On mini-superspace limit of boundary three-point function in Liouville field theory.

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Liouville field theory

Let us review basic facts on the Liouville field theory . Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \tag{1.1}$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b} \tag{1.2}$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \tag{1.3}$$

with the central charge

$$c_L = 1 + 6Q^2 \tag{1.4}$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q-\alpha) \tag{1.5}$$

Let us recall from the Liouville field zero mode properties that when momenta α_i satisfy the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = Q - nb \tag{1.6}$$

the structure constants have a pole with the residue equals to the Coulomb gas integrals (1.9):

$$\mathrm{res}_{\alpha_1+\alpha_2+\alpha_3=Q} C(\alpha_1, \alpha_2.\alpha_3) = 1\,, \tag{1.7}$$

and

$$\mathrm{res}_{\alpha_1+\alpha_2+\alpha_3=Q-nb} C(\alpha_1, \alpha_2.\alpha_3) = I_n(\alpha_1, \alpha_2, \alpha_3)\,. \tag{1.8}$$

where

$$I_n(\alpha_1,\alpha_2,\alpha_3) = \frac{\left(b^4\gamma(b^2)\pi\mu\right)^n \prod_{j=1}^{n}\gamma(-jb^2)}{\prod_{k=0}^{n-1}[\gamma(2\alpha_1 b+kb^2)\gamma(2\alpha_2 b+kb^2)\gamma(2\alpha_3 b+kb^2)]}, \qquad (1.9)$$

where $\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$. The solution to the condition (1.8) is given by the the famous DOZZ formula

$$C(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b}\times \frac{\Upsilon_b(b)\Upsilon_b(2\alpha_1)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)}\times \frac{\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}, \qquad (1.10)$$

where

$$\lambda = \pi\mu\gamma(b^2)b^{2-2b^2} \qquad (1.11)$$

An integral representation convergent in the strip $0 < \mathrm{Re}(x) < Q$ is

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \quad (1.12)$$

Important properties of $\Upsilon_b(x)$ are

1. Functional equation: $\Upsilon_b(x+b) = b^{1-2bx}\frac{\Gamma(bx)}{\Gamma(1-bx)}\Upsilon_b(x)$.
2. Analyticity: $\Upsilon_b(x)$ is entire analytic, zeros:
   $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$,
   $x = Q + nb + mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.
3. Self-duality: $\Upsilon_b(x) = \Upsilon_{1/b}(x)$.
4. $\Upsilon_b(x) = \Upsilon_b(Q-x)$
5. $\lim_{b\to 0}\Upsilon_b(bx) \to \frac{b^{1/2-x}}{\Gamma(x)}$

Taking on of the α_i in DOZZ formula to zero one can get two-point function:

$$\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\rangle = \frac{S(\alpha)}{(z_1-z_2)^{2\Delta_\alpha}(\bar{z}_1-\bar{z}_2)^{2\Delta_\alpha}} \qquad (1.13)$$

where

$$S(\alpha) = \frac{\left(\pi\mu\gamma(b^2)\right)^{\frac{(Q-2\alpha)}{b}}}{b^2}\frac{\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))}{\Gamma(b(Q-2\alpha))\Gamma(1+b^{-1}(Q-2\alpha))} \qquad (1.14)$$

Setting

$$\alpha = \frac{Q}{2} + iPb \qquad (1.15)$$

and taking $b \to 0$ we obtain:

$$S\left(\frac{Q}{2}+iPb\right) \to \left(\frac{\pi\mu}{b^2}\right)^{-2iP}\frac{\Gamma(2iP)}{\Gamma(-2iP)} \qquad (1.16)$$

Misisuper-space limit of two-point function

In the minisuperpsace limit, ignoring σ dependence and keeping only zero mode ϕ_0 we get Schrödinger equation for eigenfunctions:

$$-\frac{1}{2}\frac{\partial^2\psi}{\partial\phi_0^2}+2\pi\mu e^{2b\phi_0}\psi=2P^2b^2\psi\ . \tag{2.1}$$

The solution is given by the modified Bessel functions:

$$\psi_P(\phi_0)=2\left(\frac{\pi\mu}{b^2}\right)^{-iP}\frac{K_{2iP}\left(2\sqrt{\frac{\pi\mu}{b^2}}e^{b\phi_0}\right)}{\Gamma(-2iP)} \tag{2.2}$$

