Spin Calogero–Moser integrable systems related with the cyclic quiver

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Rational Calogero–Moser system for A_{n-1} case

- Hamiltonian of Calogero–Moser system for $W_{A_{n-1}} = S_n$:

$$H = \sum_{a=1}^{n} p_a^2 - 2\sum_{a<b} \frac{1}{(x_a - x_b)^2} \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})^{S_n} = \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}}/S_n)\,,$$

where $\mathfrak{h}_{reg} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_a \neq x_b \text{ if } a \neq b\}$.

- There exist n algebraically independent integrals of motion $H_1, \ldots, H_n \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})^{S_n}$ such that

$$\{H_k, H_\ell\} = 0\,, \qquad H_1 = \sum_{a=1}^{n} p_a\,, \qquad H_2 = H$$

- Commuting flows: $\partial_{t_k} f = \{H_k, f\}$.

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Calogero–Moser space

- *Calogero–Moser space* is a symplectic affine variety defined as

$$\mathcal{C}_n = \{(X, Y, v, w) \mid [X, Y] = 1 - vw\}/\mathrm{GL}(n, \mathbb{C}),$$

where $X, Y \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, $v \in \mathbb{C}^n$, $w \in (\mathbb{C}^n)^*$. The action of $g \in \mathrm{GL}(n, \mathbb{C})$ is $g.(X, Y, v, w) = (gXg^{-1}, gYg^{-1}, gv, wg^{-1})$.

- In a generic point of $\mathcal{C}_n$:

$$X_{ab} = \delta_{ab} x_a, \qquad Y_{ab} = \delta_{ab} p_a - (1 - \delta_{ab}) \frac{1}{x_a - x_b},$$

$$v_a = 1, \qquad w_a = 1$$

- The local Darboux coordinates on $\mathcal{C}_n$ are $(p_a, x_a)_{a=1}^n$.
- $\mathcal{C}_n$ is a completion of the symmetrised phase space $T^*\mathfrak{h}_{\mathrm{reg}}/S_n$.

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Dynamics on $\mathcal{C}_n$

- $X(t) = X + \sum_{k=1}^{n} k t_k Y^{k-1}, \quad Y, v, w = const,$ where $t = (t_1, \dots, t_n)$.
- Dynamics on $\mathcal{C}_n$ in the local coordinates $(p_a, x_a)_{a=1}^n$ gives solutions of the Calogero–Moser system: $x_a = x_a(t)$, $p_a = p_a(t)$.
- This dynamics can be given by the Poisson-commuting Hamiltonians

$$H_k = \mathrm{tr}(Y^k) \in \mathcal{O}(\mathcal{C}_n),$$

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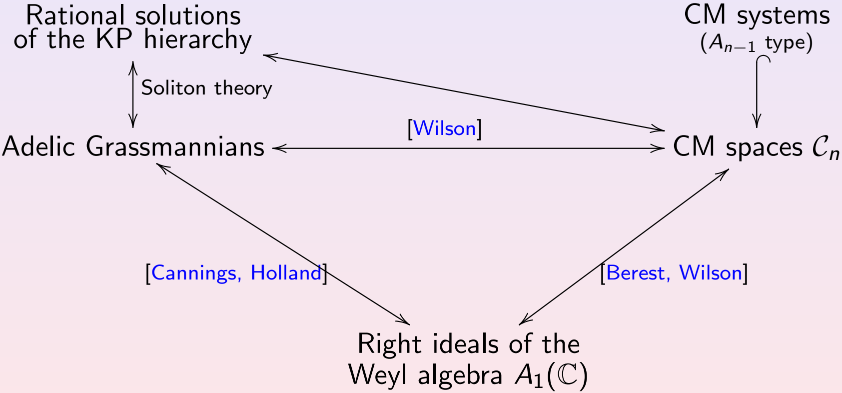
Scheme of the CM correspondence



where $A_1(\mathbb{C}) = \mathbb{C}\langle x, y\rangle / \big(xy - yx - 1 = 0\big)$.

