$\mathbb{C}^N$-Smorodinsky-Winternitz system in a constant magnetic field

Hovhannes Shmavonyan

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Content

- Symmetry algebra of $\mathbb{R}^N$-Smorodinsky-Winternitz system
- $\mathbb{C}^N$-Smorodinsky-Winternitz system in a constant magnetic field and its symmetry algebra
- Quantization of $\mathbb{C}^N$-Smorodinsky-Winternitz in a constant magnetic field
- Reduction of $\mathbb{C}^2$-SW by $U(1)$ and its symmetry algebra

$\mathbb{R}^N$-Smorodinsky-Winternitz system

$$H = \sum_{i=1}^{N} I_i \quad I_i = \frac{p_i^2}{2} + \frac{g_i^2}{2x_i^2} + \frac{\omega^2 x_i^2}{2} \quad \{I_i, I_j\} = 0$$

$$\{p_i, x_j\} = \delta_{ij} \quad \{p_i, p_j\} = 0 \quad \{x_i, x_j\} = 0$$

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Hidden constant of motion

$$I_{ij} = L_{ij}L_{ji} - \frac{g_i^2 x_j^2}{x_i^2} - \frac{g_j^2 x_i^2}{x_j^2} \quad \{I_{ij}, H\} = 0, \quad L_{ij} = p_i x_j - p_j x_i$$

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The symmetry algebra

$$\{I_i, I_{jk}\} = \delta_{ij} S_{ik} - \delta_{ik} S_{ij} \quad \{I_{ij}, I_{kl}\} = \delta_{jk} T_{ijl} + \delta_{ik} T_{jkl} - \delta_{jl} T_{ikl} - \delta_{il} T_{ijk}$$

$$S_{ij}^2 = -16(I_i I_j I_{ij} + I_i^2 g_j^2 - I_j^2 g_i^2 + \frac{\omega^2}{4} I_{ij}^2 - g_i^2 g_j^2 \omega^2)$$

$$T_{ijk}^2 = -16(I_{ij} I_{jk} I_{ik} + g_k^2 I_{ij}^2 + g_j^2 I_{ik}^2 + g_i^2 I_{jk}^2 - 4 g_i^2 g_j^2 g_k^2)$$

Redefinition

$$M_{ij} = I_{ij} \quad M_{0i} = I_i \quad M_{ii} = g_i^2 \quad M_{00} = \frac{\omega^2}{4} \quad R_{ijk} = T_{ijk} \quad R_{ij0} = S_{ij}$$

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The symmetry algebra

$$\{M_{IJ}, M_{KL}\} = \delta_{JK} R_{IJL} + \delta_{IK} R_{JKL} - \delta_{JL} R_{IKL} - \delta_{IL} R_{IJK}$$

$$R_{IJK}^2 = -16(M_{IJ} M_{JK} M_{IK} + M_{IJ}^2 M_{KK} + M_{IK}^2 M_{JJ} + M_{KL}^2 M_{II} - 4 M_{II} M_{JJ} M_{KK})$$

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Quantum solution

$$E_{n|\omega} = \hbar\omega\Big(2n + 1 + \sum_{i=1}^{N} \sqrt{\frac{1}{4} + \frac{g_i^2}{\hbar^2}}\Big), \qquad \Psi = \prod_{i=1}^{N} \psi(x_i, n_i),$$

$$\psi(x_i, n_i) = F\Big(- n_i, 1 + \sqrt{\frac{1}{4} + \frac{g_i^2}{\hbar^2}}, \frac{\omega x_i^2}{\hbar}\Big) \Big(\frac{\omega x_i^2}{\hbar}\Big)^{\frac{1+\sqrt{1+4g_i^2/\hbar^2}}{4}} e^{-\frac{\omega x_i^2}{2\hbar}}$$

$\mathbb{C}^N$-Smorodinsky-Winternitz system

$$\mathcal{H} = \sum_a I_a, \qquad I_a = \pi_a \bar{\pi}_a + \frac{g_a^2}{z^a \bar{z}^a} + \omega^2 z^a \bar{z}^a \quad \{I_a, I_b\} = 0$$

$$\{\pi_a, z^b\} = \delta_{ab} \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_{ab} \quad \{\pi_a, \bar{\pi}_b\} = \imath B \delta_{ab}$$

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Hidden constant of motion

$$I_{ab} = L_{a\bar{b}}L_{b\bar{a}} - \left(\frac{g_a^2 z^b\bar{z}^b}{z^a\bar{z}^a} + \frac{g_b^2 z^a\bar{z}^a}{z^b\bar{z}^b}\right) \quad \{I_{ab}, \mathcal{H}\} = 0 \quad a \neq b$$

$$\mathcal{H} = \sum_a I_a, \qquad I_a = \pi_a \bar{\pi}_a + \frac{g_a^2}{z^a \bar{z}^a} + \omega^2 z^a \bar{z}^a \quad \{I_a, I_b\} = 0$$

