$6j$-SYMBOLS FOR PRINCIPAL SERIES REPRESENTATIONS OF $\mathrm{SL}(2,\mathbb{C})$ GROUP

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$\mathrm{SL}(2,\mathbb{C})$ **GROUP**

The group of 2×2 matrices with complex entries

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \quad \alpha\delta - \beta\gamma = 1.$$

$s\ell(2,\mathbb{C})$ algebra generators $S_j, \bar{S}_j$, $j = 0, \pm$,

$$[S_+, S_-] = 2S_0, \qquad [S_0, S_\pm] = \pm S_\pm.$$

Explicit realization in the space of complex functions $\Phi(z,\bar{z})$:

$$S_- = -\partial_z, \qquad S_0 = z\partial_z - s, \qquad S_+ = z^2\partial_z - 2sz, \quad s\in\mathbb{C}.$$

$\bar{S}_j$ are obtained by $z \to \bar{z} = z^*$ and $s \to \bar{s} \neq s^*$.

Casimir operator

$$K = S_+S_- + S_0(S_0 - 1) = s(s+1)$$

Equivalent representations

$$s \to -1 - s, \quad \text{or} \quad a \to -a \quad \text{for} \quad a = 2s+1.$$

General representation

$$[\mathrm{T}_a(g)\,\Phi]\,(z,\bar z) = [\beta z+\delta]^{a-1}\,\Phi\left(\frac{\alpha z+\gamma}{\beta z+\delta},\frac{\bar\alpha\bar z+\bar\gamma}{\bar\beta\bar z+\bar\delta}\right),$$

where

$$[z]^a := z^a\,\bar z^{\bar a} = |z|^{2\bar a} z^{a-\bar a}$$

Restrictions on the spin variables $a,\bar a$:

1) single-valuedness $\Rightarrow a-\bar a = m\in\mathbb{Z}$

2) unitarity with respect to the scalar product

$$\int d^2z\,\overline{f_1(z,\bar z)}f_2(z,\bar z) \qquad \Rightarrow$$

$$a=\frac{m+i\nu}{2},\quad \bar a = \frac{-m+i\nu}{2} = -a^*,\quad \nu\in\mathbb{R}.$$

Then,

$$[z]^a := z^a\bar z^{\bar a} = |z|^{2\bar a}z^{a-\bar a} = z^m|z|^{i\nu-m}.$$

Infinite-dimensional unitary principal series representation.

Tensor product of two representations

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \xrightarrow{\mathrm{P}(a_1,a_2|a_3)} \mathrm{T}_{a_3}$$

The projection operator

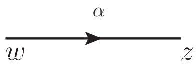
$$\Phi(z_1, z_2) \xrightarrow{\mathrm{P}(a_1,a_2|a_3)} \left[\mathrm{P}(a_1, a_2|a_3)\,\Phi\right](z_3)$$
$$= \int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2\, W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} \Phi(z_1, z_2),$$

where $d^2 z = dxdy$ (for $z = x + iy$) Naimark, 1957

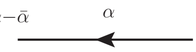
$$W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} = \frac{1}{[z_2 - z_1]^{\frac{1+a_1+a_2+a_3}{2}} [z_3 - z_1]^{\frac{1+a_1-a_2-a_3}{2}} [z_2 - z_3]^{\frac{1-a_1+a_2-a_3}{2}}},$$

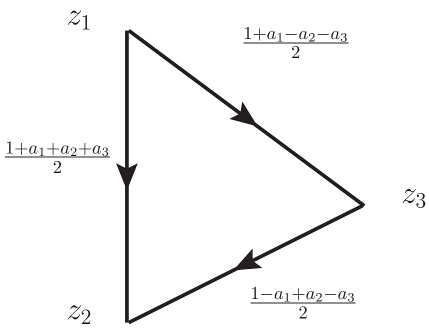
with $m_1 + m_2 + m_3 \in 2\mathbb{Z}$. This means that

$$[\beta z_3 + \delta]^{a_3 - 1} \left[\mathrm{P}(a_1, a_2|a_3)\,\Phi\right] \left(\frac{\alpha z_3 + \gamma}{\beta z_3 + \delta}\right) = \int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2$$
$$\times\, W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} [\beta z_1 + \delta]^{a_1 - 1}\, [\beta z_2 + \delta]^{a_2 - 1}\, \Phi\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right).$$