CM correspondence for the cyclic quiver (spherical case)

Spherical solutions of the generalised KP hierarchy

CM systems for $G = S_n \ltimes \mathbb{Z}_m^n$

[Etingof, Ginzburg] if $\alpha = n\delta$

[Chalykh, A.S.]

Quiver varieties $M_\lambda(\alpha, \varepsilon_0)$

[Baranovsky, Ginzburg, Kuznetsov]

Right ideals of the spherical Cherednik algebra $B_\lambda(\mathbb{Z}_m)$

where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, $\alpha \in \mathbb{Z}^m$, $\lambda \in \mathbb{C}^m$, $\varepsilon_0 = (1, 0 \ldots, 0)$, $\delta = (1, \ldots, 1)$.

CM correspondence for the cyclic quiver

More general solutions of the generalised KP hierarchy

CM systems for $G = S_n \ltimes \mathbb{Z}_m^n$ with some internal variables

if $\alpha = n\delta$

[Chalykh, A.S.]

Quiver varieties $M_\lambda(\alpha, \delta)$

Gibbons–Hermsen system

- Hamiltonian of Gibbons–Hermsen system (spin A_{n-1} Calogero–Moser system):

$$H = \sum_{a=1}^{n} p_a^2 - 2 \sum_{a<b} \frac{(\psi_a \varphi_b)(\psi_b \varphi_a)}{(x_a - x_b)^2},$$

where $\varphi_a \in \mathbb{C}^d$, $\psi_a \in (\mathbb{C}^d)^*$ such that $\psi_a \varphi_a = 1$ for any $a = 1, \dots, n$.

- There exist nd algebraically independent integrals of motion $H_{k,r}$, $k = 1, \dots, n$, $r = 1, \dots, d$:

$$\{H_{k,r}, H_{\ell,s}\} = 0\,,$$

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CM correspondence for the matrix KP hierarchy

Solutions of the $(d \times d)$ matrix KP hierarchy

Gibbons–Hermsen systems (spin A_{n-1} CM system)

[Chalykh, A.S.]

Quiver varieties $M_\lambda(n, d)$ $(m = 1)$

where $d \in \mathbb{Z}_{\geqslant 1}$.

CM correspondence for general $m, d \in \mathbb{Z}_{\geqslant 1}$

Solutions of the generalised matrix KP hierarchy

General spin CM systems for $G = S_n \ltimes \mathbb{Z}_m^n$

if $\alpha = n\delta$

[Chalykh, A.S.]

Quiver varieties $M_\lambda(\alpha, d \cdot \delta)$

Quivers and their representations

- *Quiver* is a directed graph $Q = (I, E)$, where I and E are (finite) sets of vertices and edges. Let notation $X : i \to j$ mean that the edge $X \in E$ goes from the vertex $i \in I$ to the vertex $j \in I$.
- *Representation* of the quiver Q is a family $V = (V_i, V_X)_{i \in I, X \in E}$, where V_i are vector spaces and $V_X : V_i \to V_j$ are linear operators $(X : i \to j)$.
- Let $\alpha = (\alpha_i)_{i \in I} \in \mathbb{Z}^I_{\geqslant 0}$ and $V_i = \mathbb{C}^{\alpha_i}$. Then the representations $V = (V_i, V_X)$ form the vector space $\mathrm{Rep}(Q, \alpha)$.
- The group $\mathrm{GL}(\alpha) = \prod_{i \in I} \mathrm{GL}(\alpha_i, \mathbb{C})$ acts on $\mathrm{Rep}(Q, \alpha)$.
- Two representations $V, V' \in \mathrm{Rep}(Q, \alpha)$ are isomorphic if and only if they present the same equivalency class of $\mathrm{Rep}(Q, \alpha)/\mathrm{GL}(\alpha)$.

