CLASSICAL AND QUANTUM DYNAMICS OF HIGHER-DERIVATIVE SYSTEMS

or

LIVING WITH GHOSTS

SIS-18, Dubna, August 14

based on

A.S., Int. J. Mod. Phys. A32 (2017) no.33, 1730025 [arXiv:1710.11538]

MOTIVATION:

... To myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton

- The ocean is now charted up to $E \lesssim 10^3$ GeV, $l \gtrsim 10^{-17}$ cm.
- But it extends up to $M_P \approx 10^{19}$ GeV. We have now explored its 10^{-16}-th part.

PROBLEMS IN QUANTUM (AND CLASSICAL) GRAVITY:

- Nonrenormalizability
- Non-causality. Closed time loops. Paradoxes.

TOE = strings?

- No fundamental quantum string theory
- No phenomenological successes.

An alternative (dream) solution: [A.S., 2005]

Our Universe as a soap film in a flat higher dimensional bulk. The TOE is a field theory in this bulk. Gravity etc is an effective theory living on the film, like

$$H_{\text{soap}} = \sigma \mathcal{A} = \sigma \int d^2x \sqrt{g}$$

TRY

$$S = -\frac{1}{2h^2} \int \mathrm{Tr}\{F_{MN} F_{MN}\}\, d^6x\,,$$

in $D = 6$, $M, N = 0, 1, 2, 3, 4, 5$.

- Dimensionful coupling constant, nonrenormalizable

A SECOND TRY

$$\mathcal{L}^{D=6} = \alpha \mathrm{Tr}\{F_{\mu\nu} \Box F_{\mu\nu}\} + \beta \mathrm{Tr}\{F_{\mu\nu} F_{\nu\alpha} F_{\alpha\mu}\}$$

- α, β are dimensionless, renormalizability
- Includes higher derivatives

But GHOSTS appear

- In a ghost system the Hamiltonian has no ground state. No vacuum in field theory. It is inherent for all higher-derivative theories.

OSTROGRADSKY HAMILTONIAN

M. Ostrogradsky [1801-1862] is known by

- Ostrogradsky theorem from vector analysis
- Ostrogradsky method for calculating $\int P(x)/Q(x)\,dx$.
- Ostrogradsky Hamiltonian

In the paper

[M. Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres*, Mem. Ac. St. Petersbourg **VI 4** (1850) 385.]

he reinvented the Hamiltonian formalism and applied it to higher-derivative theories.

- Consider $L(x, \dot{x}, \ddot{x})$.
- Equation of motion:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} \;=\; 0\,.$$

- Conserved energy:

$$E \;=\; \ddot{x}\frac{\partial L}{\partial \ddot{x}} + \dot{x}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right) - L\,.$$

• Treat $v = \dot{x}$ as an independent variable and define

$$p_v = \frac{\partial L}{\partial \dot{v}} = \frac{\partial L}{\partial \ddot{x}} ,$$
$$p_x = \frac{\partial L}{\partial \dot{x}} - \dot{p}_v ,$$

• Canonical Hamiltonian:

$$H(p_v, p_x; v, x) = p_v \dot{v} + p_x \dot{x} - L =$$
$$p_v a(p_v, x, v) + p_x v - L\left[a(p_v, x, v), x, v\right] ,$$

where $a(p_v, x, v)$ is the solution of the equation $\partial L(x, v, a)/\partial a = p_v$.

• Linear term $p_x v \longrightarrow$

Theorem 1: [Woodard, 2015] *The classical energy of a nondegenerate higher-derivative system can acquire an arbitrary positive ot negative value.*

also

Theorem 2: [Raidal + Veermae, 2017] *The spectrum of a Hamiltonian of a higher-derivative system is not bounded neither from below, nor from above.*

• This phenomenon is sometimes called "Ostrogradsky instability", but:

- It is not an instability,
- was noticed first not by Ostrogradsky.

Arnold's principle: *If a notion bears a personal name, then this name is not the name of the discoverer.*

(self-referential).

