha$.[2] Therefore this form is called *pseudoreal* and in terms of the Cartesian generators (18) it is denoted by $\mathfrak{osp}^*(1|\mathfrak{o}(3))$. In therms of the CW generators $H:=\imath I_3, E_\pm:=\imath I_1\mp I_1, v_+:=v_1, v_-:=v_2$ this pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ denoted also by $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, is given as follows

$$H^* \;=\; H,\quad E_\pm^* \;=\; E_\mp,\quad v_\pm^* \;=\; \pm k_* v_\mp. \tag{21}$$

[2] The condugation $j:={}^*$ together with the imaginary unit $\imath$ and the antilinear map $k:=\imath j$, equips the odd component $\mathfrak{osp}_{\mathbf{1}}^*(1|\mathfrak{su}(2))$ with the structure of a quaternionic vector space: $j^2=\imath^2=k^2=-1$.

II. *The noncompact real form* $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R})) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$.
In therms of the generators $\{I_i, v_\alpha | i = 1,2,3; \alpha = 1,2\}$ this form is defined by the following conjugation:

$$I_i^\dagger \;=\; (-1)^{i-1} I_i, \quad v_1^\dagger \;=\; k_\dagger v_1 \quad v_2^\dagger \;=\; k_\dagger v_2, \tag{22}$$

where $k_\dagger = 1$ if $(^\dagger)$ is the graded conjugation, i.e.

$$[\![x_a, x_b]\!]^\dagger \;=\; (-1)^{ab}[\![x_b^\dagger, x_a^\dagger]\!] \quad \text{(graded)} \tag{23}$$

and $k_\dagger = \imath$ if

$$[\![x_a, x_b]\!]^\dagger = [\![x_b^\dagger, x_a^\dagger]\!] \quad \text{(ungraded)} \tag{24}$$

for all homogeneous elements $x_a \in \mathfrak{g}_a$, $x_b \in \mathfrak{g}_b$ ($\mathfrak{g} := \mathfrak{osp}(1|2;\mathbb{C})$). We see that the conjugation $(^\dagger)$ prolonged to the odd generators v_α is an antilinear antiautomorphism of second order, that is $(v_\alpha^*)^* = v_\alpha$. therefore this form is called *real*, and it is denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$.
If we introduce the CW generators $H := \imath I_3, E_\pm := \imath I_1 \mp I_2, v_+ := v_1, v_- := v_2$, where the Cartesian generators $\{I_i, v_\alpha | i = 1,2,3; \alpha = 1,2\}$ satisfy the conjugation (22), then the real condition is given as follows

$$H^\dagger \;=\; -H, \quad E_\pm^\dagger \;=\; -E_\pm, \quad v_\pm^\dagger \;=\; k_\dagger v_\pm. \tag{25}$$

In terms of the given CW basis the real form $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)$ is also denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$.

We can also introduce an alternative CW basis $H', E'_{\pm}, v'_{\pm}$ in $\mathfrak{osp}^\dagger(1|\mathfrak{so}(2,1))$ which are expressed in terms of the Cartesian generators $I_i, v_\alpha (i=1,2,3; \alpha=1,2)$ and the CW generators $H, E_{\pm}, v_{\pm}$ as follows:

$$\begin{aligned} H' &= \imath I_2 = -\frac{\imath}{2}(E_+ - E_-), \\ E'_{\pm} &= \imath I_1 \pm I_3 = \mp\imath H + \frac{1}{2}(E_+ + E_-), \\ v'_+ &= \frac{1}{\sqrt{2}}(v_+ + \imath v_-), \quad v'_- = \frac{1}{\sqrt{2}}(\imath v_+ + v_-), \end{aligned} \tag{26}$$

The CW basis $H', E'_{\pm}, v'_{\pm}$ satisfy the defining relations of $\mathfrak{osp}(1|2;\mathbb{C})$, and it has the conjugation properties;

$$(H')^\dagger = H, \quad (E'_{\pm})^\dagger = -E'_{\mp}, \quad (v'_{\pm})^\dagger = -\imath k_\dagger v'_{\mp}. \tag{27}$$

The real superalgebra $\mathfrak{osp}^\dagger(1|\mathfrak{so}(2,1))$ in terms of the CW basis $H', E'_{\pm}, v'_{\pm}$ will be also denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$.
It should be noted that in the case of $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ the Cartan generator H' is compact while for the case $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2,\mathbb{R}))$ the Cartan generator H is noncompact.

