SU(4|1) supersymmetric mechanics

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The talk is based on:

Evgeny Ivanov, Olaf Lechtenfeld, Stepan Sidorov, arXiv:1609.00490 [hep-th], arXiv:1807.11804 [hep-th].

Outline

Introduction

- In the last decade, interest has grown in rigid supersymmetric theories invariant under some “curved” analogs of Poincaré supersymmetry in diverse dimensions (*e.g.*, G. Festuccia, N. Seiberg, arXiv:1105.0689 [hep-th]), since the localization method V. Pestun, arXiv:0712.2824 [hep-th] for rigid supersymmetric theories is a powerful tool allowing to compute non-perturbatively quantum objects, such as partition function, *etc.*

Introduction

- In the last decade, interest has grown in rigid supersymmetric theories invariant under some "curved" analogs of Poincaré supersymmetry in diverse dimensions (*e.g.*, G. Festuccia, N. Seiberg, **arXiv:1105.0689 [hep-th]**), since the localization method V. Pestun, **arXiv:0712.2824 [hep-th]** for rigid supersymmetric theories is a powerful tool allowing to compute non-perturbatively quantum objects, such as partition function, *etc.*

- Motivated by this interest, we proposed a new type of Supersymmeric Quantum Mechanics (SQM) based on the worldline realization of the supergroup $SU(2|1)$ in the appropriate $\mathcal{N} = 4$, $d = 1$ superspace E. Ivanov, S. Sidorov, **arXiv:1307.7690 [hep-th], 1312.6821 [hep-th]**.

SU(2|1) supersymmetric mechanics

- In the frame of SU(2|1) superfield approach, we reproduced “Weak Supersymmetry” models A. Smilga, arXiv:hep-th/0311023 based on the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ and models based on two types of the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ (S. Bellucci, A. Nersessian, arXiv:hep-th/0211070, hep-th/0401232 and C. Römelsberger, arXiv:hep-th/0510060, 0707.3702 [hep-th]).

- The SU(2|1) superfield techniques not only reproduced these known models, but also revealed new models studied in the series of papers:
  * E. Ivanov, S. Sidorov, F. Toppan, arXiv:1501.05622 [hep-th],
  * E. Ivanov, S. Sidorov, arXiv:1507.00987 [hep-th],
  * E. Ivanov, S. Sidorov, arXiv:1509.05561 [hep-th],
  * S. Fedoruk, E. Ivanov, arXiv:1610.04202 [hep-th],
  * S. Fedoruk, E. Ivanov, S. Sidorov, arXiv:1710.02130 [hep-th],
  * S. Fedoruk, E. Ivanov, O. Lechtenfeld, S. Sidorov, arXiv:1801.00206 [hep-th].

- Some further studies of SU(2|1) mechanics were given in the component level:
  * B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, D. Martelli, arXiv:1503.05537 [hep-th],
  * C.T. Asplund, F. Denef, E. Dzienkowski, arXiv:1510.04398 [hep-th],
  * N. Kozyrev, S. Krivonos, O. Lechtenfeld, A. Sutulin, arXiv:1712.09898 [hep-th].

The superalgebra $su(2|1)$

Our studies of SU(2|1) supersymmetric mechanics were based on a deformation

$$\mathcal{N} = 4,\ d = 1 \text{ Poincaré} \quad \Rightarrow \quad su(2|1),$$

where the superalgebra $su(2|1)$ is given by

$$\{Q^i, \bar{Q}_j\} = 2mI^i_j + 2\delta^i_j \tilde{H}, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[\tilde{H}, \bar{Q}_l\right] = \frac{m}{2}\,\bar{Q}_l\,, \qquad \left[\tilde{H}, Q^k\right] = -\frac{m}{2}\,Q^k.$$

The generators I^i_j form SU(2) symmetry, while the mass-dimensional generator $\tilde{H}$ is U(1) symmetry generator. In the limit $m = 0$, the generators I^i_j become the SU(2) automorphism generators of the standard $\mathcal{N} = 4$, $d = 1$ superalgebra with Hamiltonian $\tilde{H}$ becoming a central charge generator.

Deformations of $\mathcal{N} = 8$, $d = 1$ superalgebra

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A. The first option is

$$\mathcal{N}=8,\ d=1\ \text{Poincaré} \quad \Rightarrow \quad su(2|2).$$

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B. The second deformation is

$$\mathcal{N}=8,\ d=1\ \text{Poincaré} \quad \Rightarrow \quad su(4|1).$$

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$SU(2|2)$ supersymmetric mechanics

- The superalgebra $su(2|2)$ in the complex basis is written as

$$\left\{Q^{ia},\bar{Q}^{jb}\right\} = -\,2m\left(\varepsilon^{ab}L^{ij}-\varepsilon^{ij}R^{ab}\right)+2\,\varepsilon^{ab}\varepsilon^{ij}H,$$
$$\left[L^{ij},L^{kl}\right]=\varepsilon^{il}L^{kj}+\varepsilon^{jk}L^{il},\quad \left[R^{ab},R^{cd}\right]=\varepsilon^{ad}R^{bc}+\varepsilon^{bc}R^{ad},$$
$$\left[L^{ij},Q^{ka}\right]=\frac{1}{2}\left(\varepsilon^{ik}Q^{ja}+\varepsilon^{jk}Q^{ia}\right),\quad \left[R^{ab},Q^{kc}\right]=\frac{1}{2}\left(\varepsilon^{ac}Q^{kb}+\varepsilon^{bc}Q^{ka}\right),$$
$$\left[L^{ij},\bar{Q}^{ka}\right]=\frac{1}{2}\left(\varepsilon^{ik}\bar{Q}^{ja}+\varepsilon^{jk}\bar{Q}^{ia}\right),\quad \left[R^{ab},\bar{Q}^{kc}\right]=\frac{1}{2}\left(\varepsilon^{ac}\bar{Q}^{kb}+\varepsilon^{bc}\bar{Q}^{ka}\right).$$

All other (anti)commutators are vanishing. Here the superalgebra $su(2|2)$ contains the generators $L^{ij}=L^{ji}$, $R^{ab}=R^{ba}$ forming two mutually commuting $su(2)$ algebras and the central charge generator H.