PAIS-UHLENBECK OSCILLATOR [1950]

Consider

$$L = \frac{1}{2}\left[\ddot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\omega_2^2 x^2\right]$$

- Ostrogradsky Hamiltonian:

$$H = p_x v + \frac{p_v^2}{2} + \frac{(\omega_1^2 + \omega_2^2)v^2}{2} - \frac{\omega_1^2\omega_2^2 x^2}{2}$$

- Canonical transformation:

$$X_1 = \frac{1}{\omega_1}\frac{\hat{p}_x + \omega_1^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}, \qquad \hat{P}_1 \equiv -i\frac{\partial}{\partial X_1} = \omega_1 \frac{\hat{p}_v + \omega_2^2 x}{\sqrt{\omega_1^2 - \omega_2^2}},$$

$$X_2 = \frac{\hat{p}_v + \omega_1^2 x}{\sqrt{\omega_1^2 - \omega_2^2}}, \qquad \hat{P}_2 \equiv -i\frac{\partial}{\partial X_2} = \frac{\hat{p}_x + \omega_2^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}.$$

($\omega_1 > \omega_2$ was assumed).

- In these variables,

$$H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2}.$$

The spectrum is

$$E_{nm} = \left(n + \frac{1}{2}\right)\omega_1 - \left(m + \frac{1}{2}\right)\omega_2$$

with positive integer n, m.

• All states are normalizable ("pure point"). Infinite degeneracy if ω_1/ω_2 is rational. Everywhere dense if ω_1/ω_2 is irrational.

UNUSUAL BUT NOT SICK!

UNITARITY CONFUSION

Consider

$$\hat{H} = \omega_1 a_1^\dagger a_1 - \omega_2 a_2 a_2^\dagger \,.$$

• Ordinary "vacuum" $|\Phi\rangle$ with $a_1|\Phi\rangle = a_2|\Phi\rangle = 0$ is in the middle of the spectrum.

• Introduce the state $|\tilde{\Phi}\rangle$ satisfying

$$a_1|\tilde{\Phi}\rangle = a_2^\dagger|\tilde{\Phi}\rangle = 0$$

and consider the tower of states

$$|n\rangle = \frac{a_2^n}{n!}|\tilde{\Phi}\rangle$$

Then $\hat{H}|\tilde{\Phi}\rangle = 0$,

$$\hat{H}|1\rangle = \hat{H}(a_2|\tilde{\Phi}) = -\omega_2 a_2 a_2^\dagger a_2 = \omega_2 a_2|\tilde{\Phi}\rangle = \omega_2|1\rangle$$

and $\hat{H}|n\rangle = n\omega_2|n\rangle$.

• The spectrum is positive definite. One can rename $a_2 \to b_2^\dagger,\ a_2^\dagger \to b_2$.

THE PRICE

- $[b_2, b_2^\dagger] = -1$ and hence $|1\rangle = b_2^\dagger|\tilde{\Phi}\rangle$ has a negative norm:

$$\langle 1|1\rangle = -\langle\tilde{\Phi}|b_2 b_2^\dagger|\tilde{\Phi}\rangle = -\langle\tilde{\Phi}|\tilde{\Phi}\rangle = -1\,.$$

- $$|\tilde{\Phi}\rangle = \exp\left\{-\frac{\omega_1}{2}X_1^2\right\}\exp\left\{\frac{\omega_2}{2}X_2^2\right\}$$

  and the whole tower $|n\rangle$ are not in $\mathcal{L}_2$.

IT IS BETTER NOT TO THINK IN THESE TERMS!

BENDER AND MANNHEIM PROPOSAL

Consider

$$H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2} \,.$$

and assume X_1 to be real and X_2 purely imaginary.

Then the normalizable wave functions involve a factor

$$\exp\left\{-\frac{\omega_1}{2} X_1^2\right\} \exp\left\{\frac{\omega_2}{2} X_2^2\right\} \,.$$

and the spectrum is positive definite.