In this section we obtain complete classification bialgebras (classical r-matrices) for the complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and its real forms $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ using the isomorphisms: $\mathfrak{osp}(1|\mathfrak{o}(3;\mathbb{C}))\simeq\mathfrak{osp}(1|\mathfrak{sl}(2;\mathbb{C}))$, $\mathfrak{osp}^*(1|\mathfrak{o}(3))\simeq\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))\simeq\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))\simeq\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$. In particular, we explicitly find out an isomorphism between $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ bialgebras and fix on the basis $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ bialgebra in such forms which are convenient for quantizations.

By the definition any classical r-matrix of arbitrary complex or real Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$, $r\in\mathfrak{g}_{\bar 0}\wedge\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}\wedge\mathfrak{g}_{\bar 1}$, satisfy the classical Yang-Baxter equation (CYBE):

$$[r,\,r]_S \;=\; \tilde\Omega\,. \tag{1}$$

Here $[\cdot,\cdot]_S$ is the Schouten bracket which for any monomial skew-symmetric even two-tensors $r_1=x_a\wedge y_a$ and $r_2=u_b\wedge v_b$ ($x_a,y_a\in\mathfrak{g}_a$; $u_b,v_b\in\mathfrak{g}_b$; $a,b,\in\{\bar 0,\bar 1\}$) is given by

$$\begin{aligned}[x_a\wedge y_a,\,u_b\wedge v_b]_S \;:=\;& x_a\wedge\big([\![y_a,u_b]\!]\wedge v_b+(-1)^{ab}u_b\wedge[\![y_a,v_b]\!]\big)\\ &-(-1)^a y_a\wedge\big([\![x_a,u_b]\!]\wedge v_b+(-1)^{ab}u_b\wedge[\![x_a,v_b]\!]\big)\\ =\;&[u_b\wedge v_b,\,x_a\wedge y_a]_S\end{aligned} \tag{2}$$

and $\tilde\Omega$ is the $\mathfrak{g}$-invariant element, $\tilde\Omega\in(\overset{3}{\wedge}\mathfrak{g})_{\mathfrak{g}}$, that in the case of $\mathfrak{g}:=\mathfrak{osp}(1|2;\mathbb{C})$ looks as follows:

$$\begin{aligned}\tilde\Omega\;=\;\gamma\,\Omega(\mathfrak{osp}(1|2;\mathbb{C}))\;=\;&\gamma 2(2E_-\wedge H\wedge E_+-2v_-\wedge H\wedge v_+\\ &+v_-\wedge v_-\wedge E_+-E_-\wedge v_+\wedge v_+),\end{aligned} \tag{3}$$

where $\gamma\in\mathbb{C}$.

We have already mentioned that a classical r-matrix r is an even two-tensor, i.e.:

$$r \;\in\; V_{\bar{0}} := \mathfrak{osp}_{\bar{0}}(1|2;\mathbb{C}) \wedge \mathfrak{osp}_{\bar{0}}(1|2;\mathbb{C}) \oplus \wedge\mathfrak{osp}_{\bar{1}}(1|2;\mathbb{C}) \wedge \mathfrak{osp}_{\bar{1}}(1|2;\mathbb{C}). \tag{4}$$

As a basis in the linear space V_0 we can take the following two-tensors:

$$\begin{aligned} r_0 \;&:=\; E_+ \wedge E_- + 2v_+ \wedge v_-\,, \quad r_\pm \;:=\; \pm E_\pm \wedge H \pm v_\pm \wedge v_\pm\,, \\ \tilde{r}_0 \;&:=\; E_+ \wedge E_-\,, \quad \tilde{r}_\pm \;:=\; \pm E_\pm \wedge H. \end{aligned} \tag{5}$$

Theorem

The following statements are valid:
(i) Any linear combination of the elents $r_0, r_\pm$ is a classical r-matrix, nemaly, if

$$r \;\;:=\;\; \beta_+ r_+ + \beta_0 r_0 + \beta_- r_- \quad (\forall\, \beta_+, \beta_0, \beta_- \in \mathbb{C}), \tag{6}$$

then we have

$$[r,r]_S \;=\; (\beta_0^2 + \beta_+\beta_-)\Omega \;\equiv\; \gamma\Omega. \tag{7}$$