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- Employing the appropriate worldline superfield approach SU(2|2), we considered deformed analogs of $\mathcal{N} = 8$ supersymmetric quantum mechanics (Evgeny Ivanov, Olaf Lechtenfeld, Stepan Sidorov, arXiv:1609.00490 [hep-th]). We studied models of SU(2|2) supersymmetric mechanics based on the off-shell multiplets $(\mathbf{3},\mathbf{8},\mathbf{5})$, $(\mathbf{4},\mathbf{8},\mathbf{4})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$.

Deformed $\mathcal{N} = 8$ supermultiplets

- Other multiplets $(\mathbf{k}, \mathbf{8}, \mathbf{8} - \mathbf{k})$, $\mathbf{k} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}, \mathbf{8}$ of the standard $\mathcal{N} = 8$ mechanics have no deformations to $\mathrm{SU}(2|2)$ supersymmetric mechanics.

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- There are four types of $\mathcal{N} = 8$ superconformal algebras: $osp(8|2)$, $F(4)$, $su(4|1,1)$, $osp(4^*|4)$. According to S. Khodaee, F. Toppan, `arXiv:1208.3612 [hep-th]`, the relevant superconformal transformations are realized on $\mathcal{N} = 8$ multiplets as

  1) $osp(8|2)$ on $(\mathbf{0}, \mathbf{8}, \mathbf{8})$ and $(\mathbf{8}, \mathbf{8}, \mathbf{0})$,

  2) $F(4)$ on $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$,

  3) $su(4|1,1)$ on $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ and $(\mathbf{6}, \mathbf{8}, \mathbf{2})$,

  4) $osp(4^*|4)$ on $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ and $(\mathbf{5}, \mathbf{8}, \mathbf{3})$.

  The superalgebra $su(2|2)$ can be embedded only into the superconformal algebra $osp(4^*|4)$.

Deformed $\mathcal{N}=8$ supermultiplets

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  4) $osp(4^*|4)$ on $(\mathbf{3},\mathbf{8},\mathbf{5})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$.

  The superalgebra $su(2|2)$ can be embedded only into the superconformal algebra $osp(4^*|4)$.

- The multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ is exceptional: none of $\mathcal{N}=8$, $d=1$ superconformal symmetries can act on it. However, one can realize on it an $\mathcal{N}=8$ extended Heisenberg superalgebra given by S. Bellucci, E. Ivanov, A. Sutulin, `arXiv:hep-th/0504185`. This extended superalgebra contains an $su(2|2)$ superalgebra.

Deformed $\mathcal{N} = 8$ supermultiplets

- Hence, the supergroup SU(2|2) admits an action only on the multiplets $(\mathbf{3}, \mathbf{8}, \mathbf{5})$, $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ and $(\mathbf{5}, \mathbf{8}, \mathbf{3})$.

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- The superalgebra $su(4|1)$ can be embedded into the rest of superconformal algebras $osp(8|2)$, $F(4)$, $su(4|1,1)$.

Deformed $\mathcal{N} = 8$ supermultiplets

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- The superalgebra $su(4|1)$ can be embedded into the rest of superconformal algebras $osp(8|2)$, $F(4)$, $su(4|1,1)$.

- Thus, the rest of $\mathcal{N} = 8$ multiplets $(\mathbf{0}, \mathbf{8}, \mathbf{8})$, $(\mathbf{1}, \mathbf{8}, \mathbf{7})$, $(\mathbf{2}, \mathbf{8}, \mathbf{6})$, $(\mathbf{6}, \mathbf{8}, \mathbf{2})$, $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ and $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ have generalizations to SU(4|1) supersymmetric mechanics.

SU(4|1) supersymmetric mechanics

We consider the superalgebra $su(4|1)$ as a deformation of the standard $\mathcal{N}=8,\ d=1$ superalgebra:

$$\mathcal{N}=8,\ d=1 \quad \text{Poincaré} \quad \Rightarrow \quad su(4|1).$$

It is given by the following (anti)commutators:

$$\left\{Q^I, \bar{Q}_J\right\} = 2m\, L^I_J + 2\delta^I_J \mathcal{H}, \qquad \left[L^I_J, L^K_L\right] = \delta^K_J\, L^I_L - \delta^I_L L^K_J\,,$$
$$\left[L^I_J, Q^K\right] = \delta^K_J Q^I - \frac{1}{4}\,\delta^I_J Q^K, \qquad \left[L^I_J, \bar{Q}_L\right] = \frac{1}{4}\,\delta^I_J \bar{Q}_L - \delta^I_L \bar{Q}_J\,,$$
$$\left[\mathcal{H}, Q^K\right] = -\,\frac{3m}{4}\, Q^K, \qquad \left[\mathcal{H}, \bar{Q}_L\right] = \frac{3m}{4}\,\bar{Q}_L\,.$$

All other (anti)commutators are vanishing. Here, the superalgebra $su(4|1)$ contains eight supercharges and $\mathrm{SU}(4)\times\mathrm{U}(1)$ generators L^I_J, $\mathcal{H}$. It can be viewed as a deformation of the standard $\mathcal{N}=8$, $d=1$ superalgebra.