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Doubled quiver

- The *doubled quiver* for $Q=(I,E)$ is the quiver $\overline{Q}=(I,\overline{E})$, where $\overline{E}=E\sqcup\{X^*\colon j\to i\mid X\in E, X\colon i\to j\}$.
- For a vector $\boldsymbol{\lambda}=(\lambda_i)_{i\in I}\in\mathbb{C}^I$ denote by $\mu_\alpha^{-1}(\boldsymbol{\lambda})$ the set of representations $V\in\mathrm{Rep}(\overline{Q},\alpha)$ satisfying $\mu_{\alpha,i}(V)=\lambda_i\mathbf{1}_{\alpha_i}$, where $\mathbf{1}_n$ is the $n\times n$ matrix unit and

$$\mu_{\alpha,i}(V)=\sum_{\substack{X\in E,\ j\in I\\ X\colon j\to i}} V_X V_{X^*}-\sum_{\substack{X\in E,\ j\in I\\ X\colon i\to j}} V_{X^*}V_X$$

- If the orbit space $\mu_\alpha^{-1}(\boldsymbol{\lambda})/\mathrm{GL}(\alpha)$ has a structure of variety, it is called *symplectic quotient*. It has a canonical Poisson brackets inherited from $\mathrm{Rep}(\overline{Q},\alpha)=T^*\,\mathrm{Rep}(Q,\alpha)$.

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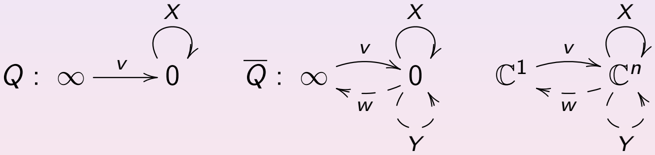
Calogero–Moser spaces as symplectic quotients

- Example: $I = \{\infty, 0\}$, $E = \{X, v\}$, $Y = X^*$, $w = v^*$:



If $\boldsymbol{\alpha} = (1, n)$, $\boldsymbol{\lambda} = (-n, 1)$ then $\mu_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{\lambda})/\mathrm{GL}(\boldsymbol{\alpha}) = \mathcal{C}_n$.

Framing of a quiver

- Let $\zeta \in \mathbb{Z}^I_{\geqslant 0}$. Framing of Q is the quiver $Q_\zeta = (I_\infty, E_\zeta)$ where

$$I_\infty = \{\infty\} \sqcup I\,, \quad E_\zeta = E \sqcup \{v_{i,r} \colon \infty \to i \mid i \in I,\ r = 1, \dots, \zeta_i\}$$

For $\alpha \in \mathbb{Z}^I_{\geqslant 0}$ and $\lambda \in \mathbb{C}^I$ we extend them to

$$\boldsymbol{\alpha} = (1, \alpha)\,, \qquad \boldsymbol{\lambda} = (-\lambda \cdot \alpha, \lambda)$$

Consider the symplectic quotient for the framed quiver Q_ζ:

$$M_\lambda(\alpha, \zeta) = \mu_{\boldsymbol{\alpha}}^{-1}(\boldsymbol{\lambda})/\mathrm{GL}(\boldsymbol{\alpha})$$

- For generic λ the quotient $M_\lambda(\alpha, \zeta)$ is a connected smooth affine variety and it is called *quiver variety*.

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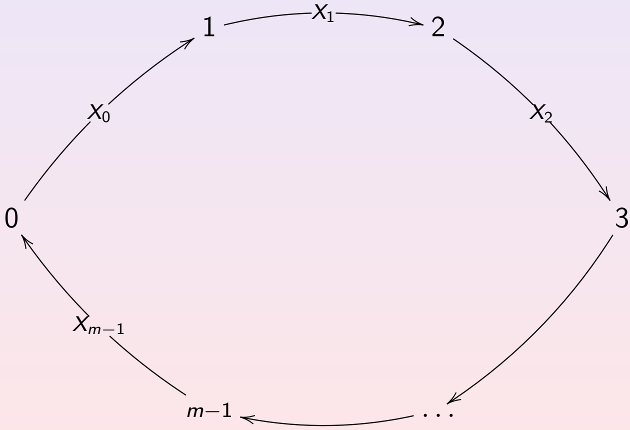
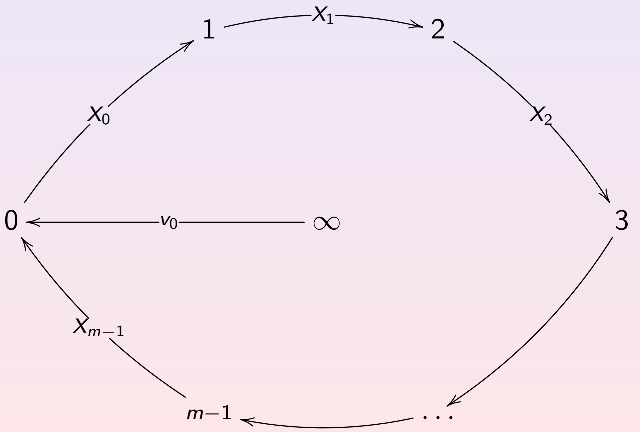
Cyclic quiver