Using the small argument behaviour of the modified Bessel function:

$$K_\nu(z)\sim 2^{-\nu-1}\Gamma(-\nu)z^\nu+2^{\nu-1}\Gamma(\nu)z^{-\nu} \tag{2.3}$$

we obtain

Misisuper-space limit of two-point function

that in the limit $\phi_0 \to -\infty$

$$\psi_P(\phi_0) \to e^{2iP\phi_0} + \left(\frac{\pi\mu}{b^2}\right)^{-2iP} \frac{\Gamma(2iP)}{\Gamma(-2iP)} e^{-2iP\phi_0} \tag{2.4}$$

Mini-superspace limit of DOZZ three-point function

Now consider DOZZ three-point function in the limit $b \to 0$ with arguments scaling according to the rule [C. Thorn, hep-th/0204142]:

$$\alpha_1 = \frac{Q}{2} + iPb \tag{2.5}$$

$$\alpha_2 = b\sigma \tag{2.6}$$

$$\alpha_3 = \frac{Q}{2} + iP'b \tag{2.7}$$

Using properties of Υ_b function one can show that:

$$C(\alpha_1, \alpha_2, \alpha_3) \to C(P, \sigma, P') \tag{2.8}$$

where

Mini-superspace limit of DOZZ three-point function

$$C(P,\sigma,P') = \frac{1}{b}\left(\frac{b^2}{\pi\mu}\right)^{iP+iP'+\sigma} \times \tag{2.9}$$
$$\frac{\Gamma(\sigma+iP+iP')\Gamma(\sigma-iP-iP')\Gamma(\sigma+iP-iP')\Gamma(\sigma-iP+iP')}{\Gamma(-2iP)\Gamma(2\sigma)\Gamma(-2iP')}$$

Consider the matrix element

$$\langle P'|e^{2\sigma b\phi_0}|P\rangle = \int_{-\infty}^{\infty} \psi_{-P'}(\phi_0)e^{2\sigma b\phi_0}\psi_P(\phi_0)d\phi_0 \tag{2.10}$$

Using the integral

$$\int_0^\infty x^{-\lambda} K_\mu(x) K_\nu(x) dx = \frac{2^{-2-\lambda}}{\Gamma(1-\lambda)} \times \tag{2.11}$$
$$\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \times$$
$$\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)$$

one can see that

$$\langle P'|e^{2\sigma b\phi_0}|P\rangle = C(P,\sigma,P') \tag{2.12}$$

Lagrangian of Boundary Liouville field theory

Let us consider the Liouville field theory on a strip $\mathbb{R} \times [0, \pi]$, parameterized by the time τ and space σ coordinates, $0 \leq \sigma \leq \pi$. The conformal invariant action has the form:

$$S = \int_{-\infty}^{\infty} d\tau \int_{0}^{\pi} d\sigma \left(\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) + \int_{-\infty}^{\infty} d\tau M_1 e^{b\phi}|_{\sigma=0} + \int_{-\infty}^{\infty} d\tau M_2 e^{b\phi}|_{\sigma=\pi} \ , \tag{3.1}$$

where M_1 and M_2 are the corresponding boundary cosmological constants.

One-point Correlation functions in BLFT

In the presence of the boundary with the cosmological constant M the primary fields V_α have the one-point functions [V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, hep-th/0001012]:

$$\langle 0|V_\alpha(z,\bar{z})|0\rangle = \frac{U_\sigma(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}} \ , \tag{3.2}$$

where

$$U_\sigma(\alpha) = \frac{2}{b}(\pi\mu\gamma(b^2))^{(Q-2\alpha)/2b} \times \tag{3.3}$$
$$\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))\cos(\pi(2\sigma-Q)(2\alpha-Q))$$

and the parameter σ is related to the boundary cosmological constant M by the relation:

$$M = \sqrt{\frac{\mu}{\sin(\pi b^2)}}\cos\pi b\,(2\sigma-Q) \ . \tag{3.4}$$

Two-point Correlation functions in BLFT

Besides the bulk primary fields, in boundary conformal field theory exist also boundary condition changing operators, parameterized by the types of the switched boundary conditions and conformal weights. In the case of the BLFT they are given by the fields $\Psi_{\beta}^{\sigma_1\sigma_2}$ with the conformal weight $\Delta_\beta = \beta(Q-\beta)$. They have the two-point function:

$$\langle 0|\Psi_{\beta_1}^{\sigma_1\sigma_2}(x)\Psi_{\beta_2}^{\sigma_2\sigma_1}(0)|0\rangle = \frac{\delta(\beta_2+\beta_1-Q)+S(\beta_1,\sigma_2,\sigma_1)\delta(\beta_2-\beta_1)}{|x|^{2\Delta_{\beta_1}}}\,, \tag{3.5}$$

where

$$S(\beta,\sigma_2,\sigma_1) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2\beta}{2b}} \times \tag{3.6}$$
$$\times\frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}\frac{S_b(\sigma_2+\sigma_1-\beta)S_b(2Q-\sigma_2-\sigma_1-\beta)}{S_b(\sigma_2-\sigma_1+\beta)S_b(\sigma_1-\sigma_2+\beta)}$$

Three-point Correlation functions in BLFT

and the three-point function [B. Ponsot and J. Teschner, hep-th/0110244]:

$$\langle 0|\Psi^{\sigma_1\sigma_3}_{\beta_3}(x_3)\Psi^{\sigma_3\sigma_2}_{\beta_2}(x_3)\Psi^{\sigma_2\sigma_1}_{\beta_1}(x_3)|0\rangle = \frac{C^{\sigma_3\sigma_2\sigma_1}_{\beta_3\beta_2\beta_1}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}\,, \tag{3.7}$$

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} \equiv C^{\sigma_3\sigma_2\sigma_1}_{Q-\beta_3,\beta_2,\beta_1}\,, \tag{3.8}$$

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} = R_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\int_{-i\infty}^{i\infty}\frac{d\tau}{i}J_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\,, \tag{3.9}$$

where

Three-point Correlation functions in BLFT

$$R_{\sigma_2,\beta_3}\left[\begin{array}{cc}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{array}\right]=(\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(\beta_3-\beta_2-\beta_1)} \tag{3.10}$$

$$\times\quad\frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(Q+\beta_2-\beta_1-\beta_3)}{\Gamma_b(2\beta_3-Q)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_1)\Gamma_b(Q)}$$

$$\times\quad\frac{\Gamma_b(Q+\beta_3-\beta_2-\beta_1)S_b(\beta_3+\sigma_1-\sigma_3)S_b(Q+\beta_3-\sigma_3-\sigma_1)}{S_b(\beta_2+\sigma_2-\sigma_3)S_b(Q+\beta_2-\sigma_3-\sigma_2)}$$

and

Three-point Correlation functions in BLFT

$$J_{\sigma_2,\beta_3}\left[\begin{array}{cc}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{array}\right]=\frac{S_b(U_1+\tau)S_b(U_2+\tau)}{S_b(V_1+\tau)S_b(V_2+\tau)}\frac{S_b(U_3+\tau)S_b(U_4+\tau)}{S_b(V_3+\tau)S_b(V_4+\tau)}$$

$$\begin{aligned}
U_1 &= \sigma_2+\sigma_1-\beta_1, & V_1 &= Q+\sigma_2+\beta_3-\beta_1-\sigma_3\ , \quad (3.11)\\
U_2 &= Q+\sigma_2-\beta_1-\sigma_1, & V_2 &= 2Q+\sigma_2-\beta_3-\sigma_3-\beta_1\ ,\\
U_3 &= \sigma_2+\beta_2-\sigma_3, & V_3 &= 2\sigma_2\ ,\\
U_4 &= Q+\sigma_2-\beta_2-\sigma_3, & V_4 &= Q\ .
\end{aligned}$$

Double Gamma and Sine functions

$\Gamma_b(x)$ can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t}\Big[\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\Big]. \quad (3.12)$$

Important properties of $\Gamma_b(x)$ are

1. Functional equation: $\Gamma_b(x+b) = \sqrt{2\pi}\, b^{bx-\frac{1}{2}} \Gamma^{-1}(bx) \Gamma_b(x)$.
2. Analyticity: $\Gamma_b(x)$ is meromorphic, poles: $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.
3. Self-duality: $\Gamma_b(x) = \Gamma_{1/b}(x)$.