SU(4|1), $d = 1$ superspace

The SU(4|1), $d = 1$ superspace is defined as the coset superspace

$$\frac{\mathrm{SU}(4|1)}{\mathrm{SU}(4)} \sim \frac{\{Q^I, \bar{Q}_J, L^I_J, \mathcal{H}\}}{\{L^I_J\}}\,,$$

where its parameters define the superspace coordinates:

$$\zeta = \left\{t, \theta_I, \bar{\theta}^J\right\}, \qquad \overline{(\theta_I)} = \bar{\theta}^I.$$

Realization of the supergroup for the fermionic coset SU(n|1)/U(n) was studied by E. Ivanov, L. Mezincescu, A. Pashnev, P.K. Townsend, arXiv:hep-th/0301241. Extending this realization by time coordinate, we obtain the odd transformations:

$$\delta\theta_I = \epsilon_I + 2m\,\bar{\epsilon}^K\theta_K\theta_I\,, \qquad \delta\bar{\theta}^J = \bar{\epsilon}^J - 2m\,\epsilon_K\bar{\theta}^K\bar{\theta}^J, \qquad \delta t = i\left(\bar{\epsilon}^K\theta_K + \epsilon_K\bar{\theta}^K\right).$$

Relation to matrix models

- Berenstein, Maldacena and Nastase (BMN) proposed a matrix model associated with M-theory on pp-wave background D. Berenstein, J. Maldacena, H. Nastase, arXiv:hep-th/0202021 with 16 supersymmetries corresponding to the mass-deformed world-line supersymmetry group SU(4|2). Their (on-shell) multiplet is given by

$$\boxed{\left(y^{IJ}, v^{ij}, \chi^{Ii}, \bar{\chi}_{Ii}\right), \qquad y^{IJ} \equiv y^{[IJ]}, \qquad v^{ij} \equiv v^{(ij)},}$$

$$\overline{(y^{IJ})} = y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,y^{KL}, \qquad \overline{(v^{ij})} = v_{ij}, \qquad \overline{(\chi^{Ii})} = \bar{\chi}_{Ii}\,,$$

The indices $I = 1, 2, 3, 4$ and $i = 1, 2$ are indices of the subgroup $\mathrm{SU}(4) \times \mathrm{SU}(2)$, respectively.

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  The indices $I = 1,2,3,4$ and $i = 1,2$ are indices of the subgroup $\mathrm{SU}(4)\times\mathrm{SU}(2)$, respectively.

- It spurred investigations of massive matrix models of SQM with 8 supersymmetries corresponding to the groups SU(2|2), SU(4|1) and with 4 supersymmetries corresponding to the group SU(2|1).

SU(4|1) multiplets $(\mathbf{8}, \mathbf{8}, \mathbf{0})$

- SU(4|1) supersymmetric model corresponding to the first multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ were obtained from BMN matrix model by L. Motl, A. Neitzke, M.M. Sheikh-Jabbari, `arXiv:hep-th/0306051`.

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- Our aim here is to consider $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets of the deformed supersymmetric mechanics with respect to the appropriate worldline realization of the supergroup SU(4|1):

$$\boxed{\left(\phi\,,\bar{\phi}\,,y^{IJ},\chi^{I},\bar{\chi}_{I}\right), \qquad \left(z^{I},\bar{z}_{I}\,,\chi\,,\bar{\chi}\,,\chi^{IJ}\right).}$$

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$$\Big(\phi\,,\bar{\phi}\,,y^{IJ},\chi^{I},\bar{\chi}_{I}\Big)\,,\qquad \Big(z^{I},\bar{z}_{I}\,,\chi\,,\bar{\chi}\,,\chi^{IJ}\Big)\,.$$

- Two $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets have “inverted” SU(4) contents (“mirroring”): the contents of bosons and fermions of the first version coincide with those of fermions and bosons in the second one.

The multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$

The first version of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is defined by the SU(4|1) covariant constraints

$$
\begin{aligned}
&\bar{\mathcal{D}}_J\,\Phi = 0\,, \qquad \mathcal{D}^I\,\bar{\Phi} = 0\,, \qquad \bar{\mathcal{D}}_I\bar{\mathcal{D}}_J\,\bar{\Phi} = \frac{1}{2}\,\varepsilon_{IJKL}\,\mathcal{D}^K\mathcal{D}^L\,\Phi\,,\\
&\sqrt{2}\,\mathcal{D}^I\,Y^{JK} = -\,\varepsilon^{IJKL}\,\bar{\mathcal{D}}_L\,\bar{\Phi}\,, \qquad \sqrt{2}\,\bar{\mathcal{D}}_J\,Y_{KL} = \varepsilon_{IJKL}\,\mathcal{D}^I\,\Phi\,,\\
&\overline{(Y^{IJ})} = Y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,Y^{KL}, \qquad \overline{(\Phi)} = \bar{\Phi}\,,
\end{aligned}
$$

where Φ is a chiral superfield and Y^{IJ} is an antisymmetric tensor superfield. Note that in the flat limit, when $m \to 0$ and

$$
D^I = \frac{\partial}{\partial\theta_I} - i\bar{\theta}^I\partial_t\,, \qquad \bar{D}_J = -\,\frac{\partial}{\partial\bar{\theta}^J} + i\theta_J\partial_t\,,
$$

this set of constraints becomes the set of superfield constraints defining the standard $\mathcal{N}=8$, $d=1$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$, such that only $\mathrm{SU}(4)\subset\mathrm{SO}(8)$ is manifest.

Chiral superspace description

The supergroup SU(4|1) admits two mutually conjugated complex supercosets which can be identified with the left and right chiral subspaces:

$$\zeta_{\rm L} = (t_{\rm L}, \theta_I)\,, \qquad \zeta_{\rm R} = \left(t_{\rm R}, \bar{\theta}^J\right).$$

The left even coordinate $t_{\rm L}$ is related to the real time coordinate t via

$$t_{\rm L} = t + \frac{i}{2m}\,\log\left(1 + 2m\,\bar{\theta}^K\theta_K\right).$$

Then we obtain the left chiral space $\zeta_{\rm L}$ closed under the supersymmetry transformations

$$\delta\theta_I = \epsilon_I + 2m\,\bar{\epsilon}^K\theta_K\theta_I\,, \qquad \delta t_{\rm L} = 2i\bar{\epsilon}^K\theta_K\,.$$

The left chiral measure is defined as

$$d\zeta_{\rm L} := dt_{\rm L}\,d^4\theta\,e^{-3imt_{\rm L}}, \qquad \delta\left(d\zeta_{\rm L}\right) = 0\,,$$
$$\int d\zeta_{\rm L}\,\theta_I\theta_J\theta_K\theta_L\,e^{3imt_{\rm L}} = \varepsilon_{IJKL}\,.$$