- Quiver Q:



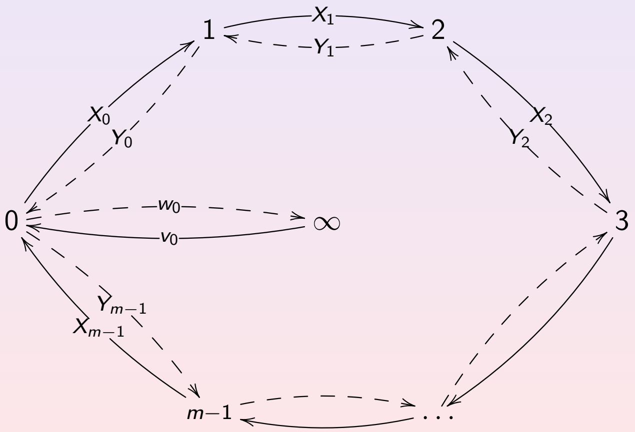
Framing by $\zeta = \varepsilon_0 = (1, 0, \ldots, 0)$

- Quiver Q_{ε_0}, where $\varepsilon_0 = (1, 0, \ldots, 0)$:



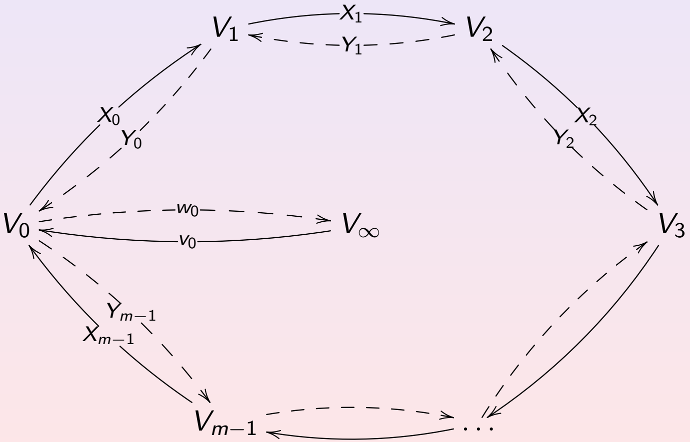
Quiver $\overline{Q}_{\varepsilon_0}$

- Quiver $\overline{Q}_{\varepsilon_0}$, where $Y_i = X_i^*$, $w_0 = v_0^*$:



Representations of the quiver $\overline{Q}_{\varepsilon_0}$

- $V \in \mathrm{Rep}(\overline{Q}_{\varepsilon_0}, \boldsymbol{\alpha})$, $\boldsymbol{\alpha} = (1, \alpha_0, \dots, \alpha_{m-1})$, $V_i = \mathbb{C}^{\alpha_i}$, $V_\infty = \mathbb{C}^1$:



Quiver varieties $M_\lambda(\alpha, \varepsilon_0)$

- Let $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{C}^m$ such that $\sum_{i=0}^{m-1} \lambda_i = 1$.
- Then $M_\lambda(\alpha, \varepsilon_0) = \{(X_i, Y_i, v_0, w_0)\}/\mathrm{GL}(\alpha)$, where matrices X_i, Y_i, vector v_0 and covector w_0 satisfy

$$X_{m-1}Y_{m-1} - Y_0X_0 + v_0w_0 = \lambda_0 \mathbf{1}_{\alpha_0}\,,$$
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$$H_k = w_0(Y_0Y_1 \cdots Y_{m-1})^k v_0, \qquad \{H_k, H_\ell\} = 0$$