(ii) A linear combination of the elements $\tilde{r}_0, \tilde{r}_\pm$:

$$\tilde{r} \;\;:=\;\; \beta_+\tilde{r}_+ + \beta_0\tilde{r}_0 + \beta_-\tilde{r}_- \quad \text{for } \beta_0^2 + \beta_+\beta_- = 0, \tag{8}$$

satisfy the homogeneous CYBE, i.e. $[\tilde{r},\tilde{r}]_S = 0$.
(iii) Any classical r-matrix is presented only in the form (18) or (19).

There are two types of explicite $\mathfrak{osp}(1|2;\mathbb{C})$-automorphisms. First type connecting the classical r-matrices with zero γ-characteristic is given by the formulas:

$$\begin{aligned}
\varphi_0(E_+) &= \chi(\tilde{\beta}_+ E_+ - 2\tilde{\beta}_0\, H + \tilde{\beta}_- E_-),\\
\varphi_0(E_-) &= \chi^{-1}(\tilde{\beta}_- E_+ - 2\kappa\tilde{\beta}_0\, H + \tilde{\beta}_+ E_-),\\
\varphi_0(H) &= \tilde{\beta}_0\, E_+ + (\kappa\tilde{\beta}_+ + \tilde{\beta}_-)\, H + \kappa\tilde{\beta}_0\, E_-\,,\\
\varphi_0(v_+) &= \sqrt{\chi}\Big(\sqrt{\tilde{\beta}_+}\, v_+ + \sqrt{\tilde{\beta}_-}\, v_-\Big),\\
\varphi_0(v_-) &= \sqrt{\chi^{-1}}\Big(\sqrt{\tilde{\beta}_-}\, v_+ + \sqrt{\tilde{\beta}_+}\, v_-\Big),
\end{aligned} \tag{9}$$

where χ is a non-zero rescaling parameter (including $\chi = 1$), κ takes two values $+1$ or -1, and the parameters $\tilde{\beta}_i$ $(i = +, 0, -)$ satisfy the conditions:

$$\gamma \;:=\; \tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- \;=\; 0, \qquad \kappa\tilde{\beta}_+ - \tilde{\beta}_- = 1. \tag{10}$$

Let us consider two general r-matrices with zero γ-characteristics:

$$\begin{aligned}
r &:= \beta_+ r_+ + \beta_0 r_0 + \beta_- r_-\,,\\
r' &:= \beta'_+ r_+ + \beta'_0 r_0 + \beta'_- r_-\,,
\end{aligned} \tag{11}$$

where $\gamma = \beta_0^2 + \beta_+\beta_- = 0$ and $\gamma = \beta_0'^2 + \beta'_+\beta'_- = 0$. Moreover, we suppose that the parameters $\beta_\pm$ and $\beta'_\pm$ satisfy the additional relations:

$$\kappa\beta_+ - \beta_- = \chi\beta'_+ - \chi^{-1}\kappa\beta'_- \neq 0, \tag{12}$$

where the parameters κ and χ are the same as in (9).

One can check that the following formula is valid:

$$r \;=\; (\varphi_0\otimes\varphi_0)r', \qquad (13)$$

i.e.

$$\beta_+r_+ + \beta_0r_0 + \beta_-r_- = \beta'_+(\varphi_0\otimes\varphi_0)r_+ + \beta'_0(\varphi_0\otimes\varphi_0)r_0 + \beta'_-(\varphi_0\otimes\varphi_0)r_-, \qquad (14)$$

where φ_0 is the $\mathfrak{osp}(1|2;\mathbb{C})$-automorphism (9) with the following parameters:

$$\begin{aligned}
\tilde{\beta}_0 &= \frac{\beta_0(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-) - \beta'_0(\kappa\beta_+ + \beta_-)}{(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)}\,, \\
\tilde{\beta}_+ &= \frac{\kappa(\kappa\beta_+ + \beta_-)(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-) + 4\beta_0\beta'_0}{2(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)} + \frac{\kappa}{2}\,, \qquad (15)\\
\tilde{\beta}_- &= \frac{(\kappa\beta_+ + \beta_-)(\chi\beta'_+ + \chi^{-1}\kappa\beta'_-) + 4\kappa\beta_0\beta'_0}{2(\kappa\beta_+ - \beta_-)(\chi\beta'_+ - \chi^{-1}\kappa\beta'_-)} - \frac{1}{2}\,.
\end{aligned}$$