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$SU(N)$ generators

$$L_{a\bar{b}} = \imath(\pi_a z^b - \bar{\pi}_b \bar{z}^a) - B z^b \bar{z}^a \quad \{L_{a\bar{a}}, \mathcal{H}\} = 0$$

$$\{L_{a\bar{a}}, I_b\} = \{L_{a\bar{a}}, I_{bc}\} = \{L_{a\bar{a}}, L_{b\bar{b}}\} = 0$$

The symmetry algebra of $\mathbb{C}^N$-Smorodinsky-Winternitz

Non-trivial commutators

$$\{I_a, I_{bc}\} = \delta_{ab} S_{ac} - \delta_{ac} S_{ab}$$

$$\{I_{ab}, I_{cd}\} = \delta_{bc} T_{abd} + \delta_{ac} T_{bcd} - \delta_{bd} T_{acd} - \delta_{ad} T_{abc}$$

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where

$$S^2_{ab} = 4I_{ab}I_aI_b - (L_{a\bar{a}}I_b + L_{b\bar{b}}I_a)^2 - 4g_a^2 I_b^2 - 4g_b^2 I_a^2 - 4\omega^2 I_{ab}(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})$$
$$+4\omega^2 g_b^2 L^2_{a\bar{a}} + 4g_a^2\omega^2 L^2_{b\bar{b}} + 16g_a^2 g_b^2\omega^2 - 2B(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})(L_{a\bar{a}}I_b + L_{b\bar{b}}I_a)$$
$$-B^2(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})^2 + 4B(g_b^2 I_a L_{a\bar{a}} + g_a^2 I_b L_{b\bar{b}}) + 4B^2 g_a^2 g_b^2$$

Non-trivial commutators

$$\{I_a, I_{bc}\} = \delta_{ab} S_{ac} - \delta_{ac} S_{ab}$$

$$\{I_{ab}, I_{cd}\} = \delta_{bc} T_{abd} + \delta_{ac} T_{bcd} - \delta_{bd} T_{acd} - \delta_{ad} T_{abc}$$

where

$$S_{ab}^2 = 4I_{ab}I_aI_b - (L_{a\bar{a}}I_b + L_{b\bar{b}}I_a)^2 - 4g_a^2I_b^2 - 4g_b^2I_a^2 - 4\omega^2 I_{ab}(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})$$
$$+4\omega^2 g_b^2 L_{a\bar{a}}^2 + 4g_a^2\omega^2 L_{b\bar{b}}^2 + 16g_a^2g_b^2\omega^2 - 2B(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})(L_{a\bar{a}}I_b + L_{b\bar{b}}I_a)$$
$$-B^2(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})^2 + 4B(g_b^2 I_a L_{a\bar{a}} + g_a^2 I_b L_{b\bar{b}}) + 4B^2 g_a^2 g_b^2$$

$$T_{abc}^2 = 2(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})(I_{bc} - L_{b\bar{b}}L_{c\bar{c}})(I_{ac} - L_{a\bar{a}}L_{c\bar{c}}) + 2I_{ab}I_{ac}I_{bc} + L_{a\bar{a}}^2L_{b\bar{b}}^2L_{c\bar{c}}^2$$
$$-(I_{bc}^2L_{a\bar{a}}^2 + I_{ab}^2L_{c\bar{c}}^2 + I_{ac}^2L_{b\bar{b}}^2) + 4g_b^2g_c^2L_{a\bar{a}}^2 + 4g_a^2g_c^2L_{b\bar{b}}^2 + 4g_a^2g_b^2L_{c\bar{c}}^2 + 16g_a^2g_b^2g_c^2$$
$$-4(g_c^2I_{ab}(I_{ab} - L_{a\bar{a}}L_{b\bar{b}}) + g_a^2I_{bc}(I_{bc} - L_{b\bar{b}}L_{c\bar{c}}) + g_b^2I_{ac}(I_{ac} - L_{a\bar{a}}L_{c\bar{c}}))$$

Redefinition of symmetry generators for $\mathbb{C}^N$-SW

$$M_{aa} = L_{a\bar{a}}^2 + 4g_a^2 \quad M_{ab} = I_{ab} - \frac{1}{2}L_{a\bar{a}}L_{b\bar{b}} \quad M_{a0} = I_a - \frac{B}{2}L_{a\bar{a}}$$

Redefinition of symmetry generators for $\mathbb{C}^N$-SW

$$M_{aa} = L^2_{a\bar{a}} + 4g_a^2 \quad M_{ab} = I_{ab} - \frac{1}{2}L_{a\bar{a}}L_{b\bar{b}} \quad M_{a0} = I_a - \frac{B}{2}L_{a\bar{a}}$$