Chiral superfield

We consider the chiral superfield Φ given by the general θ-expansion

$$\Phi\left(t_{\rm L},\theta_{I}\right) = \phi+\sqrt{2}\,\theta_{K}\chi^{K}e^{3imt_{\rm L}/4}+\theta_{I}\theta_{J}A^{IJ}e^{3imt_{\rm L}/2}+\frac{\sqrt{2}}{3}\,\theta_{I}\theta_{J}\theta_{K}\xi^{IJK}e^{9imt_{\rm L}/4} + \frac{1}{4}\,\varepsilon^{IJKL}\,\theta_{I}\theta_{J}\theta_{K}\theta_{L}B\,e^{3imt_{\rm L}}, \quad A^{IJ}\equiv A^{[IJ]}, \quad \xi^{IJK}\equiv\xi^{[IJK]}.$$

The superfield Φ transforms as a singlet of the stability subgroup SU(4), *i.e.* $\delta\Phi=0$. Its components transformations:

$$\begin{aligned}
&\delta\phi = -\sqrt{2}\,\epsilon_{K}\chi^{K}e^{3imt/4},\\
&\delta\chi^{I} = \sqrt{2}\,\bar{\epsilon}^{I}\left(i\dot{\phi}\right)e^{-3imt/4}-\sqrt{2}\,\epsilon_{K}A^{IK}e^{3imt/4},\\
&\delta A^{IJ} = 2\sqrt{2}\,\bar{\epsilon}^{[I}\left(i\dot{\chi}^{J]}+\frac{m}{4}\,\chi^{J]}\right)e^{-3imt/4}-\sqrt{2}\,\epsilon_{K}\xi^{IJK}e^{3imt/4},\\
&\frac{\sqrt{2}}{3}\,\delta\xi^{IJK} = 2\,\bar{\epsilon}^{[K}\left(i\dot{A}^{IJ]}+\frac{m}{2}\,A^{IJ]}\right)e^{-3imt/4}-\varepsilon^{IJKL}\,\epsilon_{L}B\,e^{3imt/4},\\
&\varepsilon^{IJKL}\,\delta B = \frac{8\sqrt{2}}{3}\,\bar{\epsilon}^{[L}\left(i\dot{\xi}^{IJK]}+\frac{3m}{4}\,\xi^{IJK]}\right)e^{-3imt/4}.
\end{aligned}$$

Additional constraints

As the next step, we give the rest of constraints in the component level requiring the field content to be $(\mathbf{8},\mathbf{8},\mathbf{0})$. Components of the chiral superfield Φ are subjected to the additional constraints

$$A^{IJ} = \sqrt{2}\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right), \qquad \overline{(y^{IJ})} = y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,y^{KL},$$
$$\xi^{IJK} = -\,\varepsilon^{IJKL}\left(i\dot{\bar{\chi}}_L - \frac{5m}{4}\,\bar{\chi}_L\right), \qquad \overline{(\chi^I)} = \bar{\chi}_I\,,$$
$$B = \frac{2}{3}\left(\ddot{\bar{\phi}} + 2im\dot{\bar{\phi}}\right).$$

It gives the following transformations:

$$\delta\phi = -\sqrt{2}\,\epsilon_I\chi^I e^{3imt/4}, \qquad \delta\bar{\phi} = \sqrt{2}\,\bar{\epsilon}^I\,\bar{\chi}_I\,e^{-3imt/4},$$
$$\delta y^{IJ} = -\,2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L\,e^{3imt/4},$$
$$\delta\chi^I = \sqrt{2}\,\bar{\epsilon}^I\left(i\dot{\phi}\right)e^{-3imt/4} - 2\,\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{3imt/4},$$
$$\delta\bar{\chi}_I = -\sqrt{2}\,\epsilon_I\left(i\dot{\bar{\phi}}\right)e^{3imt/4} + 2\,\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-3imt/4}.$$

The SU(4|1) invariant superfield action of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is written as

$$S_{\rm SK}=\int dt\,\mathcal{L}_{\rm SK}=-\,\frac{1}{4}\left[\int d\zeta_{\rm L}\,K\left(\Phi\right)+\int d\zeta_{\rm R}\,\bar{K}\left(\bar{\Phi}\right)\right].$$

Its component Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\rm SK} \;\;=\;\; & g_1\left[\dot{\phi}\dot{\bar{\phi}}+\frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ}+\frac{i}{2}\left(\chi^{K}\dot{\bar{\chi}}_{K}-\dot{\chi}^{K}\bar{\chi}_{K}\right)-\frac{5m}{4}\,\chi^{K}\bar{\chi}_{K}-\frac{m^2}{8}\,y^{IJ}y_{IJ}\right]\\ & -\frac{im}{4}\left(\dot{\phi}\,\partial_{\phi}g_1-\dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_1\right)y^{IJ}y_{IJ}+2im\left(\dot{\phi}\,\partial_{\bar{\phi}}\bar{K}-\dot{\bar{\phi}}\,\partial_{\phi}K\right)\\ & +\frac{1}{\sqrt{2}}\left(i\dot{y}_{IJ}-\frac{m}{2}\,y_{IJ}\right)\chi^{I}\chi^{J}\,\partial_{\phi}g_1+\frac{1}{\sqrt{2}}\left(i\dot{y}^{IJ}+\frac{m}{2}\,y^{IJ}\right)\bar{\chi}_{I}\bar{\chi}_{J}\,\partial_{\bar{\phi}}g_1\\ & -\frac{i}{2}\left(\dot{\phi}\,\partial_{\phi}g_1-\dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_1\right)\chi^{K}\bar{\chi}_{K}-\frac{1}{24}\,\varepsilon^{IJKL}\,\bar{\chi}_{I}\bar{\chi}_{J}\bar{\chi}_{K}\bar{\chi}_{L}\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_1\\ & -\frac{1}{24}\,\varepsilon_{IJKL}\,\chi^{I}\chi^{J}\chi^{K}\chi^{L}\,\partial_{\phi}\partial_{\phi}g_1\,.\end{aligned}$$

The complex fields ϕ is corresponding coordinate of Special Kähler manifold with the metric

$$g_1\left(\phi,\bar{\phi}\right)=\partial_{\phi}\partial_{\phi}K\left(\phi\right)+\partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}\left(\bar{\phi}\right).$$

Harmonic superspace description

- The multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ has a description also in terms of the harmonic superfield $Y^{(+2)}$ defined on $\mathrm{SU}(4)/[\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)]$ type harmonic space (E. Ivanov, S. Kalitzin, A.V. Nguyen, V. Ogievetsky, J. Phys. A **18** (1985) 3433).