It is easy to check that expected as the formulas (15) satisfy the conditions $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.
Let us assume in (14) that the parameters β'_0 and β'_- are equal to zero. Then the general classical r-matrix r, satisfying the homogeneous CYBE, is reduced to usual Jordanian form by the authomorphism φ_0 with the parameters:

$$\tilde{\beta}_0 = \frac{\beta_0}{\kappa\beta_+ - \beta_-}\,, \quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\kappa\beta_+ - \beta_-}\,. \qquad (16)$$

Second type of $\mathfrak{osp}(1|2;\mathbb{C})$-automorphism connecting the classical r-matrices with non-zero γ-characteristic is given as follows

$$\begin{aligned}
\varphi_1(E_+) &= \frac{\chi}{2}\Big((\tilde{\beta}_0+1)\,E_+ + 2\tilde{\beta}_- H - \frac{\tilde{\beta}_-^2}{\tilde{\beta}_0+1}\,E_-\Big),\\
\varphi_1(E_-) &= \frac{\chi^{-1}}{2}\Big(\frac{-\tilde{\beta}_+^2}{\tilde{\beta}_0+1}\,E_+ + 2\tilde{\beta}_+ H + (\tilde{\beta}_0+1)E_-\Big),\\
\varphi_1(H) &= \frac{1}{2}(-\tilde{\beta}_+ E_+ + 2\tilde{\beta}_0 H - \tilde{\beta}_- E_-),\\
\varphi_1(v_+) &= \sqrt{\frac{\chi}{2}}\Big(\sqrt{\tilde{\beta}_0+1}\,v_+ + \frac{\tilde{\beta}_-}{\sqrt{\beta_0+1}}\,v_-\Big),\\
\varphi_1(v_-) &= \sqrt{\frac{\chi^{-1}}{2}}\Big(\frac{\tilde{\beta}_+}{\sqrt{\beta_0+1}}\,v_+ + \sqrt{\tilde{\beta}_0+1}\,v_-\Big),
\end{aligned} \tag{17}$$

where χ is a non-zero rescaling parameter, and $\tilde{\beta}_0^2+\tilde{\beta}_+\tilde{\beta}_-=1$.
Let us consider two general r-matrices with non-zero γ-characteristics:

$$\begin{aligned}
r &:= \beta_+ r_+ + \beta_0 r_0 + \beta_- r_-,\\
r' &:= \beta'_+ r_+ + \beta'_0 r_0 + \beta'_- r_-,
\end{aligned} \tag{18}$$

where the parameters $\beta_\pm$, β_0 and $\beta'_\pm$, β'_0 can be equal to zero provided that $\gamma=\beta_0^2+\beta_+\beta_-=\gamma'=(\beta'_0)^2+\beta'_+\beta'_-\neq 0$, i.e. both r-matrices r and r' have the same non-zero γ-characteristic $\gamma=\gamma'\neq 0$.

One can check the following relation:

$$r \;=\; (\varphi_1\otimes\varphi_1)r', \tag{19}$$

i.e.

$$\beta_+ r_+ + \beta_0 r_0 + \beta_- r_- = \beta'_+(\varphi_1\otimes\varphi_1)r_+ + \beta'_0(\varphi_1\otimes\varphi_1)r_0 + \beta'_-(\varphi_1\otimes\varphi_1)r_-\,, \tag{20}$$

where φ_1 is the $\mathfrak{osp}(1|2;\mathbb{C})$-automorphism (17) with the parameters:

$$\begin{aligned}\tilde{\beta}_0 &= \frac{(\beta_0+\beta'_0)^2-(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}{(\beta_0+\beta'_0)^2+(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}\,,\\ \tilde{\beta}_\pm &= \frac{2(\beta_0+\beta'_0)(\beta_\pm-\chi^{\pm1}\beta'_\pm)}{(\beta_0+\beta'_0)^2+(\beta_+-\chi\beta'_+)(\beta_- -\chi^{-1}\beta'_-)}\,.\end{aligned} \tag{21}$$