- The flow $\partial_{t_k} f = \{H_k, f\}$ defined by the Hamiltonian H_k can be written explicitly:

$$X_i(t_k) = X_i + kt_k Y_{i+1}Y_{i+2} \cdots Y_{i+mk-1}\,, \quad Y_i, v_0, w_0 = const$$

Quiver varieties $M_\lambda(\alpha, \varepsilon_0)$

- Let $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{C}^m$ such that $\sum_{i=0}^{m-1} \lambda_i = 1$.
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Darboux coordinates on $M_\lambda(n\delta, \varepsilon_0)$

- Let $\alpha_1 = \ldots = \alpha_{m-1} = n$, i.e. $\alpha = n\delta$, where $\delta = (1, \ldots, 1)$. Then $X_i, Y_i \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, $v_0 \in \mathbb{C}^n$, $w_0 \in (\mathbb{C}^n)^*$.
- $\dim M_\lambda(n\delta, \varepsilon_0) = 2n$.
- Generic point:

$$(X_i)_{ab} = x_a \delta_{ab}\,, \qquad (v_0)_a = 1\,, \qquad (w_0)_a = 1\,,$$

$$(Y_i)_{aa} = \frac{1}{m} p_a - \frac{1}{x_a}\left(\sum_{j=0}^{i} \lambda_j + \kappa(\lambda)\right), \qquad (Y_i)_{ab} = -\frac{x_a^{m-1-i} x_b^i}{x_a^m - x_b^m}\,,$$

where $i = 0, \ldots, m-1$, $\kappa(\lambda) = \sum_{j=0}^{m-1} \frac{j-m}{m} \lambda_j$.

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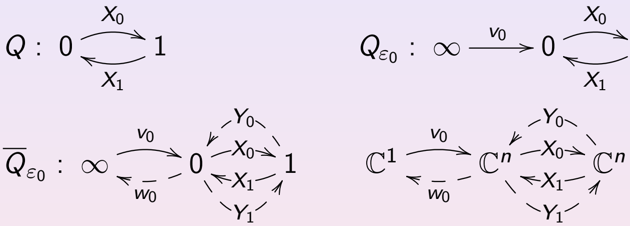
Calogero–Moser systems for B_n and $S_n \ltimes \mathbb{Z}_m^n$

- For $m = 2$ the quivers are



- The Hamiltonian is

$$H_1 = w_0 Y_0 Y_1 v_0 = \frac{1}{4}\sum_{a=1}^{n}\left(p_a^2 - \frac{\lambda_1^2}{x_a^2}\right) - \sum_{a<b}\frac{x_a^2 + x_b^2}{(x_a^2 - x_b^2)^2}$$

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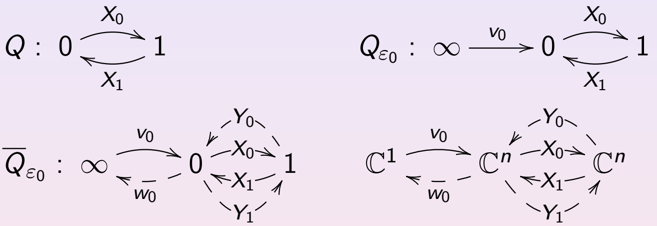

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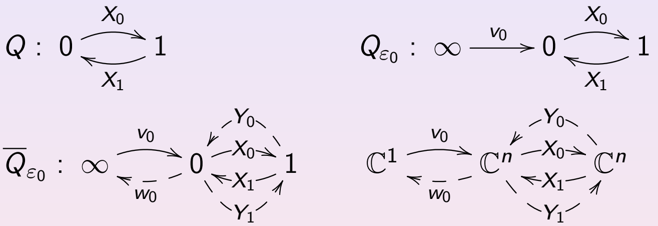

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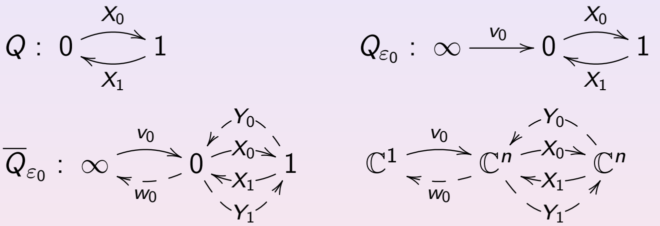