- This is a different spectral problem.
- Just no need to do this.

PATH INTEGRAL CONFUSION [Hawking + Hertog, 2002]

• The Minkowski Lagrangian path integral

$$\sim \int \prod_t dx(t) \exp\left\{ i \int dt\, L(\ddot{x}, \dot{x}, x) \right\} \tag{1}$$

• The corresponding Hamiltonian path integral

$$\sim \int \prod_t dx(t) dv(t) dp_x(t) dp_v(t)$$

$$\exp\left\{ i \int dt\, [p_v \dot{v} + p_x \dot{x} - H(p_v, p_x; v, x)] \right\} . \tag{2}$$

Substitute here the Ostrogradsky Hamiltonian and integrate over $\prod_t dp_x(t)$. We obtain the factor

$$\prod_t \delta[v(t) - \dot{x}(t)] .$$

Integrating further over $\prod_t dv(t) dp_v(t)$, we derive (**??**).

• The Euclidean rotation $t \to -i\tau$ in the Hamiltonian integral (**??**) is impossible. The integral

$$\prod_\tau \int_{-\infty}^{\infty} dp_x(\tau) \exp\left\{ \int d\tau\, p_x(\tau) \left[i \frac{dx(\tau)}{d\tau} - v(\tau) \right] \right\}$$

diverges.

• Euclidean rotation $t \to -i\tau$ in the Lagrangian integral (**??**) is possible, the integral may converge, but its analytic continuation into Minkowski space does not give a unitary evolution.

CONCLUSION:

Euclidean path integrals (in constrast to Minkowski ones) are not defined for higher-derivative systems.

INCLUDING INTERACTIONS

Consider the Lagrangian [A.S., 2005]

$$L = \frac{1}{2}\left[\ddot{x}^2 - 2\omega^2\dot{x}^2 + \omega^4 x^2\right] - \frac{1}{4}\alpha x^4 .$$

Equation of motion:

$$\left(\frac{d^2}{dt^2} + \omega^2\right)^2 x - \alpha x^3 = 0 .$$

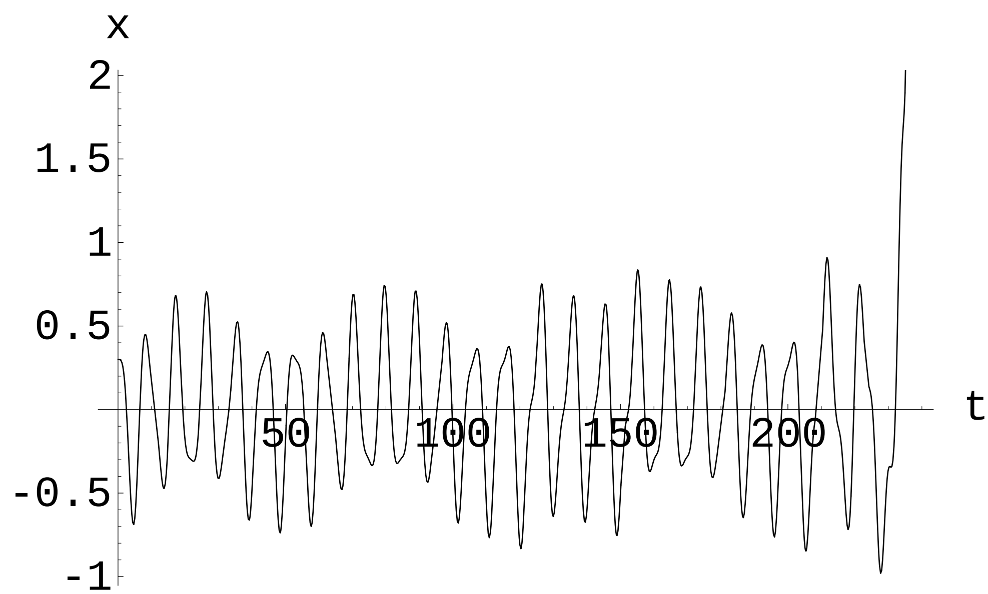
fixed point:

$$x(0) = \dot{x}(0) = \ddot{x}(0) = x^{(3)}(0) = 0 .$$

- Stable trajectories for small deviations (the island of stability).