It is easy to check that the formulas (21) satisfy the condition $\tilde{\beta}_0^2+\tilde{\beta}_+\tilde{\beta}_- = 1$.
If we assume in (20) that the parameters $\beta'_\pm$ are equal to zero then the general classical r-matrix r, satisfying the non-homogeneous CYBE, is reduced to the usual standard form by the automorphism φ_1, (17), with the following parameters:

$$\tilde{\beta}_0 = \frac{\beta_0}{\beta'_0}\,,\quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\beta'_0}\,. \tag{22}$$

Finally for $\mathfrak{osp}(2,\mathbb{C})$ we obtain the well-known result:

Theorem

For the complex Lie algebra $\mathfrak{osp}(1|2,\mathbb{C})$ there exists up to $\mathfrak{osp}(1|2,\mathbb{C})$ automorphisms three solutions of CYBE, namely Jordanian r_J, super-Jordanian r_{sJ} and standard $r_J r_{st}$:

$$\begin{aligned} r_J &= \beta E_+ \wedge H, & [r_J, r_J]_S &= 0, \\ r_{sJ} &= \beta_1(E_+ \wedge H + v_+ \wedge v_+), & [r_{sJ}, r_{sJ}]_S &= 0, \\ r_{st} &= \beta_0(E_+ \wedge E_- + 2v_+ \wedge v_-), & [r_{st}, r_{st}]_S &= \beta_0^2\Omega, \end{aligned} \tag{23}$$

where the complex parameters β and β_1 can be removed by the rescaling $\mathfrak{osp}(1|2,\mathbb{C})$-automorphism: $\varphi(E_+) = \beta^{-1}E_+$, $\varphi(E_-) = \beta E_-$, $\varphi(v_+) = \sqrt{\beta^{-1}}\, v_+$, $\varphi(v_-) = \sqrt{\beta}\, v_-$, $\varphi(H) = H$; the parameter $\beta_0 = e^{\imath\phi}|\beta_0|$ for $|\phi| \le \frac{\pi}{2}$ is effective.

The general non-reduced expression of the r-matrix r is convenient for the application of reality conditions:

$$r^{\circledast\otimes\circledast} \;:=\; \beta_+^* r_+^{\circledast\otimes\circledast} + \beta_0^* r_0^{\circledast\otimes\circledast} \wedge + \beta_-^* r_-^{\circledast\otimes\circledast} = -r, \tag{24}$$

where $\circledast$ is the conjugation associated with corresponding real form ($\circledast = *, \dagger$), and β_i^* ($i = +, 0, -$) means the complex conjugation of the number β_i. Moreover, if r-matrix is *anti-real* (anti-Hermitian), i.e. it satisfies the condition (24), then its γ-characteristic is real. Indeed, applying the conjugation $\circledast$ to CYBE we have for the left-side: $[[r,r]]^\circledast = -[[r^\circledast, r^\circledast]] = -[[r,r]]$ and for the right-side: $(\gamma\Omega)^\circledast = -\gamma^*\Omega$ for all real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, $\mathfrak{su}(2;\mathbb{R})$. It follows that the parameter γ is real, $\gamma^* = \gamma$.

Theorem

(i) For the compact pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ with graded conjugation $(^*)$, $k_* = 1$, there exists up to the $\mathfrak{osp}^*(1|\mathfrak{su}(2))$-automorphisms only one solution of CYBE and this solution is the usual standard supersymmetric classical r-matrix r_{st}:

$$r_{st} \;:=\; \alpha(E_+ \wedge E_- + 2v_+ \wedge v_-), \quad [[r_{st}, r_{st}]] \;=\; \gamma\Omega, \tag{25}$$

where the effective parameter α is a positive number, and $\gamma = \alpha^2$.

(ii) For the compact pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ with non-graded conjugation $(^*)$, $k_* = \imath$, there does not exists any solution of CYBE.