$= [z-w]^{-\alpha} \quad = (-1)^{\alpha-\bar\alpha}$





Feynman diagrams for the propagator and $3j$-symbol W,

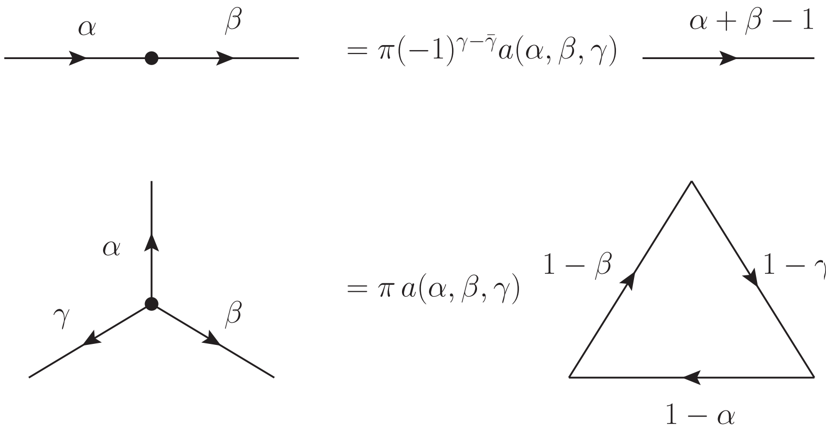
$$\frac{1}{[z-w]^\alpha} \equiv \frac{1}{(z-w)^\alpha(\bar z-\bar w)^{\bar\alpha}} = \frac{(\bar z-\bar w)^{\alpha-\bar\alpha}}{|z-w|^{2\alpha}} = \frac{(-1)^{\alpha-\bar\alpha}}{[w-z]^\alpha}.$$

Denote

$$a(\alpha) = \frac{\Gamma(1-\bar\alpha)}{\Gamma(\alpha)}, \quad a(\alpha,\beta,\gamma,\ldots) := a(\alpha)a(\beta)a(\gamma)\ldots$$

The chain rule and the star-triangle relation:

$$\int \frac{d^2w}{[z_1-w]^\alpha[w-z_2]^\beta} = \frac{\pi a(\alpha,\beta,\gamma)}{[z_2-z_1]^{\alpha+\beta-1}},$$