Double Gamma and Sine functions

The Υ_b and S_b may be defined in terms of Γ_b as follows

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)} . \quad S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} . \tag{3.13}$$

For $b \to 0$ the double Gamma $\Gamma_b(x)$ and double Sine $S_b(x)$ functions have the asymptotic behaviour :

$$S_b(bx) \to (2\pi b^2)^{x-\frac{1}{2}}\Gamma(x) , \quad S_b\left(\frac{1}{2b} + bx\right) \to 2^{x-\frac{1}{2}} , \tag{3.14}$$

$$S_b\left(\frac{1}{b} + bx\right) \to \frac{2\pi(2\pi b^2)^{x-\frac{1}{2}}}{\Gamma(1-x)} , \quad \Gamma_b(bx) \to (2\pi b^3)^{\frac{1}{2}(x-\frac{1}{2})}\Gamma(x) , \tag{3.15}$$

$$\Gamma_b(Q-bx) \to \sqrt{2\pi}(2\pi b)^{\frac{1}{2}(\frac{1}{2}-x)} . \tag{3.16}$$

Mini-superspace limit of boundary two-point function

Consider the minisuperspace limit of the boundary two-point function (3.6). Let us take the limit $b \to 0$ and scale the parameters β and σ in the following way [Z. Bajnok, C. Rim, Al. Zamolodchikov, arXiv:0710.4789]:

$$\beta = \frac{Q}{2} + ikb \tag{3.17}$$

and

$$\sigma_1 = \frac{1}{4b} + \rho_1 b \, , \quad \sigma_2 = \frac{1}{4b} + \rho_2 b \; . \tag{3.18}$$

Using asymptotic properties of the double Gamma and Sine functions one can obtain easily:

$$S(\beta, \sigma_2, \sigma_1) \to \left(\frac{4\pi\mu}{b^2} \right)^{-ik} \frac{\Gamma(2ik)}{\Gamma(-2ik)} \frac{\Gamma\left(\rho_1 + \rho_2 - \frac{1}{2} - ik\right)}{\Gamma\left(\rho_1 + \rho_2 - \frac{1}{2} + ik\right)} \; . \tag{3.19}$$

Mini-superpspace limit of boundary three-point function

To compute the mini-superspace limit of the boundary three-point function we will use the ansatz (3.18) for all the three boundary condition parameters:

$$\sigma_1 = \frac{1}{4b} + \rho_1 b \ , \quad \sigma_2 = \frac{1}{4b} + \rho_2 b \ , \quad \sigma_3 = \frac{1}{4b} + \rho_3 b \ . \quad (3.20)$$

For the primary fields parameters we will use the ansatz used for calculation of the mini-superspace limit of the bulk three-point function:

$$\beta_1 = \frac{Q}{2} + ik_1 b \ , \quad \beta_2 = \eta b \ , \quad \beta_3 = \frac{Q}{2} + ik_2 b \ . \quad (3.21)$$

It is convenient to denote

$$\rho_1 + \rho_2 = 1 - \lambda \ , \quad \rho_2 - \rho_3 = \xi \ , \quad \rho_1 + \rho_3 = 1 - \lambda - \xi \ . \quad (3.22)$$

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} \to C^{\lambda\xi}_{k_2|\eta k_1} = \left(\frac{4\pi\mu}{b^2}\right)^{(ik_2-ik_1-\eta)/2} \times \tag{3.23}$$

$$\frac{\Gamma(\frac{1}{2}+\lambda-ik_1)}{\sin\pi(ik_1+\frac{1}{2}+\lambda)\Gamma(2ik_2)\Gamma(-2ik_1)\Gamma(\frac{1}{2}-ik_2-\lambda-\xi)} \times$$

$$\left[\frac{\Gamma(2ik_2)\Gamma(ik_1-ik_2+\eta)\Gamma(-ik_1-ik_2+\eta)\Gamma(\frac{1}{2}-ik_2-\lambda-\xi)}{\Gamma(-ik_2+\frac{1}{2}+\lambda+\eta)\Gamma(-ik_2+\frac{1}{2}-\lambda+\eta)\Gamma(ik_2+\frac{1}{2}+\lambda-\eta)} \times\right.$$

$${}_3F_2\left(\begin{array}{c} -ik_1-ik_2+\eta, ik_1-ik_2+\eta, \frac{1}{2}+\lambda+\xi-ik_2; \\ 1-2ik_2, \frac{1}{2}+\lambda-ik_2+\eta : 1 \end{array}\right) +$$

$$\frac{\Gamma(-2ik_2)\Gamma(ik_1+ik_2+\eta)\Gamma(-ik_1+ik_2+\eta)\Gamma(\frac{1}{2}+ik_2-\lambda-\xi)}{\Gamma(ik_2+\frac{1}{2}+\lambda+\eta)\Gamma(ik_2+\frac{1}{2}-\lambda+\eta)\Gamma(-ik_2+\frac{1}{2}+\lambda-\eta)} \times$$

$$\left.{}_3F_2\left(\begin{array}{c} ik_1+ik_2+\eta, -ik_1+ik_2+\eta, \frac{1}{2}+\lambda+\xi+ik_2; \\ 1+2ik_2, \frac{1}{2}+\lambda+ik_2+\eta : 1 \end{array}\right)\right] .$$