- Collapse (the system runs to the infinity at finite time) for large enough deviations.

ON THE SHORE OF THE STABILITY ISLAND



A similar stability island for another HD system in

[S.N. Carrol, M. Hoffman, and M. Trodden, PR **D68** (2003) 023509]

AN EXAMPLE OF COLLAPSE: FALLING TO THE CENTER

Consider

$$V(r) = -\frac{\kappa}{r^2}$$

with $m\kappa > 1/8$.

- Spectrum is not bounded from below.
- Schrödinger problem is not well defined.

If one smoothes the singularity,

$$\begin{aligned} V(r) &= -\frac{\kappa}{r^2}, \qquad r > a\,, \\ V(r) &= -\frac{\kappa}{a^2}, \qquad r \le a\,, \end{aligned}$$

the spectrum is bounded, but depends on a.

- Violation of unitarity (probability “leaks” into the singularity).

AN OBSERVATION:

- If quantum theory is sick, so is its classical counterpart. If classical theory is benign, so is its quantum counterpart.

INTERACTING SYSTEMS WITH BENIGN GHOSTS

[D. Robert + A.S., 2006]

$$S \;=\; \int dt d\bar{\theta} d\theta \left[\frac{i}{2} \bar{\mathcal{D}} \Phi \frac{d}{dt} \mathcal{D} \Phi + V(\Phi) \right] \,,$$

with the real (0+1)-dimensional superfield

$$\Phi = \; \phi + \theta \bar{\psi} + \psi \bar{\theta} + D \theta \bar{\theta}$$

- An extra time derivative.

The Hamiltonian

$$H = pP - DV'(\phi) + \text{fermion term}$$

is not positive definite.

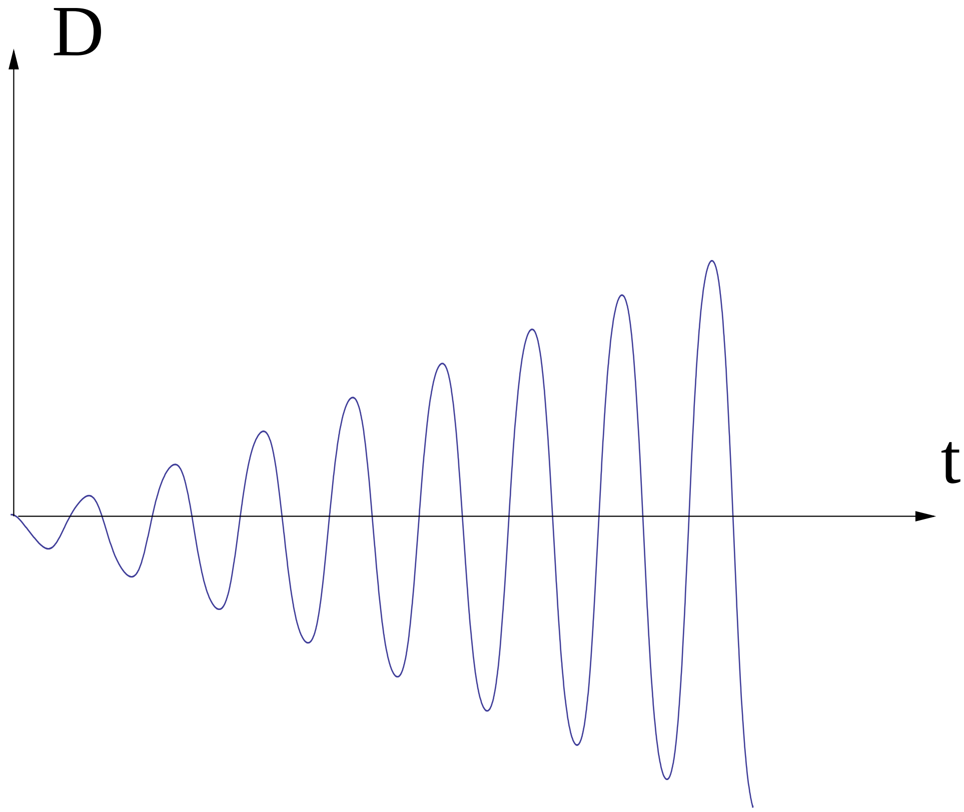
- 4-dimensional phase space $(p, \phi), (P, D)$.
- Two integrals of motion: H and