Theorem

(i) For the non-compact real form $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ *with graded conjugation* $(^\dagger)$, $k_\dagger = 1$*,there exists up to* $\mathfrak{osp}^\dagger 1|\mathfrak{sl}(2;\mathbb{R}))$*-automorphisms four solutions of CYBE, namely Jordanian* r_J*, super-Jordanian* r_{sJ}*, standard* r_{st} *and quasi-standard* r_{qst}*:*

$$\begin{aligned}
r_J &= \imath\beta E_+ \wedge H, \qquad [r_J, r_J]_S = 0,\\
r_{sJ} &= \imath\beta(E_+ \wedge H + v_+ \wedge v_+), \quad [r_{sJ}, r_{sJ}]_S = 0,\\
r_{st} &= \imath\alpha(E_+ \wedge E_- + 2v_+ \wedge v_-), \quad [r_{st}, r_{st}]_S = -\alpha^2\Omega,\\
r_{qst} &= \alpha(E_+ \wedge H + v_+ \wedge v_+ - H \wedge E_- - v_- \wedge v_-), \quad [r_{qst}, r_{qst}]_S = \alpha^2\Omega,
\end{aligned} \tag{26}$$

where α *effectively is a positive number.*

(ii) For the non-compact real form $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ *with non-graded conjugation* $(^\dagger)$, $k_\dagger = \imath$*, there exists only one solution of CYBE and this solution is the usual Jordanian classical r-matrix*

$$r_J = \imath\beta E_+ \wedge H, \qquad [r_J, r_J]_S = 0, \tag{27}$$

Theorem

(i) For the non-compact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ *with graded conjugation* $(^{\dagger})$, $k_{\dagger}=\imath$, *there exists up to* $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$*-automorphisms four solutions of CYBE, namely quasi-Jordanian* r_{qJ}, *quasi-super-Jordanian* r_{qsJ}, *standard* r_{st} *and quasi-standard* r_{qst}*:*

$$\begin{aligned}
r'_{qJ} &= \imath\beta E'_{+}\wedge H'-H'\wedge E'_{-}), \qquad [r'_{J},r'_{J}]_{S} = 0,\\
r'_{qsJ} &= \beta(E'_{+}\wedge H'+v'_{+}\wedge v'_{+}-H'\wedge E'_{-}-v'_{-}\wedge v'_{-}), \quad [r'_{sJ},r'_{sJ}]_{S}=0,\\
r'_{st} &= \alpha(E'_{+}\wedge E'_{-}+2v'_{+}\wedge v'_{-}), \quad [r'_{st},r'_{st}]_{S} = \alpha^{2}\Omega,\\
r'_{qst}, &= \imath\alpha(E'_{+}\wedge H'+v'_{+}\wedge v'_{+}+H'\wedge E'_{-}+v'_{-}\wedge v'_{-}), \;\; [r'_{qst},r'_{qst}]_{S}=-\alpha^{2}\Omega,
\end{aligned} \tag{28}$$

where β and α *are positive numbers.*

(ii) For the non-compact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1.1))$ *with non-graded conjugation* $(^{\dagger})$, $k_{\dagger}=1$, *there exists only one solution of CYBE and this solution is the usual quasi-Jordanian classical r-matrix* r_{qJ}*:*

$$r_{qJ}=\beta(E'_{+}\wedge H'-H'\wedge E'_{-}), \qquad [r'_{qJ},r'_{qJ}]_{S}=0, \tag{29}$$

where β *is positive number.*

Using the formulas of connections between CW and the Cartesian bases we can express the classical $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ r-matrices in terms of the $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ Cartesian basis. We get the following results.

1. The $\mathfrak{osp}^{\dagger}(1|;\mathbb{R}))$ case.

(i) For the non-compact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ with graded conjugation:

$$\begin{aligned} r_J &= \imath\beta E_+ \wedge H = -\beta(\imath I_1 - I_2)\wedge I_3, \\ [r_J, r_J]_S &= 0, \end{aligned} \tag{1}$$

$$\begin{aligned} r_{sJ} &= \imath\beta(E_+ \wedge H + v_+ \wedge v_+) = -\beta(\imath I_1 - I_2)\wedge I_3 + v_1 \wedge v_1), \\ [r_{sJ}, r_{sJ}]_S &= 0, \end{aligned} \tag{2}$$

$$\begin{aligned} r_{st} &= \imath\alpha(E_+ \wedge E_- + 2v_+ \wedge v_-) = -2\alpha(I_1 \wedge I_2 + 2v_1 \wedge v_2), \\ [r_{st}, r_{st}]_S &= -\alpha^2\Omega, \end{aligned} \tag{3}$$

$$\begin{aligned} r_{qst} &= \alpha(E_+ \wedge H + v_+ \wedge v_+ - H \wedge E_- - v_- \wedge v_-) \\ &= -2\imath\alpha(I_1 \wedge I_3 + v_1 \wedge v_1 - v_2 \wedge v_2), \\ [r_{qst}, r_{qst}]_S &= \alpha^2\Omega, \end{aligned} \tag{4}$$