Here ${}_3F_2$ is the generalized hypergeometric function:

$$ {}_3F_2\left(\begin{array}{c} a, b, c; \\ d, e : x \end{array}\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{x^n}{n!} \ , $$

where

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3.24} $$

is the Pochhammer symbol.

For $\xi = -\eta$ eq.(3.23) simplifies and takes the form:

$$C^{\lambda(-\eta)}_{k_2|\eta k_1} = \left(\frac{4\pi\mu}{b^2}\right)^{(ik_2-ik_1-\eta)/2} \times \tag{3.25}$$

$$\frac{\Gamma(\frac{1}{2}+\lambda-ik_1)}{\sin\pi(ik_1+\frac{1}{2}+\lambda)\Gamma(2ik_2)\Gamma(-2ik_1)\Gamma(\frac{1}{2}-ik_2-\lambda+\eta)} \times$$

$$\left[\frac{\Gamma(2ik_2)\Gamma(ik_1-ik_2+\eta)\Gamma(-ik_1-ik_2+\eta)}{\Gamma(-ik_2+\frac{1}{2}+\lambda+\eta)\Gamma(ik_2+\frac{1}{2}+\lambda-\eta)} \times\right.$$

$${}_3F_2\left(\begin{array}{c} -ik_1-ik_2+\eta, ik_1-ik_2+\eta, \frac{1}{2}+\lambda-\eta-ik_2; \\ 1-2ik_2, \frac{1}{2}+\lambda-ik_2+\eta : 1 \end{array}\right) +$$

$$\frac{\Gamma(-2ik_2)\Gamma(ik_1+ik_2+\eta)\Gamma(-ik_1+ik_2+\eta)}{\Gamma(ik_2+\frac{1}{2}+\lambda+\eta)\Gamma(-ik_2+\frac{1}{2}+\lambda-\eta)} \times$$

$$\left.{}_3F_2\left(\begin{array}{c} ik_1+ik_2+\eta, -ik_1+ik_2+\eta, \frac{1}{2}+\lambda-\eta+ik_2; \\ 1+2ik_2, \frac{1}{2}+\lambda+ik_2+\eta : 1 \end{array}\right)\right] .$$

Let us consider the limit $\beta_2 \to 0$ and correspondingly $\eta \to 0$.
It is straightforward to show that:

$$\lim_{\eta\to 0} C^{\lambda(-\eta)}_{k_2|\eta k_1} = \delta(k_1 - k_2) + \tag{3.26}$$

$$\left(\frac{4\pi\mu}{b^2}\right)^{-ik_1} \frac{\Gamma(2ik_1)}{\Gamma(-2ik_1)} \frac{\Gamma\left(\frac{1}{2} - \lambda - ik_1\right)}{\Gamma\left(\frac{1}{2} - \lambda + ik_1\right)} \delta(k_1 + k_2)$$

in agreement with (3.19).

Morse potential Eigenfunctions

In the mini-superspace limit the boundary Liouville field theory is described by the Hamiltonian with the Morse potential [Z. Bajnok, C. Rim, Al. Zamolodchikov, arXiv:0710.4789 and H. Dorn, G. Jorjadze, arXiv:0801.3206]. The corresponding eigenfuntions satisfy the Schrödinger equation:

$$-\frac{\partial^2\psi}{\partial\phi_0^2}+\pi\mu e^{2b\phi_0}\psi+(M_1+M_2)e^{b\phi_0}\psi=k^2b^2\psi\ . \qquad (4.1)$$

The relation between parameters M_i, appearing in the Schrödinger equation, and parameters ρ_i, used in the previous section, can be found using (3.20) and (3.4) and taking the limit $b\to 0$:

$$M_i=\sqrt{\frac{\mu}{\sin(\pi b^2)}}\sin\pi b^2(2\rho_i-1)\to\pm(\mu\pi)^{1/2}b(2\rho_i-1)\ . \qquad (4.2)$$