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- The analytic superspace

$$\zeta_{\mathrm{A}} = \left\{ t_{\mathrm{A}}, \theta^{(+)}_a, \bar\theta^{(+)i}, u^{(+)I}_a, u^{(+)i}_I, u^{(-)a}_I, u^{(-)I}_i \right\}.$$

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- The unitarity and unimodularity conditions are written as

$$u^{(+)i}_K u^{(-)K}_j = \delta^i_j\,, \qquad u^{(-)a}_K u^{(+)K}_b = \delta^a_b\,, \qquad u^{(-)a}_J u^{(+)I}_a + u^{(+)i}_J u^{(-)I}_i = \delta^I_J\,,$$
$$u^{(-)a}_K u^{(-)K}_j = u^{(+)i}_K u^{(+)K}_b = 0\,, \quad \varepsilon^{IJKL}\varepsilon_{ij}\, u^{(+)i}_K u^{(+)j}_L + 2\,\varepsilon^{ab} u^{(+)I}_a u^{(+)J}_b = 0\,.$$

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$$u^{(-)a}_K u^{(-)K}_j=u^{(+)i}_K u^{(+)K}_b=0\,,\quad \varepsilon^{IJKL}\varepsilon_{ij}\,u^{(+)i}_K u^{(+)j}_L+2\,\varepsilon^{ab}u^{(+)I}_a u^{(+)J}_b=0\,.$$

- The relevant analytic harmonic superfield is defined as

$$\mathcal{D}^{(+2)i}_a Y^{(+2)}=0\,,\qquad \mathcal{D}^i_j Y^{(+2)}=\mathcal{D}^a_b Y^{(+2)}=0\,.$$

  The rest of constraints can be given by requiring the field content to be $(\mathbf{8},\mathbf{8},\mathbf{0})$ at the component level.

SU(2|1) superfield approach

- For more general construction of SU(4|1) invariant actions, it is convenient to employ SU(2|1) superfield approach. So, we split the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ into SU(2|1) multiplets as a sum of the ordinary multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and the "mirror" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ (E. Ivanov, S. Sidorov, arXiv:1507.00987 [hep-th]).

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- One can consider reducing of the SU(4|1) superspace to the SU(2|1) one. The SU(2|1) superspace coordinates as

$$\left\{t, \theta_i, \bar{\theta}^i\right\}, \qquad \overline{(\theta_i)} = \bar{\theta}^i, \quad i = 1, 2.$$

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$$\left\{t,\theta_i,\bar{\theta}^i\right\}, \qquad \overline{(\theta_i)}=\bar{\theta}^i, \quad i=1,2.$$

- Choosing ϵ_1 and ϵ_2 transformations, we obtain the SU(2|1) supersymmetric transformations:

$$\delta\theta_i=\epsilon_i+2m\,\bar{\epsilon}^k\theta_k\theta_i\,, \quad \delta\bar{\theta}^j=\bar{\epsilon}^j-2m\,\epsilon_k\bar{\theta}^k\bar{\theta}^j, \quad \delta t=i\left(\bar{\epsilon}^k\theta_k+\epsilon_k\bar{\theta}^k\right).$$

Subalgebra

The superalgebra $su(4|1)$ contains as subalgebra the extended $su(2|1)\niplus u(1)$ superalgebra:

$$\left\{Q^i,\bar Q_j\right\} = 2m\,I^i_j + m\,\delta^i_j F + 2\delta^i_j\mathcal{H}, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j,$$
$$\left[I^i_j, Q^k\right] = \delta^k_k Q^i - \frac{1}{2}\,\delta^i_j Q^k, \qquad \left[I^i_j,\bar Q_l\right] = \frac{1}{2}\,\delta^i_j \bar Q_l - \delta^i_l \bar Q_j\,,$$
$$\left[\mathcal{H}, Q^k\right] = -\,\frac{3m}{4}\,Q^k, \qquad [\mathcal{H},\bar Q_l] = \frac{3m}{4}\,\bar Q_l\,,$$
$$\left[F, Q^k\right] = \frac{1}{2}\,Q^k, \qquad [F,\bar Q_l] = -\,\frac{1}{2}\,\bar Q_l\,.$$

Here, SU(2) generators of $su(2|1)$ are defined as

$$I^i_j = L^i_j - \frac{1}{2}\,\delta^i_j F,$$

an internal U(1) generator of $su(2|1)$ by the combination

$$\tilde H = \mathcal{H} + \frac{m}{2}\,F,$$

where F is an external U(1) generator.

Splitting $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0})$

- The ordinary multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is described the superfield q^{ia} that obeys the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}q^{i)a} = \bar{\mathcal{D}}^{(k}q^{i)a} = 0\,, \qquad \tilde{F}q^{ia} = 0\,, \qquad \overline{(q^{ia})} = q_{ia}\,.$$

Here, $\mathcal{D}^k$ and $\bar{\mathcal{D}}^k$ are SU(2|1) covariant derivatives. The indices $i = 1, 2$ and $a = 1, 2$ correspond to the fundamental indices of the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset \mathrm{SU}(4)$, respectively.