(ii) For the non-compact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ with non-graded conjugation:

$$\begin{aligned} r_J &= \imath\beta E_+ \wedge H = -\beta(\imath I_1 - I_2)\wedge I_3, \\ [r_J, r_J]_S &= 0, \end{aligned} \tag{5}$$

2. The $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1.1))$ case.

(i) For the noncompact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ with graded conjugation:

$$\begin{aligned} r'_{qJ} &= \imath\beta(E'_{+}+E'_{-})\wedge H' = -\beta(\imath I_1 - I_2)\wedge I_3, \\ [r'_{qJ}, r'_{qJ}]_S &= 0, \end{aligned} \tag{6}$$

$$\begin{aligned} r'_{qsJ} &= \imath\beta(E'_{+}\wedge H' + v'_{+}\wedge v'_{+} - H'\wedge E'_{-} - v'_{-}\wedge v'_{-}) \\ &= -\beta(\imath I_1 - I_2)\wedge I_3 + v_1\wedge v_1), \\ [r'_{qsJ}, r'_{qsJ}]_S &= 0, \end{aligned} \tag{7}$$

$$\begin{aligned} r'_{qst} &= \alpha(E'_{+}\wedge H' + v'_{+}\wedge v'_{+} - H'\wedge E'_{-} - v'_{-}\wedge v'_{-}) \\ &= -2\alpha(I_1\wedge I_2 + 2v_1\wedge v_2), \\ [r'_{qst}, r'_{qst}]_S &= -\alpha^2\Omega, \end{aligned} \tag{8}$$

$$\begin{aligned} r'_{st} &= \imath\alpha(E'_{+}\wedge E'_{-} + 2v'_{+}\wedge v'_{-}), \\ &= -2\imath\alpha(I_1\wedge I_3 + v_1\wedge v_1 - v_2\wedge v_2), \\ [r'_{st}, r'_{st}]_S &= \alpha^2\Omega, \end{aligned} \tag{9}$$

(ii) For the noncompact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ with ungraded conjugation:

$$\begin{aligned} r'_{qJ} &= \imath\beta(E'_{+}+E'_{-})\wedge H' = -\beta(\imath I_1 - I_2)\wedge I_3, \\ [r'_{qJ}, r'_{qJ}]_S &= 0, \end{aligned} \tag{10}$$

Comparing the r-matrix expressions (1)–(4) with (6)-(9) we obtain that

$$r_J \;=\; r'_{qJ} \;=\; -\alpha\big((\imath I_1 - I_2)\wedge I_3\big), \tag{11}$$

$$r_{sJ} \;=\; r'_{qsJ} \;=\; -\alpha\big((\imath I_1 - I_2)\wedge I_3 + v_1\wedge v_1\big), \tag{12}$$

$$r_{st} \;=\; r'_{qst} \;=\; -2\imath\alpha(I_1\wedge I_3 + 2v_1\wedge v_2), \tag{13}$$

$$r_{qst} \;=\; r'_{st} \;=\; -2\alpha(I_1\wedge I_2 + v_1\wedge v_1 - v_2\wedge v_2). \tag{14}$$

We see the following.
(a) The Jordanian r-matrix r_J in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis is the same as the quasi-Jordanian r-matrix r'_{qJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.
(b) The super-Jordanian r-matrix r_{sJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis is the same as the super-quasi-Jordanian r-matrix r'_{qsJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.
(c) The standard r-matrix r_{st} in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis becomes the quasi-standard r-matrix r'_{qst} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1.1))$ basis.
(d) Conversely, the quasi-standard r-matrix r_{qst} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ basis is the same as the standard r-matrix r'_{st} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.