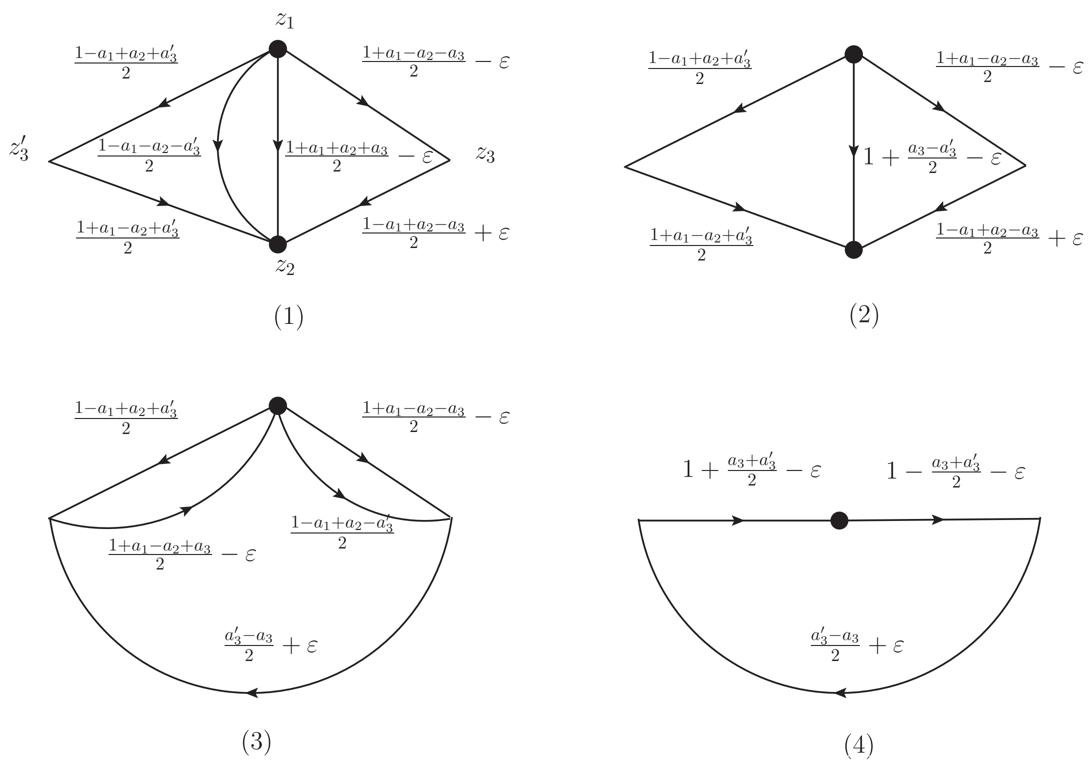


$$\int \frac{d^2 w}{[z_1 - w]^{\alpha}[z_2 - w]^{\beta}[z_3 - w]^{\gamma}} = \frac{\pi a(\alpha,\beta,\gamma)}{[z_2 - z_1]^{1-\gamma}[z_1 - z_3]^{1-\beta}[z_3 - z_2]^{1-\alpha}},$$

where $\alpha+\beta+\gamma = \bar{\alpha}+\bar{\beta}+\bar{\gamma} = 2.$

Blobs = integrations over the vertex coordinates.

$$\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2\, W\begin{pmatrix} -a_1, -a_2, -a_3' \\ z_1, z_2, z_3' \end{pmatrix} W_{\varepsilon}\begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix}$$
$$= (-1)^{\frac{m_1-m_2-m_3}{2}} \pi^2\, a\Big(\frac{1-a_1+a_2-a_3}{2}+\varepsilon,\ \frac{1+a_1-a_2+a_3'}{2}\Big)$$
$$\times a\Big(1+\frac{a_3-a_3'}{2}-\varepsilon, 1+\frac{a_3+a_3'}{2}-\varepsilon, 1-\frac{a_3+a_3'}{2}-\varepsilon, 2\varepsilon\Big).$$



Diagrammatic computation of the integral.

Application of the limiting relations

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{x^2+\varepsilon^2}\to\pi\delta(x),\qquad \lim_{\varepsilon\to 0}\frac{\varepsilon}{[z_3-z'_3]^{1-\varepsilon}}=\pi\delta^2(z_3-z'_3)$$

yields the biorthogonality relation:

$$\int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2\, W \begin{pmatrix} -a_1, -a_2, -a_3' \\ z_1, z_2, z_3' \end{pmatrix} W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix}$$

$$= \rho^{-1}(a_3)\delta_R(a_3 - a_3')\, \delta^2(z_3 - z_3') + B(a_1, a_2, a_3)\, \frac{\delta_R(a_3 + a_3')}{[z_3 - z_3']^{1-a_3}},$$

where

$$\delta_R(a - a') = \delta_{m\,m'}\delta\left(\sigma - \sigma'\right), \quad a = \frac{m}{2} + i\sigma, \quad a' = \frac{m'}{2} + i\sigma',$$

$$\rho(a_3) = -\frac{a_3\bar{a}_3}{4\pi^4}, \qquad B(a_1, a_2, a_3) = 4\pi^3 \frac{a\left(\frac{1-a_1+a_2-a_3}{2}, 1 + a_3\right)}{a\left(\frac{1-a_1+a_2+a_3}{2}\right)}.$$

Completeness relation ($a = \frac{m}{2} + i\sigma$) Naimark, 1957

$$\sum_{m\in 2\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}\sigma \int_{\mathbb{C}} \mathrm{d}^2 z\, \frac{\rho(a)}{2}\, W \begin{pmatrix} -a_1, -a_2, -a \\ z_3, z_4, z \end{pmatrix} W \begin{pmatrix} a_1, a_2, a \\ z_1, z_2, z \end{pmatrix}$$

$$= \delta^2(z_1 - z_3)\, \delta^2(z_2 - z_4), \quad m_1 + m_2 \in 2\mathbb{Z}.$$

A triple tensor product decomposition

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(a_1,a_2|c')} \mathrm{T}_{c'} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(c',a_3|\ell)} \mathrm{T}_{\ell},$$

realized as

$$\Phi(z_1, z_2, z_3) \xrightarrow{\mathrm{P}(c',a_3|\ell)\mathrm{P}(a_1,a_2|c')} \left[\mathrm{P}(c', a_3|\ell)\mathrm{P}(a_1, a_2|c')\, \Phi\right](z)$$

$$= \int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3 \int \mathrm{d}^2 z_0\, W \begin{pmatrix} a_1, a_2, c' \\ z_1, z_2, z_0 \end{pmatrix} W \begin{pmatrix} c', a_3, \ell \\ z_0, z_3, z \end{pmatrix} \Phi(z_1, z_2, z_3).$$