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- Corresponding SU(2|1) constraints defining the "mirror" $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet are written as

$$\bar{\mathcal{D}}^{i}Z=\bar{\mathcal{D}}^{i}Y=0\,,\qquad \mathcal{D}^{i}\bar{Z}=\mathcal{D}^{i}\bar{Y}=0\,,$$
$$\mathcal{D}^{i}Z=-\bar{\mathcal{D}}^{i}\bar{Y},\quad \mathcal{D}^{i}Y=\bar{\mathcal{D}}^{i}\bar{Z},\qquad \tilde{F}Z=0\,,\quad \tilde{F}Y=Y.$$

SU(2|1) superfield approach

Alternatively, we can employ the construction of SU(4|1) invariant actions in the framework of the SU(2|1) superfields q^{ia}, Y, Z. The general SU(2|1) superfield action is given by

$$S = \int dt\, d^2\theta\, d^2\bar\theta \left(1 + 2m\,\bar\theta^k\theta_k\right) f\left(Z, \bar Z, Y\bar Y, q^{ia}q_{ia}\right).$$

The metric g of target space is defined according to E. Ivanov, O. Lechtenfeld, A. Sutulin, `arXiv:0705.3064 [hep-th]` as

$$\begin{aligned}
& g = \Delta_2 f = -\Delta_1 f\,, \qquad f = f\left(z,\bar z, y\bar y, x^{ia}x_{ia}\right), \qquad g = g\left(z,\bar z, y\bar y, x^{ia}x_{ia}\right),\\
& \Delta_1 f + \Delta_2 f = 0 \quad \Rightarrow \quad \Delta_1 g + \Delta_2 g = 0\,,\\
& \Delta_1 = \varepsilon^{ik}\varepsilon^{ab}\partial_{ia}\partial_{kb}\,, \qquad \Delta_2 = 2\left(\partial_z\partial_{\bar z} + \partial_y\partial_{\bar y}\right).
\end{aligned}$$

Since SU(4) and SU(2|1) transformations are closed on SU(4|1) transformations, we require SU(4) invariance of the corresponding component action. Then we obtain the equation

$$m\left(\bar y g + 2\partial_y f + x^{ia}\partial_{ia}\partial_y f\right) = 0 \quad \Rightarrow \quad m\left(x_{ia}\partial_y - \bar y\partial_{ia}\right) g = 0\,, \qquad \text{c.c.}$$

Solutions

The equation gives three solutions:

1) Special Kähler manifold metric (Chiral superfield solution)

$$f_1 = \frac{1}{2}\left[\bar{z}\,\partial_z K\left(z\right) + z\,\partial_{\bar{z}}\bar{K}\left(\bar{z}\right)\right] - \left(\frac{x^{ia}x_{ia}}{16} + \frac{y\bar{y}}{4}\right)\left[\partial_z\partial_z K\left(z\right) + \partial_{\bar{z}}\partial_{\bar{z}}\bar{K}\left(\bar{z}\right)\right],$$

$$g_1 = \frac{1}{2}\left[\partial_z\partial_z K\left(z\right) + \partial_{\bar{z}}\partial_{\bar{z}}\bar{K}\left(\bar{z}\right)\right] \quad \Longrightarrow \quad g_1 = \partial_\phi\partial_\phi K\left(\phi\right) + \partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}\left(\bar{\phi}\right).$$

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2) SO(6)-invariant metric (Harmonic superfield solution)

$$f_2 = \frac{1}{4}\left(x^{ia}x_{ia}\right)^{-1}\log\left(2y\bar{y} + x^{ia}x_{ia}\right),$$

$$g_2 = \left(2y\bar{y} + x^{ia}x_{ia}\right)^{-2} \quad \Longrightarrow \quad g_2 = \left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-2}.$$

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3) SO(8)-invariant metric (OSp(8|2) superconformal solution)

$$f_3 = -\frac{1}{8}\left(x^{ia}x_{ia}\right)^{-1}\left(2z\bar z + 2y\bar y + x^{ia}x_{ia}\right)^{-1},$$

$$g_3 = \left(2z\bar z + 2y\bar y + x^{ia}x_{ia}\right)^{-3} \quad\Longrightarrow\quad g_3 = \left[\phi\bar\phi + \frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-3}.$$

OSp(8|2) superconformal Lagrangian

OSp(8|2) superconformal Lagrangian of the trigonometric type contains only m^2 deformed terms:

$$
\begin{aligned}
\mathcal{L}_{\rm con} \;=\;& g_3\left[\dot\phi\dot{\bar\phi} + \frac{1}{2}\,\dot y^{IJ}\dot y_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar\chi}_K - \dot\chi^K\bar\chi_K\right) - \frac{m^2}{4}\left(\phi\bar\phi + \frac{1}{2}\,y^{IJ}y_{IJ}\right)\right] \\
& - \frac{i}{\sqrt{2}}\,\dot{\bar\phi}\,\partial_{IJ}g_3\,\chi^I\chi^J - \frac{i}{\sqrt{2}}\,\dot\phi\,\partial^{IJ}g_3\,\bar\chi_I\bar\chi_J - \frac{i}{2}\left(\dot\phi\,\partial_\phi g_3 - \dot{\bar\phi}\,\partial_{\bar\phi}g_3\right)\chi^K\bar\chi_K \\
& + \frac{i}{\sqrt{2}}\left(\dot y_{IJ}\,\chi^I\chi^J\,\partial_\phi g_3 + \dot y^{IJ}\,\bar\chi_I\bar\chi_J\,\partial_{\bar\phi}g_3\right) \\
& + i\left(\dot y_{IK}\,\partial^{JK}g_3 - \dot y^{JK}\,\partial_{IK}g_3\right)\chi^I\bar\chi_J - \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_3\,\chi^I\chi^J\bar\chi_K\bar\chi_L \\
& - \frac{1}{24}\left(\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_\phi\partial_\phi g_3 + \varepsilon^{IJKL}\,\bar\chi_I\bar\chi_J\bar\chi_K\bar\chi_L\,\partial_{\bar\phi}\partial_{\bar\phi}g_3\right) \\
& - \frac{1}{\sqrt{2}}\left(\chi^I\chi^J\,\partial_{IJ}\partial_\phi g_3 + \bar\chi_I\bar\chi_J\,\partial^{IJ}\partial_{\bar\phi}g_3\right)\chi^K\bar\chi_K + \frac{1}{2}\,\partial_\phi\partial_{\bar\phi}g_3\,\chi^I\bar\chi_I\chi^J\bar\chi_J\,.
\end{aligned}
$$