The relations (8)–(10) show that the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ bialgebras are isomorphic. This result finally resolves the doubts about isomorphisms of these two bialgebras.

Using the isomorphisms of the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ bialgebras we take as basic r-matrices for the $N=1$, $D=3$ Lorentz superalgebra $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ the following ones that

$$r_J = -\alpha\big((\imath I_1 - I_2)\wedge I_3\big) = \imath\beta E_+\wedge H, \tag{15}$$

$$r_{sJ} = -\alpha\big((\imath I_1 - I_2)\wedge I_3 + v_1\wedge v_1\big) = \imath\beta(E_+\wedge H + v_+\wedge v_+), \tag{16}$$

$$r_{st} = -2\imath\alpha(I_1\wedge I_3 + 2v_1\wedge v_2) = \imath\alpha(E_+\wedge E_- + 2v_+\wedge v_-), \tag{17}$$

$$r'_{st} = -2\alpha(I_1\wedge I_2 + v_1\wedge v_1 - v_2\wedge v_2) = \imath\alpha(E'_+\wedge E'_- + 2v'_+\wedge v'_-). \tag{18}$$

The first two r-matrices r_J and r_{sJ} present the Jordanian and super-Jordanian twist deformations of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, the third and fourth r-matrices r_{st} and r'_{st} correspond to the q-analogs of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ real algebras.

SUMMARY. A general picture of all basic (non-isomorphic) quasitriangular Hopf quantizations of the complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C}))$ and its 'real' forms: $\mathfrak{osp}^*(1|\mathfrak{so}(3))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{so}(2,1))$.

- The complex superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$.
  Three quantum deformations: the Jordanian and super-Jordanian twist deformations and q-analog (standart trigonometric deformation).
- The compact pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{so}(3))$ (has quartenionic structure on the odd sector $\mathfrak{osp}_1^*(1|\mathfrak{so}(3))$).
  (i) For the graded conjugation: the q-analog quantum deformation only (in terms of $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ q-analog).
  (ii) For the ungraded conjugation: nothing (null set).
- The noncompact real form $\mathfrak{osp}^\dagger(1|\mathfrak{so}(2,1))$ ($N=1, D=3$ Lorentz SUSY).
  (i) For the graded conjugation: the Jordanian and super-Jordanian twist deformations and two q-analogs (in terms of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ q-analogs).
  (ii) For the ungraded conjugation: the Jordanian twist deformation only (in terms of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ Jordanian twist)

COMMENT. A natural example of the real form $\mathfrak{osp}^\dagger(1|\mathfrak{so}(2,1))$ with the ungraded conjugation.

One bosonic Weyl algebra with the generators $b_\pm, 1$ and the defining relations: $[b_+, b_-] = 1$, $b_\pm^\dagger = b_\mp$. Bosonic realization of $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$:

$$
\begin{aligned}
E'_+ \;&:=\; \{b_+, b_+\} \quad E'_- \;:=\; -\{b_-, b_-\} \quad H' \;:=\; -\{b_+, b_-\},\\
v'_+ \;&:=\; \frac{1}{\sqrt{2}}\, b_+, \quad v'_- \;:=\; \frac{1}{\sqrt{2}}\, b_-\\
(E'_\pm)^\dagger \;&:= -E'_\mp, \quad (H')^\dagger \;:=\; H', \quad (v'_\pm)^\dagger \;:= v'_\mp
\end{aligned}
\tag{19}
$$

Bosonic realization of $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$:

$$
\begin{aligned}
H \;&=\; -\frac{\imath}{2}(E'_+ - E'_-), \quad E_\pm \;=\; \mp\imath H' + \frac{1}{2}(E'_+ + E'_-),\\
v_+ \;&=\; \frac{1}{\sqrt{2}}(v'_+ - \imath v'_-), \quad v_- \;=\; \frac{1}{\sqrt{2}}(-\imath v'_+ + v'_-),\\
(E_\pm)^\dagger \;&:= -E_\pm, \quad (H)^\dagger \;:=\; -H, \quad (v_\pm)^\dagger \;:= \imath v_\pm
\end{aligned}
\tag{20}
$$

O. Ogievetsky's twist functions: $F = \exp(H \otimes \sigma)$, where $\sigma = \ln(1 + \beta E_+)$.

THANK YOU FOR YOUR ATTENTION