The solution of the eq. (4.1) is given by the Whittaker functions $W_{\mu,\nu}(y)$:

$$\psi = \mathcal{N} \times \tag{4.3}$$
$$\left[e^{-\frac{y}{2}} y^{ik} \frac{\Gamma(-2ik)}{\Gamma\left(\frac{1}{2} - ik + \frac{M_1+M_2}{2b\sqrt{\pi\mu}}\right)} {}_1F_1\left(\frac{1}{2} + ik + \frac{M_1+M_2}{2b\sqrt{\pi\mu}}, 1+2ik, y\right) + \right.$$
$$\left. e^{-\frac{y}{2}} y^{-ik} \frac{\Gamma(2ik)}{\Gamma\left(\frac{1}{2} + ik + \frac{M_1+M_2}{2b\sqrt{\pi\mu}}\right)} {}_1F_1\left(\frac{1}{2} - ik + \frac{M_1+M_2}{2b\sqrt{\pi\mu}}, 1-2ik, y\right) \right]$$
$$\equiv \mathcal{N}\, W_{-\frac{M_1+M_2}{2b\sqrt{\pi\mu}}, ik}(y) y^{-\frac{1}{2}} \ ,$$

where

$$y = \frac{2\sqrt{\pi\mu}}{b} e^{b\phi_0} \ , \tag{4.4}$$

$\mathcal{N}$ is the normalization and ${}_1F_1(a, c, z)$ is the confluent hypergeometric function:

$$ {}_1F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} \ . \qquad (4.5) $$

Now we wish to compute matrix elements of the "vertex operator" $e^{\eta b \phi_0}$, between the wave functions corresponding to the boundary condition changing operators. According to this solution to the operator $\Psi^{\sigma_2\sigma_1}_{\beta_1}$ corresponds the wave function $\mathcal{N}_1 W_{\chi_1, ik_1}(y) y^{-\frac{1}{2}}$ with

$$\chi_1 = -\frac{M_1 + M_2}{2b\sqrt{\pi\mu}} = \pm\lambda \tag{4.6}$$

and to the operator $\Psi_{\beta_3}^{\sigma_1\sigma_3}$ corresponds the wave function $\mathcal{N}_2 W_{\chi_2, ik_2}(y) y^{-\frac{1}{2}}$ with

$$\chi_2 = -\frac{M_1 + M_3}{2b\sqrt{\pi\mu}} = \pm(\lambda + \xi) \ . \tag{4.7}$$

The corresponding integral has the form:

$$\mathcal{M}^{\chi_1\chi_2}_{\eta k_1 k_2} = \mathcal{N}_1\mathcal{N}_2^* \int_{-\infty}^{\infty} W_{\chi_1,ik_1}(y)y^{-\frac{1}{2}} W_{\chi_2,-ik_2}(y)y^{-\frac{1}{2}} e^{\eta b\phi_0} d\phi_0 = (4.8$$

$$= \frac{\mathcal{N}_1\mathcal{N}_2^*}{b}\left(\frac{4\pi\mu}{b^2}\right)^{-\eta/2} \int_0^{\infty} W_{\chi_1,ik_1}(y) W_{\chi_2,-ik_2}(y) y^{\eta-2} dy =$$

$$\mathcal{N}_1\mathcal{N}_2^*(4\pi\mu b^{-2})^{-\eta/2} b^{-1} \times$$

$$\Bigg[\frac{\Gamma(ik_1 - ik_2 + \eta)\Gamma(-ik_1 - ik_2 + \eta)\Gamma(2ik_2)}{\Gamma(\frac{1}{2} - \chi_2 + ik_2)\Gamma(\frac{1}{2} - \chi_1 - ik_2 + \eta)} \times$$

$${}_3F_2\left(\begin{array}{c} -ik_1 - ik_2 + \eta, ik_1 - ik_2 + \eta, \frac{1}{2} - \chi_2 - ik_2; \\ 1 - 2ik_2, \frac{1}{2} - \chi_1 - ik_2 + \eta : 1 \end{array}\right) +$$

$$\frac{\Gamma(ik_1 + ik_2 + \eta)\Gamma(-ik_1 + ik_2 + \eta)\Gamma(-2ik_2)}{\Gamma(\frac{1}{2} - \chi_2 - ik_2)\Gamma(\frac{1}{2} - \chi_1 + ik_2 + \eta)} \times$$

$${}_3F_2\left(\begin{array}{c} ik_1 + ik_2 + \eta, -ik_1 + ik_2 + \eta, \frac{1}{2} - \chi_2 + ik_2; \\ 1 + 2ik_2, \frac{1}{2} - \chi_1 + ik_2 + \eta : 1 \end{array}\right)\Bigg].$$