We have eliminated all deformed terms proportional to m in Lagrangian of the third solution by redefining the component fields as

$$
\phi \to \phi\,e^{-imt/2},\qquad \chi^I \to \chi^I\,e^{-imt/4},\qquad \bar\phi \to \bar\phi\,e^{imt/2},\qquad \bar\chi_I \to \bar\chi_I\,e^{imt/4}.
$$

Superconformal transformations

Since this Lagrangian is an even function of m, it is invariant under two types of SU(4|1) transformations with the deformation parameters m and $-m$:

m

$$\begin{aligned}
&\delta\phi = -\sqrt{2}\,\epsilon_I\chi^I e^{imt}, \qquad \delta\bar\phi = \sqrt{2}\,\bar\epsilon^I\bar\chi_I\, e^{-imt},\\
&\delta y^{IJ} = -2\,\bar\epsilon^{[I}\chi^{J]}e^{-imt} + \varepsilon^{IJKL}\epsilon_K\bar\chi_L\, e^{imt},\\
&\delta\chi^I = \sqrt{2}\,\bar\epsilon^I\left(i\dot\phi + \frac{m}{2}\,\phi\right)e^{-imt} - 2\,\epsilon_J\left(i\dot y^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{imt},\\
&\delta\bar\chi_I = -\sqrt{2}\,\epsilon_I\left(i\dot{\bar\phi} - \frac{m}{2}\,\bar\phi\right)e^{imt} + 2\,\bar\epsilon^J\left(i\dot y_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-imt},
\end{aligned}$$

$-m$

$$\begin{aligned}
&\delta\phi = -\sqrt{2}\,\eta_I\chi^I e^{-imt}, \qquad \delta\bar\phi = \sqrt{2}\,\bar\eta^I\bar\chi_I\, e^{imt},\\
&\delta y^{IJ} = -2\,\bar\eta^{[I}\chi^{J]}e^{imt} + \varepsilon^{IJKL}\eta_K\bar\chi_L\, e^{-imt},\\
&\delta\chi^I = \sqrt{2}\,\bar\eta^I\left(i\dot\phi - \frac{m}{2}\,\phi\right)e^{imt} - 2\,\eta_J\left(i\dot y^{IJ} + \frac{m}{2}\,y^{IJ}\right)e^{-imt},\\
&\delta\bar\chi_I = -\sqrt{2}\,\eta_I\left(i\dot{\bar\phi} + \frac{m}{2}\,\bar\phi\right)e^{-imt} + 2\,\bar\eta^J\left(i\dot y_{IJ} - \frac{m}{2}\,y_{IJ}\right)e^{imt}.
\end{aligned}$$

In the closure of these transformations, we obtain superconformal algebra $osp(8|2)$ composed of 16 supercharges and 31 bosonic generators. This property with respect to the deformed $su(2|1)$ and superconformal $d(2,1;\alpha)$ algebras was marked in E. Ivanov, S. Sidorov, F. Toppan, arXiv:1501.05622 [hep-th].

The multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$: “mirror” counterpart

- The mirror version of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is described by a complex bosonic superfield V^I satisfying

$$\mathcal{D}^I V^J = \frac{1}{2}\,\varepsilon^{IJKL}\,\bar{\mathcal{D}}_K \bar{V}_L\,, \qquad \mathcal{D}^{(I}\,V^{J)} = 0\,, \qquad \bar{\mathcal{D}}_{(K}\,\bar{V}_{L)} = 0\,,$$
$$\mathcal{D}^I \bar{V}_J = \frac{1}{4}\,\delta^I_J \mathcal{D}^K \bar{V}_K \qquad \bar{\mathcal{D}}_J\,V^I = \frac{1}{4}\,\delta^I_J \bar{\mathcal{D}}_K V^K \qquad \overline{(V^I)} = \bar{V}_I\,.$$

In the flat limit $m \to 0$, these constraints correspond to the $\mathrm{SU}(4)$ covariant constraints defining another form of the standard $\mathcal{N} = 8$, $d = 1$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$.

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- Avoiding calculation of the deformed covariant derivatives $\mathcal{D}^I$ and $\bar{\mathcal{D}}_J$, we consider instead harmonization of these constraints corresponding to the harmonic space $\mathrm{SU}(4)/[\mathrm{SU}(3)\times \mathrm{U}(1)]$ (E. Ivanov, S. Kalitzin, A.V. Nguyen, V. Ogievetsky, J. Phys. A **18** (1985) 3433) with additional constraints given in the component level. The relevant harmonic superfield $\bar{V}^{(+3)}$ is defined on the analytic harmonic subspace

$$\left\{t_{\mathrm{A}}, \theta^{(+3)}, \bar{\theta}^{(+)\alpha}, u^{(+)\alpha}_I, u^{(+3)I}, u^{(-)I}_{\beta}, u^{(-3)}_I\right\}.$$

Harmonic superspace description

The superfield $\bar{V}^{(+3)}$ satisfies the harmonic constraints

$$\mathcal{D}^{(+4)\alpha}\bar{V}^{(+3)} = 0\,, \qquad \mathcal{D}^{\alpha}_{\beta}\bar{V}^{(+3)} = 0\,, \qquad \mathcal{D}^{0}\bar{V}^{(+3)} = 3\bar{V}^{(+3)}.$$