$$N = \frac{P^2}{2} + V(\phi)\,.$$

- Exactly solvable.
- Take

$$V = \frac{\omega^2\phi^2}{2} + \frac{\lambda\phi^4}{4}$$

• The solutions to the classical equations of motion are expressed via elliptic functions.

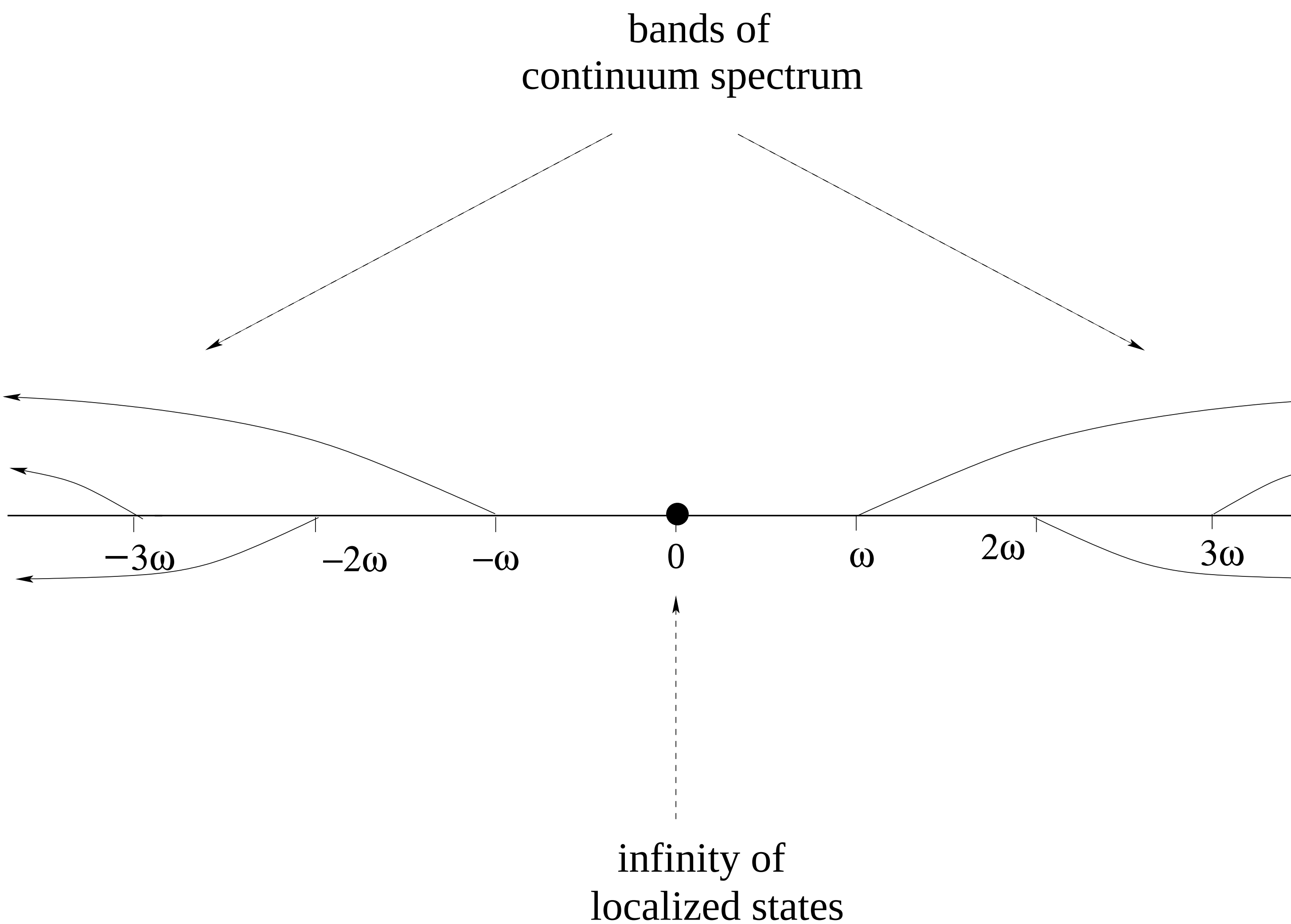


• Linear growth for $D(t)$; $\phi(t)$ is bounded. No collapse.

• Other benign ghost systems:
[Pavšič, 2013; Ilhan+Kovner, 2013]

QUANTUM PROBLEM

is also exactly solvable.



Spectrum of the Hamiltonian $H = pP + DV'(\phi)$.

Mixed model

$$L = \int d\bar{\theta} d\theta \left[\frac{i}{2} (\bar{\mathcal{D}}\Phi) \frac{d}{dt} (\mathcal{D}\Phi) + \frac{\gamma}{2} \bar{\mathcal{D}}\Phi \mathcal{D}\Phi + V(\Phi) \right] .$$