Symmetry algebra

$$\{M_{ab}, M_{cd}\} = \delta_{bc}T_{abd} + \delta_{ac}T_{bcd} - \delta_{bd}T_{acd} - \delta_{ad}T_{abc}$$

$$\{M_{a0}, M_{ab}\} = \delta_{ab}S_{ac} - \delta_{ac}S_{ab}$$

$$S^2_{ab} = 4M_{ab}M_{a0}M_{b0} + \left(\omega^2 + \frac{B^2}{4}\right)(M_{aa}M_{bb} - 4M^2_{ab}) - M^2_{b0}M_{aa} - M^2_{a0}M_{bb}$$

$$T^2_{abc} = 4M_{ab}M_{bc}M_{ac} - M^2_{ab}M_{cc} - M^2_{ac}M_{bb} - M^2_{bc}M_{aa} + \frac{1}{4}M_{aa}M_{bb}M_{cc}$$

Redefinition of symmetry generators for $\mathbb{C}^N$-SW

$$M_{aa} = L_{a\bar{a}}^2 + 4g_a^2 \quad M_{ab} = I_{ab} - \frac{1}{2}L_{a\bar{a}}L_{b\bar{b}} \quad M_{a0} = I_a - \frac{B}{2}L_{a\bar{a}}$$

Symmetry algebra

$$\{M_{ab}, M_{cd}\} = \delta_{bc}T_{abd} + \delta_{ac}T_{bcd} - \delta_{bd}T_{acd} - \delta_{ad}T_{abc}$$

$$\{M_{a0}, M_{ab}\} = \delta_{ab}S_{ac} - \delta_{ac}S_{ab}$$

$$S_{ab}^2 = 4M_{ab}M_{a0}M_{b0} + \left(\omega^2 + \frac{B^2}{4}\right)(M_{aa}M_{bb} - 4M_{ab}^2) - M_{b0}^2 M_{aa} - M_{a0}^2 M_{bb}$$

$$T_{abc}^2 = 4M_{ab}M_{bc}M_{ac} - M_{ab}^2 M_{cc} - M_{ac}^2 M_{bb} - M_{bc}^2 M_{aa} + \frac{1}{4}M_{aa}M_{bb}M_{cc}$$

Introducing $M_{00} = 4\omega^2 + B^2$

$$\{M_{AB}, M_{CD}\} = \delta_{BC}R_{ABD} + \delta_{AC}R_{BCD} - \delta_{BD}R_{ACD} - \delta_{AD}R_{ABC}$$

$$R_{ABC}^2 = 4M_{AB}M_{BC}M_{AC} - M_{AB}^2 M_{CC} - M_{AC}^2 M_{BB} - M_{BC}^2 M_{AA} + \frac{1}{4}M_{AA}M_{BB}M_{CC}$$

Can be reduced to $2D$ problem

$$\hat{I}_a \Psi_a(z_a, \bar{z}_a) = E_a \Psi_a(z_a, \bar{z}_a) \quad \hat{H}\Psi_{tot} = E_{tot}\Psi_{tot}$$

$$\Psi_{tot} = \prod_{a=1}^{N} \Psi_a(z_a, \bar{z}_a) \quad E_{tot} = \sum_{a}^{N} E_a$$

Quantization of $\mathbb{C}^N$-Smorodinsky-Winternitz

Can be reduced to $2D$ problem

$$\hat{I}_a\Psi_a(z_a,\bar{z}_a) = E_a\Psi_a(z_a,\bar{z}_a) \quad \hat{H}\Psi_{tot} = E_{tot}\Psi_{tot}$$

$$\Psi_{tot} = \prod_{a=1}^{N}\Psi_a(z_a,\bar{z}_a) \quad E_{tot} = \sum_a^N E_a$$

$$\Psi(r,\phi) = R(r)\Phi(\phi) \quad \hat{L}\Phi = \hbar m\Phi$$

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$$\Psi(r,\phi) = R(r)\Phi(\phi) \quad \hat{L}\Phi = \hbar m \Phi$$