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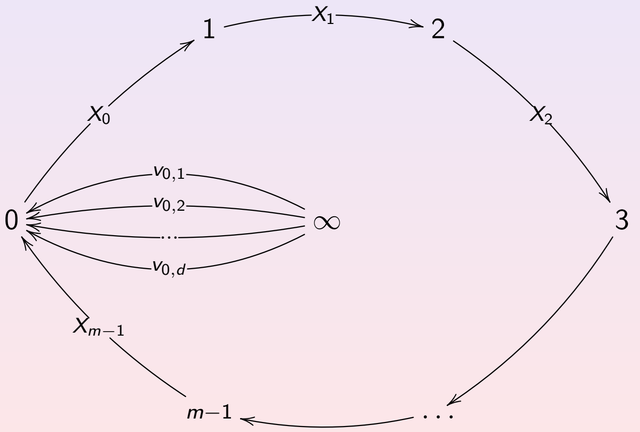
Framing by $\zeta = d\varepsilon_0 = (d, 0, \ldots, 0)$

- Quiver $Q_{d\varepsilon_0}$, where $d\varepsilon_0 = (d, 0, \ldots, 0)$, $d \in \mathbb{Z}_{\geqslant 1}$:



Quiver $\overline{Q}_{d\varepsilon_0}$

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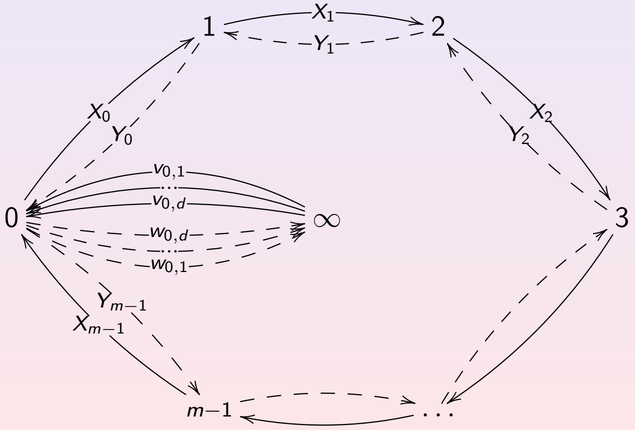
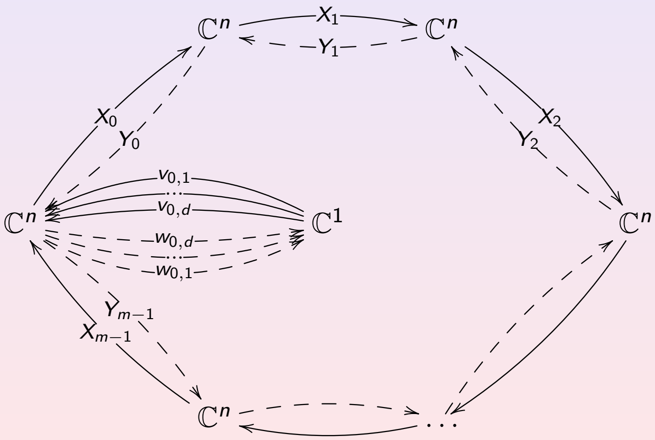


Representations of the quiver $\overline{Q}_{d\varepsilon_0}$

- $V \in \mathrm{Rep}(\overline{Q}_{d\varepsilon_0}, \boldsymbol{\alpha})$, $\boldsymbol{\alpha} = (1, n, \ldots, n)$, i.e. $\alpha = n\delta$



Quiver variety $M_\lambda(n\delta, d\varepsilon_0)$

- $M_\lambda(n\delta, d\varepsilon_0) = \{(X_i, Y_i, v_{0,r}, w_{0,r})\}/\mathrm{GL}(n\delta)$, where

$$X_{m-1}Y_{m-1} - Y_0X_0 + \sum_{r=1}^{d} v_{0,r}w_{0,r} = \lambda_0 \mathbf{1}_n\,,$$
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Darboux coordinates on $M_\lambda(n\delta, d\varepsilon_0)$