Physics is similar to the model with $\gamma = 0$, but

- Not integrable anymore.
- No linear growth for $D(t)$.

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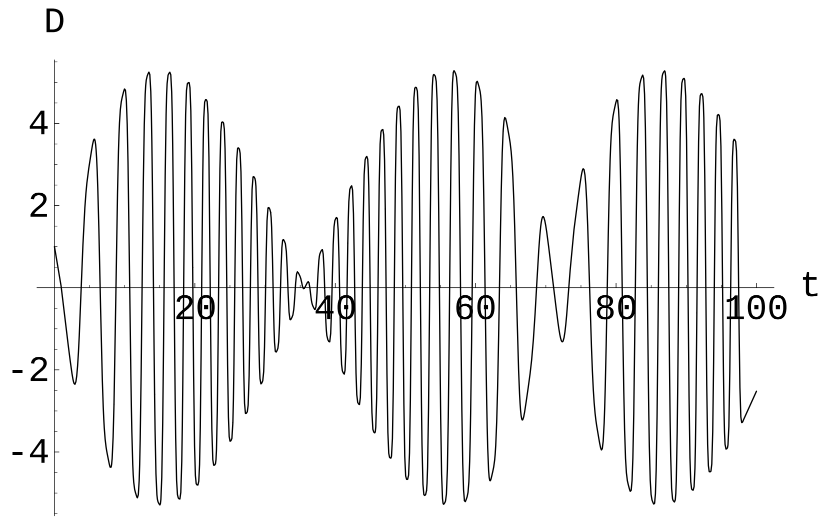


Figure 1: The function $D(t)$ for a deformed system ($\omega = 0, \lambda = 1, \gamma = .1$).

UNUSUAL ALGEBRAIC STRUCTURES

- canonical Nöther supercharges

$$
\begin{aligned}
Q &= \psi[p + iV'(x)] - \left(\bar{\chi} + \frac{\gamma}{2}\psi\right)(P - iD) \ , \\
\bar{Q} &= \left(\bar{\psi} - \frac{\gamma}{2}\chi\right)(P + iD) - \chi[p - iV'(x)] \ .
\end{aligned}
$$