Another option

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(a_2,a_3|c)} \mathrm{T}_{a_1} \otimes \mathrm{T}_{c} \xrightarrow{\mathrm{P}(a_1,c|\ell)} \mathrm{T}_{\ell},$$

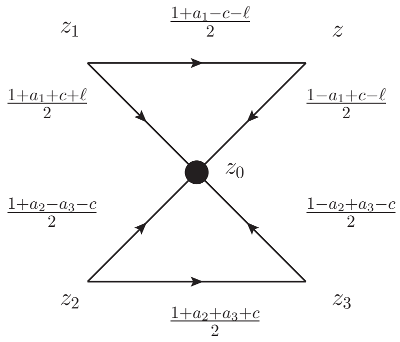
realized as

$$\Phi(z_1, z_2, z_3) \xrightarrow{\mathrm{P}(a_1,c|\ell)\mathrm{P}(a_2,a_3|c)} \left[\mathrm{P}(a_1, c|\ell)\mathrm{P}(a_2, a_3|c)\, \Phi\right](z)$$

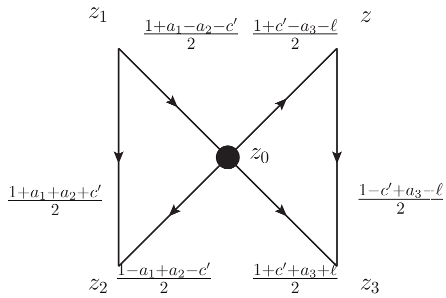
$$= \int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3 \int \mathrm{d}^2 z_0\, W \begin{pmatrix} a_2, a_3, c \\ z_2, z_3, z_0 \end{pmatrix} W \begin{pmatrix} a_1, c, \ell \\ z_1, z_0, z \end{pmatrix} \Phi(z_1, z_2, z_3).$$

Definition of the Racah coefficients, or $6j$-symbols

$$\mathrm{P}(a_1, c|\ell)\mathrm{P}(a_2, a_3|c) = \int \mathrm{D}_R c'\, \frac{\rho(c')}{2}\, \mathrm{R}_{\ell}(c, c')\, \mathrm{P}(c', a_3|\ell)\mathrm{P}(a_1, a_2|c'),$$



$= \int D_R c' \frac{\rho(c')}{2} \mathrm{R}_\ell(c, c')$



where $c' = m/2 + i\sigma$ and

$$\int \mathrm{D}_R c' = \sum_{m \in 2\mathbb{Z} \text{ or } 2\mathbb{Z}+1} \int_{\mathbb{R}} \mathrm{d}\sigma$$

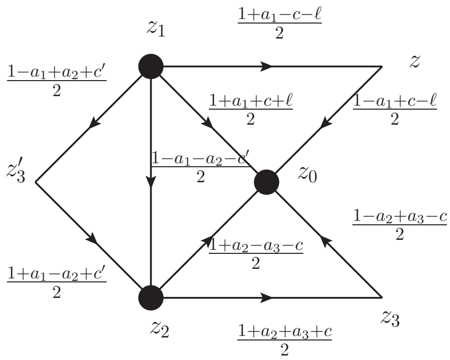
depending on whether $m_1 + m_2$ is even or odd.

This integral equation is shown on the diagram above.

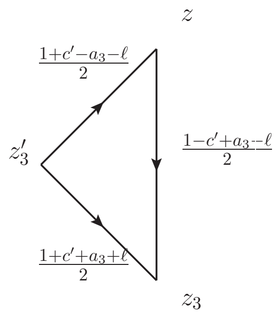
Multiply by suitable W-function, integrate and use the biorthogonality relation $\Rightarrow$

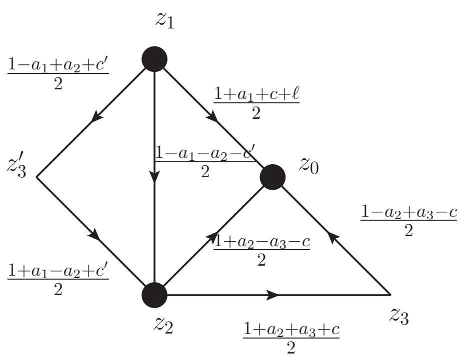
$$\int \mathrm{d}^2 z_0 \, \mathrm{d}^2 z_1 \, \mathrm{d}^2 z_2 \, W \begin{pmatrix} -a_1, -a_2, -c' \\ z_1, z_2, z_3' \end{pmatrix} W \begin{pmatrix} a_2, a_3, c \\ z_2, z_3, z_0 \end{pmatrix} W \begin{pmatrix} a_1, c, \ell \\ z_1, z_0, z \end{pmatrix}$$
$$= \mathrm{R}_\ell(c, c') W \begin{pmatrix} c', a_3, \ell \\ z_3', z_3, z \end{pmatrix}.$$

The diagram for this equation is given below.