Here, $\bar{V}^{(+3)}$ is considered as an unconstrained deformed harmonic superfield. The rest of constraints can be given in the component level requiring the field content to be $(\mathbf{8},\mathbf{8},\mathbf{0})$. Skipping details, the deformed transformations are written as

$$\begin{aligned}
\delta z^{I} &= 2\,\epsilon_{K}\chi^{IK}\,e^{3imt/4} + \sqrt{2}\,\bar{\epsilon}^{I}\,\bar{\chi}\,e^{-3imt/4}\,,\\
\delta \bar{z}_{J} &= -\,2\,\bar{\epsilon}^{K}\chi_{JK}\,e^{-3imt/4} - \sqrt{2}\,\epsilon_{J}\chi\,e^{3imt/4},\\
\delta\chi &= \sqrt{2}\,\bar{\epsilon}^{K}\left(i\dot{\bar{z}}_{K} + \frac{3m}{4}\,\bar{z}_{K}\right)e^{-3imt/4},\\
\delta\bar{\chi} &= -\,\sqrt{2}\,\epsilon_{K}\left(i\dot{z}^{K} - \frac{3m}{4}\,z^{K}\right)e^{3imt/4},\\
\delta\chi^{IJ} &= 2\,\bar{\epsilon}^{[I}\left(i\dot{z}^{J]} + \frac{m}{4}\,z^{J]}\right)e^{-3imt/4} - \varepsilon^{IJKL}\epsilon_{K}\left(i\dot{\bar{z}}_{L} - \frac{m}{4}\,\bar{z}_{L}\right)e^{3imt/4},
\end{aligned}$$

where

$$\overline{(z^{I})} = \bar{z}_{I}\,, \qquad \overline{(\chi)} = \bar{\chi}\,, \qquad \overline{(\chi^{IJ})} = \chi_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,\chi^{KL}.$$

SU(2|1) superfields

- Again, we split the given multiplet into SU(2|1) multiplets as $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0})$. The first multiplet is described by the superfield q^{iA} satisfying the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}q^{i)A} = 0\,, \quad \bar{\mathcal{D}}^{(k}q^{i)A} = 0\,, \qquad \tilde{F}q^{iA} = -\frac{1}{2}\,(\sigma_3)^{A}_{B}\,q^{iB}\,, \qquad \overline{(q^{iA})} = q_{iA}\,.$$

SU(2|1) superfields

- Again, we split the given multiplet into SU(2|1) multiplets as $(\mathbf{4},\mathbf{4},\mathbf{0})\oplus(\mathbf{4},\mathbf{4},\mathbf{0})$. The first multiplet is described by the superfield q^{iA} satisfying the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}q^{i)A}=0\,,\quad \bar{\mathcal{D}}^{(k}q^{i)A}=0\,,\qquad \tilde{F}q^{iA}=-\frac{1}{2}\,(\sigma_3)^A_B\,q^{iB}\,,\qquad \overline{(q^{iA})}=q_{iA}\,.$$

- SU(2|1) constraints defining the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet are written as

$$\bar{\mathcal{D}}^{i}Y^{a}=\mathcal{D}^{i}\bar{Y}^{a}=0\,,\qquad \mathcal{D}^{i}Y^{a}=\bar{\mathcal{D}}^{i}\bar{Y}^{a},$$
$$\tilde{F}Y^{a}=\frac{1}{2}\,Y^{a},\qquad \tilde{F}\bar{Y}^{a}=-\,\frac{1}{2}\,\bar{Y}^{a},\qquad \overline{(Y^{a})}=\bar{Y}_{a}\,.$$

Invariant action

The general SU(2|1) invariant action is written as

$$S = \int dt\, \mathcal{L} = \frac{1}{2}\int dt\, d^2\theta\, d^2\bar\theta \left(1 + 2m\,\bar\theta^k\theta_k\right) f\left(Y^a\bar Y_a, q^{iA}q_{iA}\right),$$

where the target metric G is defined as

$$\Delta_y = -\,2\,\varepsilon^{ab}\partial_a\bar\partial_b\,, \qquad \Delta_x = \varepsilon^{ij}\varepsilon^{AB}\partial_{iA}\partial_{jB}\,,$$
$$G := \Delta_y f = -\,\Delta_x f \quad \Rightarrow \quad (\Delta_y + \Delta_x)\, G = 0\,,$$

Then we require SU(4) invariance of this action that gives the following conditions:

$$\boxed{m\left(2\partial_a f + \bar y_a G + x^{iA}\partial_{iA}\partial_a f\right) = 0 \;\;\Rightarrow\;\; m\left(\bar y_a\partial_{iA} - x_{iA}\partial_a\right) G = 0\,, \qquad \text{c.c.}}$$

The only solution of these equations is given by

$$f = \frac{1}{4}\left(y^a\bar y_a\right)^{-1}\left(y^a\bar y_a + \frac{1}{2}\,x^{iA}x_{iA}\right)^{-1} \quad \Rightarrow$$
$$\Rightarrow G = \left(y^a\bar y_a + \frac{1}{2}\,x^{iA}x_{iA}\right)^{-3} = \left(z^I\bar z_I\right)^{-3}.$$

Superconformal symmetry

- The metric is SO(8)-invariant and corresponds to OSp(8|2) superconformal solution. Indeed, this solution gives the same OSp(8|2) superconformal Lagrangian and superconformal transformations are equivalent for both multiplets.

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- One can see that OSp(8|2) superconformal Lagrangians have conformally flat metrics

$$g_3 = \left[\phi\bar{\phi} + \frac{1}{2}\, y^{IJ} y_{IJ}\right]^{-3},$$
$$G = \left(z^I \bar{z}_I\right)^{-3},$$

both depending on quadratic SO(8) invariants of the same power -3.

Superconformal symmetry

- It can be shown that the fields z^I and $\bar{z}_J$ can be reexpressed, by a linear transformation, through the bosonic fields $y^{I'J'}$, ϕ and $\bar{\phi}$ of the first multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$, where I' and J' are indices of the fundamental representation of a different $\mathrm{SU}(4)'$ subgroup of $\mathrm{SO}(8)$ symmetry.

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- It means that superconformal Lagrangian of the $\mathrm{SU}(4|1)$ "mirror" multiplet is equivalent to superconformal Lagrangian of the first multiplet for another $\mathrm{SU}(4|1)'$ superfield approach. Both supergroups are subgroups of $\mathrm{OSp}(8|2)$.

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- We have shown the existence of two non-equivalent "root" multiplets $(\mathbf{8},\mathbf{8},\mathbf{0})$ of the deformed $\mathcal{N}=8$ supersymmetry associated with the supergroup $\mathrm{SU}(4|1)$.

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Thank you for your attention!