- $\dim M_\lambda(n\delta, d\varepsilon_0) = 2nd$.
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where $0 \leqslant i \leqslant m-1$, $1 \leqslant r \leqslant d$, $1 \leqslant a, b \leqslant n$ and $\varphi_a \in \mathbb{C}^d$, $\psi_a \in (\mathbb{C}^d)^*$ are such that $\psi_a\varphi_a = 1$ for any $a = 1, \ldots, n$.
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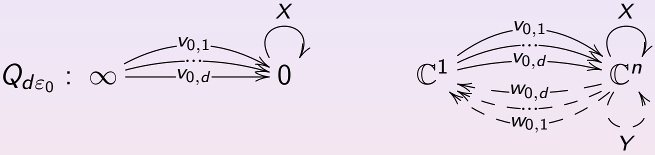
A_{n-1} case

- Let $m = 1$:



- Then $H_{k,r}$ are integrals of motion for the Gibbons–Hermsen system:

$$\sum_{r=1}^{d} H_{2,r} = \sum_{a=1}^{n} p_a^2 - 2 \sum_{a<b} \frac{1}{(x_a - x_b)^2} (\psi_a \varphi_b)(\psi_b \varphi_a)$$

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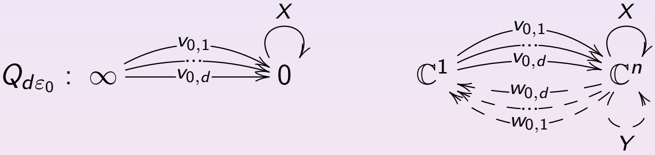

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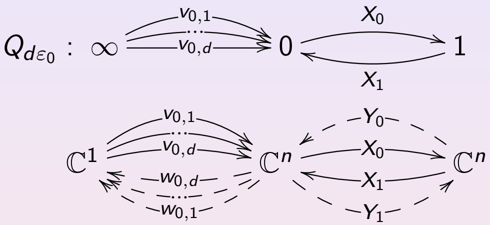
B_n case

- Let $m = 2$:



- Then the Poisson-commuting functions $H_{k,r}$ define the B_n analogue of the Gibbons–Hermsen system with the Hamiltonian

$$\sum_{r=1}^{d} H_{2,r} = \frac{1}{4}\sum_{a=1}^{n}\left(p_a^2 - \frac{\lambda_1^2}{x_a^2}\right) - \sum_{a<b}\frac{x_a^2 + x_b^2}{(x_a^2 - x_b^2)^2}(\psi_a\varphi_b)(\psi_b\varphi_a)$$

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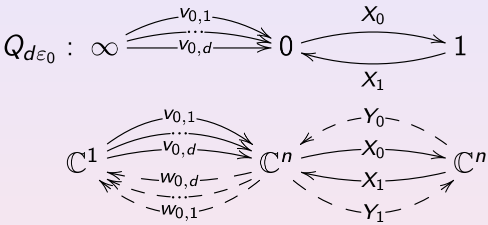

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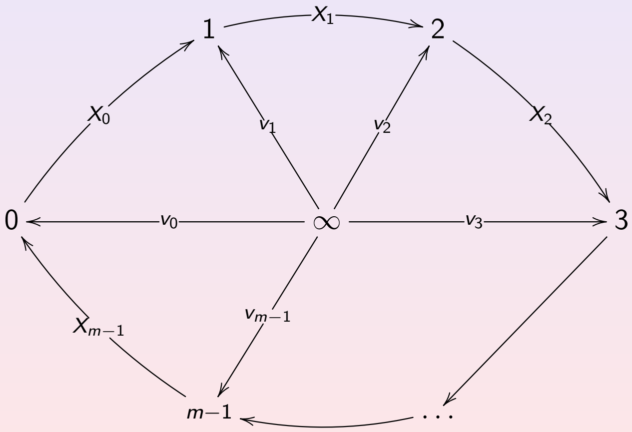
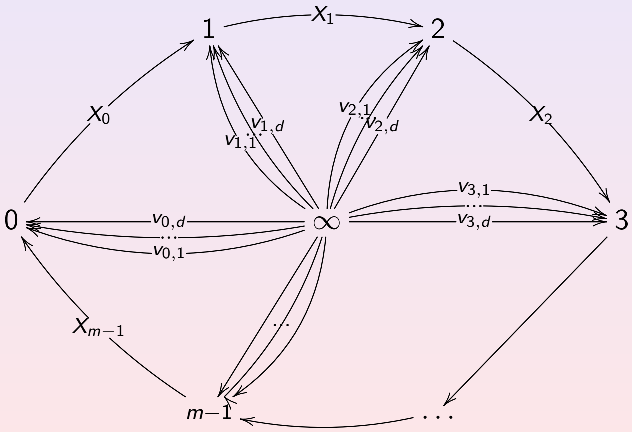
Framing by $\zeta = \delta = (1, 1, \ldots, 1)$