- and the extra pair

$$
\begin{aligned}
T &= \psi[p - iV'(x)] + \left(\bar{\chi} + \frac{\gamma}{2}\psi\right)(P + iD) \ , \\
\bar{T} &= \left(\bar{\psi} + \frac{\gamma}{2}\chi\right)(P - iD) + \chi[p + iV'(x)] \ .
\end{aligned}
$$

• When $\gamma = 0$, we have a semidirect product of the standard $\mathcal{N} = 4$ SUSY algebra

$$\{Q, \bar{Q}\} = \{T, \bar{T}\} = 2H$$

(but $\bar{Q} \neq Q^\dagger, \; \bar{T} \neq T^\dagger$!)

and the Abelian Lie algebra generated by

$$\begin{aligned} N &= \frac{P^2}{2} - V(\phi)\,, \\ F &= \psi\bar{\psi} - \chi\bar{\chi} \end{aligned}$$

Nonvanishing commutators

$$\{Q, \bar{Q}\} = \{T, \bar{T}\} = 2H;$$
$$[\bar{Q}, F] = \bar{Q}, \; [Q, F] = -Q, \; [T, F] = -T, \; [\bar{T}, F] = \bar{T};$$
$$[Q, N] = [T, N] = \frac{Q - T}{2}, \quad -[\bar{Q}, N] = [\bar{T}, N] = \frac{\bar{Q} + \bar{T}}{2} \; .$$

- When $\gamma \neq 0$, the algebra is deformed:
- Let $H = H_0 - \gamma F/2$ and introduce $F_+ = \bar{\chi}\psi, \quad F_- = \bar{\psi}\chi$

then

$$
\begin{aligned}
[F_\pm, F] = \mp 2F_\pm, \quad [F_+, F_-] = F \ , \\
[Q, H_0] = -\frac{\gamma}{2}Q, \ [\bar{Q}, H_0] = \frac{\gamma}{2}\bar{Q}, \\
[T, H_0] = \frac{\gamma}{2}T, \ [\bar{T}, H_0] = -\frac{\gamma}{2}\bar{T}\,, \\
[Q, F] = -Q, \quad [\bar{Q}, F] = \bar{Q}, \\
[T, F] = T, \quad [\bar{T}, F] = -\bar{T}\,, \\
[Q, F_-] = \bar{T}, \ [\bar{Q}, F_+] = -T, \\
[T, F_-] = -\bar{Q}, \quad [\bar{T}, F_+] = Q\,, \\
\{Q, \bar{Q}\} = 2H_0 - \gamma F, \quad \{T, \bar{T}\} = 2H_0 + \gamma F, \\
\{Q, T\} = 2\gamma F_+, \quad \{\bar{Q}, \bar{T}\} = 2\gamma F_- \ .
\end{aligned}
$$

- This is $osp(2,2)$ algebra.
- a close relative of weak supersymmetry algebra [A.S., PLB 585 (2004) 173].

(1+1) FIELD THEORY

- Let Φ depend on t and x. Choose

$$S = \int dt dx d\bar{\theta} d\theta \left[-2i\mathcal{D}\Phi\partial_+\mathcal{D}\Phi + V(\Phi)\right] ,$$

where $\partial_\pm = (\partial_t \pm \partial_x)/2$ and

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\theta\partial_-, \qquad \bar{\mathcal{D}} = \frac{\partial}{\partial\bar{\theta}} - i\bar{\theta}\partial_+$$

Bosonic Lagrangian

$$\mathcal{L}_B = \partial_\mu\phi\partial_\mu D + DV'(\phi)$$

with

$$V(\phi) = \frac{\omega^2\phi^2}{2} + \frac{\lambda\phi^4}{4}, \qquad \lambda > 0 .$$

Equations of motion:

$$\begin{aligned} \Box\phi + \omega^2\phi + \lambda\phi^3 &= 0 \\ \Box D + D(\omega^2 + 3\lambda\phi^2) &= 0 . \end{aligned} \qquad (3)$$