Solution

$$E_{tot} = \sum_{a=1}^{N} E_{n_a,m_a} = \hbar\sqrt{\omega^2 + \frac{B^2}{4}}\Big(2n+1+\sum_{a=1}^{N}\sqrt{m_a^2 + \frac{4g^2}{\hbar}}\Big)+\frac{B}{2}\sum_{a=1}^{N} m_a$$

$$\Psi_{tot}(z,\bar{z},n,m) = \prod_{a} \frac{1}{\sqrt{2\pi}}(\sqrt{z_a/\bar{z}_a})^m F\Big(-n_a, \sqrt{m_a^2 + \frac{4g_a^2}{\hbar}}+1, \frac{2\sqrt{\omega^2 + \frac{B^2}{4}}}{\hbar} z_a\bar{z}_a\Big)$$

$$\times\Big(\frac{2\sqrt{\omega^2+\frac{B^2}{4}}}{\hbar}z_a\bar{z}_a\Big)^{1/2\sqrt{m_a^2+\frac{4g_a^2}{\hbar}}} e^{\frac{2\sqrt{\omega^2+\frac{B^2}{4}}}{\hbar}z_a\bar{z}_a}$$

$U(1)$ generator

$$J_0 = L_{11} + L_{22} = \imath(z\pi - \bar{z}\bar{\pi}) - Bz\bar{z} = 2s$$

Reduction of $\mathbb{C}^2$-SW by $U(1)$

$U(1)$ generator

$$J_0 = L_{11} + L_{22} = \imath(z\pi - \bar{z}\bar{\pi}) - Bz\bar{z} = 2s$$

$$q_k = z\sigma_k\bar{z}, \quad p_k = \frac{z\sigma_k\pi + \bar{\pi}\sigma_k\bar{z}}{2z\bar{z}}, \quad k = 1, 2, 3$$

$$\{q_k, q_l\} = 0, \quad \{p_k, q_l\} = \delta_{kl}, \quad \{p_k, p_l\} = s\epsilon_{klm}\frac{q_m}{|q|^3}$$

Reduction of $\mathbb{C}^2$-SW by $U(1)$

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$$\gamma \equiv \frac{E_{SW} - Bs}{2} \quad \mathcal{E} \equiv -\frac{\omega^2 + B^2/4}{2}$$

Reduction of $\mathbb{C}^2$-SW by $U(1)$

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$$\gamma \equiv \frac{E_{SW} - Bs}{2} \quad \mathcal{E} \equiv -\frac{\omega^2 + B^2/4}{2}$$

Hamiltonian of the reduced (generalized MICZ-Kepler) system

$$\mathcal{H}_{gMICZ} = \frac{p^2}{2} + \frac{s^2}{2|q|^2} + \frac{g_1^2}{|q|(|q| + q_3)} + \frac{g_2^2}{|q|(|q| - q_3)} - \frac{\gamma}{|q|}$$

The symmetry algebra of reduced system

Introduce $SO(3)$ geberators

$$J_k = \epsilon_{klm} p_l q_m - s\frac{q_k}{|q|} = \frac{i}{2}(z\sigma_k\pi - \bar{\pi}\sigma_k\bar{z}) - \frac{Bz\sigma_k\bar{z}}{2}$$

Reduction of constants of motion

$$\mathcal{I} = \frac{I_1 - I_2}{2} + \frac{B}{4}(L_{22} - L_{11}) = p_1 J_2 - p_2 J_1 + \frac{x_3\gamma}{r} + \frac{g_1^2(r - x_3)}{r(r + x_3)} - \frac{g_2^2(r + x_3)}{r(r - x_3)}$$

$$\mathcal{L} = J_3 \quad \mathcal{J} = I_{12} = J_1^2 + J_2^2 + \frac{g_1^2(r - q_3)}{r + q_3} + \frac{g_2^2(r + q_3)}{r - q_3}$$

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symmetry algebra

$$\{\mathcal{I}, \mathcal{J}\} = S \quad \{\mathcal{L}, \mathcal{J}\} = \{\mathcal{L}, \mathcal{I}\} = 0$$

$$S^2 = 2\mathcal{H}_{gMICZ}\left[4\left(\mathcal{J} + \frac{1}{2}\left(\mathcal{L}^2 - s^2\right)\right)^2 - \left(4g_2^2 + (\mathcal{L}+s)^2\right)\left(4g_1^2 + (\mathcal{L}-s)^2\right)\right]$$
$$-\left(4g_2^2 + (\mathcal{L}+s)^2\right)\left(\mathcal{I}+\gamma\right)^2 - \left(4g_1^2 + (\mathcal{L}-s)^2\right)\left(\mathcal{I}-\gamma\right)^2$$
$$-4\left(\mathcal{J} + \frac{1}{2}(\mathcal{L}^2 - s^2)\right)\left(\mathcal{I}-\gamma\right)\left(\mathcal{I}+\gamma\right)$$

Conclusions

- The spectrum of $\mathbb{C}^N$-SW depends on $N+1$ quantum numbers
- Presence of constant magnetic field in the initial problem does not affect the reduced system
- Supersymmetrization is straightforward
- Generalizations on $\mathbb{CP}^N$
- Quaternionic generalizations and SU(2) instanton

Thank You for attention!