- Quiver Q_δ, where $\delta = (1, 1, \ldots, 1)$:



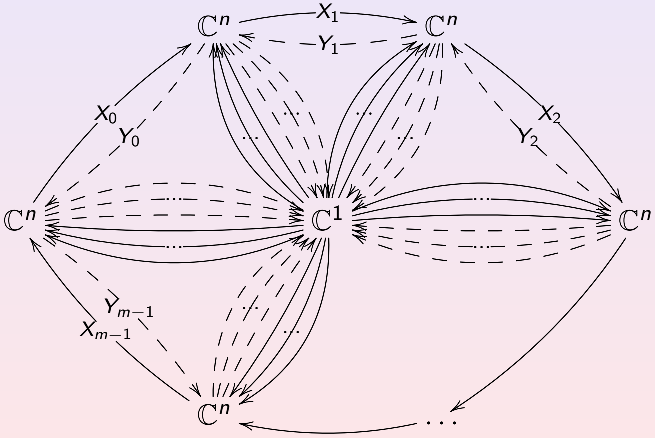
Framing by $\zeta = d \cdot \delta = (d, d, \dots, d)$

- Quiver $Q_{d\delta}$, where $d\delta = (d, d, \dots, d)$:



Representations of the quiver $\overline{Q}_{d\delta}$

- $V \in \mathrm{Rep}(\overline{Q}_{d\delta}, \boldsymbol{\alpha})$, $\boldsymbol{\alpha} = (1, n, \ldots, n)$:



Integrable system on $M_\lambda(n\delta, d\delta)$

- $M_\lambda(n\delta, d\delta) = \{(X_i, Y_i, v_{i,r}, w_{i,r})\}/\mathrm{GL}(n\delta)$, where

$$X_{i-1}Y_{i-1} - Y_i X_i + \sum_{r=1}^{d} v_{i,r} w_{i,r} = \lambda_i \mathbf{1}_n\,, \quad i = 0, \dots, m-1$$

- The integrals $H_{k,r} \in \mathcal{O}\big(M_\lambda(n\delta, d\delta)\big)$:

$$H_{k,r} = \sum_{i=0}^{m-1} w_{i,r} Y_i Y_{i+1} \cdots Y_{i+k} v_{i+k,r}\,, \quad \{H_{k,r}, H_{\ell,r}\} = 0$$

- $\dim M_\lambda(n\delta, d\delta) = 2nmd$.
- The functions $H_{k,r}$, $k = 1, \dots, nm$, $r = 1, \dots, d$, are algebraically independent. They define complete flows $\partial_{t_{k,r}} f = \{H_{k,r}, f\}$ on the variety $M_\lambda(n\delta, d\delta)$.

Integrable system on $M_\lambda(n\delta, d\delta)$

- $M_\lambda(n\delta, d\delta) = \{(X_i, Y_i, v_{i,r}, w_{i,r})\}/\mathrm{GL}(n\delta)$, where

$$X_{i-1}Y_{i-1} - Y_iX_i + \sum_{r=1}^{d} v_{i,r}w_{i,r} = \lambda_i \mathbf{1}_n\,, \quad i = 0, \ldots, m-1$$

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Thank you for your attention