- Only two integrals of motion: the energy and

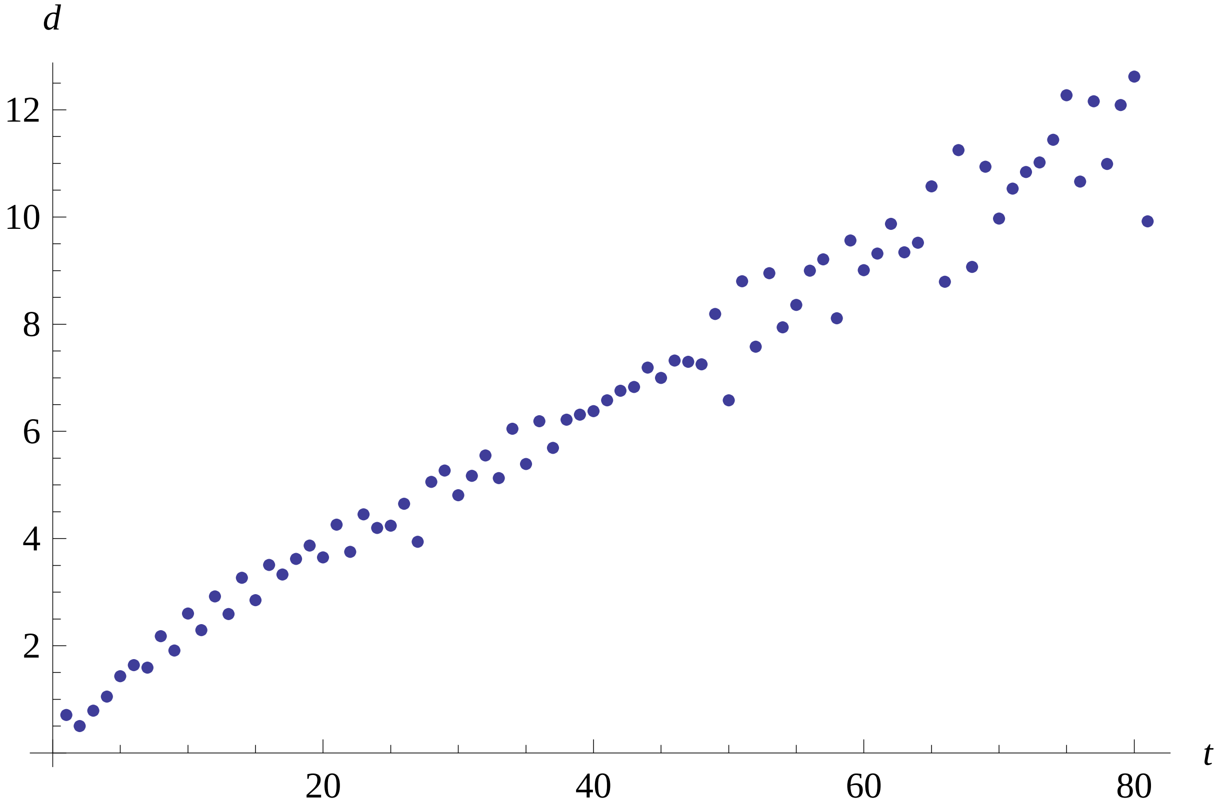
$$N = \int dx \left\{ \frac{1}{2}\left[\dot{\phi}^2 + (\partial_x\phi)^2\right] + \frac{\omega^2\phi^2}{2} + \frac{\lambda\phi^4}{4} \right\} .$$

- Not exactly solvable.

• Stochasticity. Solved numerically.

• Finite spatial box. Different initial conditions.

TYPICAL BEHAVIOR:



Dispersion $d = \sqrt{\langle D^2 \rangle_x}$ as a function of time.

SWEET DREAM

The TOE is a higher-derivative field theory with benign ghosts living in a higher-dimensional flat space-time. Our Universe represents a 3-brane — a solitonic solution extended in three spatial and the time directions and localized in the extra dimensions. Gravity arises as effective theory on the world volume of this brane.