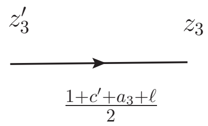


$= \mathrm{R}_\ell(c, c')$



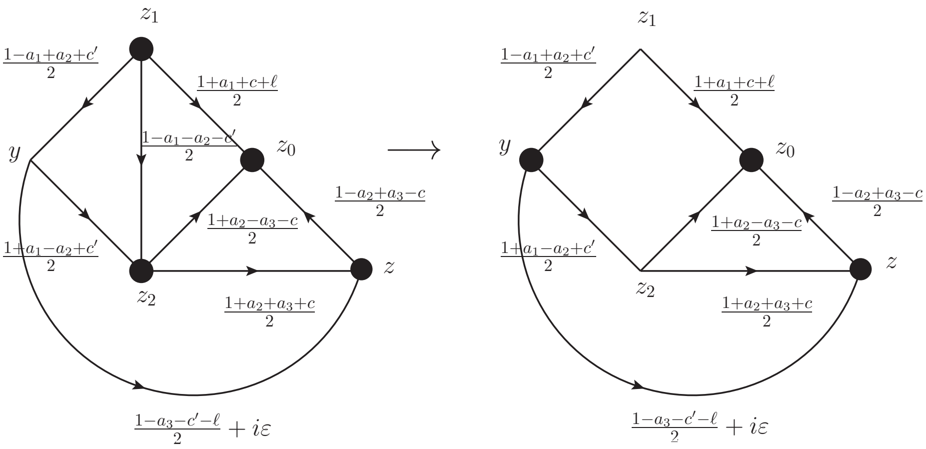


$= \mathrm{R}_\ell(c, c')$



For $z \to \infty$ we obtain

$$\int \frac{\mathrm{d}^2 z_0\, \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2}{[z_0 - z_1]^{\frac{1+a_1+c+\ell}{2}}}\, W \begin{pmatrix} -a_1, -a_2, -c' \\ z_1, z_2, z_3' \end{pmatrix} W \begin{pmatrix} a_2, a_3, c \\ z_2, z_3, z_0 \end{pmatrix} = \frac{\mathrm{R}_\ell(c, c')}{[z_3 - z_3']^{\frac{1+c'+a_3+\ell}{2}}}$$



Multiply this result by the propagator connecting $y := z_3'$ and $z := z_3$ with the index $i\varepsilon + (1 - a_3 - c' - \ell)/2$, $\varepsilon \in \mathbb{R}$ and integrate over $z \Rightarrow$

$$\mathrm{R}_\ell(c, c') \int \mathrm{d}^2 z \frac{1}{[z-y]^{\frac{1+c'+a_3+\ell}{2}}} \frac{1}{[z-y]^{i\varepsilon + \frac{1-a_3-c'-\ell}{2}}} = \mathrm{R}_\ell(c, c') 2\pi^2 \delta(\varepsilon).$$

Equivalent to (Gorishny, Isaev, 1995)

$$\int_{x \in \mathbb{R}} e^{ipx}\, dx = 2\pi \delta(p).$$

All diagrams obtained from this diagram after removal of any line yield the same coefficient times a propagator.

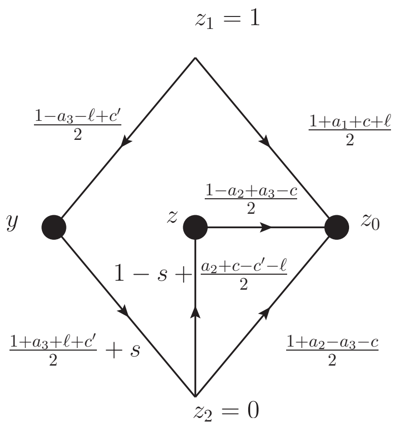
Remove the line connecting z_1 and z_2 and set $z_1 = 1,\ z_2 = 0 \Rightarrow$

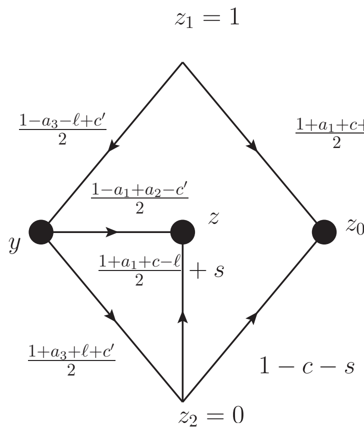
$$\mathrm{R}_\ell(c, c') = \int \mathrm{d}^2 z\, \Phi_2(a_1, a_2, a_3|\ell, c, z)\overline{\Phi_1(a_1, a_2, a_3|\ell, c', z)},$$

$$\overline{\Phi_1} = \int \frac{\mathrm{d}^2 y}{[y-1]^{\frac{1-a_1+a_2+c'}{2}}[-y]^{\frac{1+a_1-a_2+c'}{2}}[z-y]^{\frac{1-a_3-\ell+c'}{2}}},$$

$$\Phi_2 = \frac{1}{[z]^{\frac{1+a_2+a_3+c}{2}}} \int \frac{\mathrm{d}^2 z_0}{[z_0-1]^{\frac{1+a_1+\ell+c}{2}}[-z_0]^{\frac{1+a_2-a_3-c}{2}}[z_1-z]^{\frac{1-a_2+a_3-c}{2}}}.$$

Ismagilov's 2006 result: $\quad c' \to -c', \quad m_j \in 2\mathbb{Z}.$





Mellin-Barnes representation.

A $2d$ Fourier transform $(s = (n + i\nu)/2)$

$$\frac{1}{[z-y]^{\alpha}} = \frac{1}{4\pi a(1-\alpha)} \sum_{n\in\mathbb{Z}} \int_L d\nu \, \frac{a(1-s, 1+s-\alpha)}{[z]^{\alpha-s}[-y]^{s}},$$

where L is any contour lying in the strip $\mathrm{Im}(\nu) \in]0, -1[$.

Apply to the propagator connecting y and z on the diagram.

The final Mellin-Barnes type representation

$$\mathrm{R}_\ell(c,c') = (-1)^{c'-\bar{c}'}\,\frac{\pi^2}{4}\frac{a\left(\frac{1-a_3-\ell+c'}{2},\frac{1+a_1+c+\ell}{2}\right)}{a\left(\frac{1+a_1-a_2+c'}{2},\frac{1+a_2-a_3+c}{2}\right)}\sum_{n\in\mathbb{Z}}\int_L d\nu$$

$$\times\frac{a\left(\frac{1+a_1-a_2+c'}{2}+s,\frac{1-a_1-a_2+c'}{2}+s,\frac{1+a_3+\ell+c'}{2}+s,\frac{1-a_3+\ell+c'}{2}+s\right)}{a\left(s,c'+s,\frac{c'+\ell-a_2-c}{2}+s,\frac{c+c'+\ell-a_2}{2}+s\right)},$$

where $s=\frac{1}{2}(n+i\nu)$.

Residue calculus $\Rightarrow$ Racah type expression:

$$\mathrm{R}_\ell(c,c') \propto \sum_{4\text{ terms}} {}_4F_3(\ldots)\times{}_4F_3(\ldots).$$

Euler-Gauss hypergeometric function

${}_2F_1$-series:

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n, \qquad |x|<1, $$

$(a)_n = a(a+1)\dots(a+n-1)$ the Pochhammer symbol

The derived $6j$-symbols $\mathrm{R}_\ell(c,c')$ should be embeddable to the following **elliptic analogue** of ${}_2F_1$ function (VS, 2003):

$$ V(t_1,\dots,t_8;p,q) = \int_{\mathbb{T}} \prod_{k=\pm1} \frac{\prod_{j=1}^{8}\Gamma(t_j z^k;p,q)}{\Gamma(z^{2k};p,q)} \frac{dz}{z}, $$

$|t_j|, |p|, |q| < 1$ and $\prod_{j=1}^{8} t_j = p^2 q^2$.

The elliptic gamma function

$$ \Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^j q^k}, \qquad |p|,|q|<1. $$

This is the superconformal index of $\mathcal{N}=1, d=4$ SUSY field theory on $S^3\times S^1$ with $G=SU(2), F=SU(8)$ with fundamental chiral fields.