Comparing (4.8) with (3.25) we see that they coincide if we set:

$$\chi_1 = -\lambda \ , \tag{4.9}$$

$$\chi_2 = -\lambda + \eta \ , \tag{4.10}$$

$$\mathcal{N}_1 = \frac{(4\pi\mu b^{-2})^{-ik_1/2} b^{1/2}}{\sin\pi\left(\frac{1}{2}+ik_1+\lambda\right)} \frac{\Gamma\left(\frac{1}{2}+\lambda-ik_1\right)}{\Gamma(-2ik_1)} \ , \tag{4.11}$$

$$\mathcal{N}_2 = \frac{1}{\pi}(4\pi\mu b^{-2})^{-ik_2/2} b^{1/2} \sin\pi\left(\frac{1}{2}+ik_2-\lambda+\eta\right) \frac{\Gamma\left(\frac{1}{2}+\lambda-\eta-ik_2\right)}{\Gamma(-2ik_2)} \tag{4.12}$$

This result leads us to the following conclusion on a role of the exponential operator $e^{\eta b\phi_0}$.
Combining (4.6) and (4.7) with lower signes, as indicating in (4.9) and (4.10), and also remembering (3.20) and (3.22) one has

$$\frac{M_3 - M_2}{2\sqrt{\pi\mu}} = b\xi = -b\eta = \sigma_2 - \sigma_3 \ . \tag{4.13}$$

Therefore recalling also that the exponential operator $e^{\eta b\phi_0}$ should correspond to a boundary condition changing operator $\Psi^{\sigma_3\sigma_2}_{\beta_2}$, one concludes that the operator $e^{\eta b\phi_0}$ in the semiclassical limit produces change of the boundary condition given by (4.13).

It is instructive to compare the normalizations of the wave functions found here with those used in [Z. Bajnok, C. Rim, Al. Zamolodchikov, arXiv:0710.4789].

Let us compute for this purpose the matrix element (4.8) for $\eta \to 0$ and $\chi_1 = \chi_2$. In this limit we obtain:

$$\mathcal{M}^{\chi_1\chi_1}_{0k_1k_2} = \frac{\mathcal{N}_1\mathcal{N}_2^* b^{-1}\Gamma(2ik_1)\Gamma(-2ik_1)}{\Gamma(\frac{1}{2}-\chi_1+ik_1)\Gamma(\frac{1}{2}-\chi_1-ik_1)}\delta(k_1-k_2)+ \quad (4.14)$$

$$\frac{\mathcal{N}_1\mathcal{N}_2^* b^{-1}\Gamma(2ik_1)\Gamma(-2ik_1)}{\Gamma(\frac{1}{2}-\chi_1-ik_1)\Gamma(\frac{1}{2}-\chi_1+ik_1)}\delta(k_1+k_2)\ .$$

For χ_1, χ_2, $\mathcal{N}_1$, $\mathcal{N}_2$, chosen as in (4.9)-(4.12), with $\eta = 0$, the expression (4.14) surely coincides with the two-point function (3.26). But note that for

$$\chi_1 = \lambda\,, \quad \chi_2 = \lambda\,, \tag{4.15}$$

$$\mathcal{N}_1 = (4\pi\mu b^{-2})^{-ik_1/2} b^{1/2} \frac{\Gamma\left(\frac{1}{2} - \lambda - ik_1\right)}{\Gamma(-2ik_1)}\,, \tag{4.16}$$

$$\mathcal{N}_2 = (4\pi\mu b^{-2})^{-ik_2/2} b^{1/2} \frac{\Gamma\left(\frac{1}{2} - \lambda - ik_2\right)}{\Gamma(-2ik_2)}\,, \tag{4.17}$$

the expression (4.14) again coincides with the two-point function (3.26). This was established in [Z. Bajnok, C. Rim, Al. Zamolodchikov, arXiv:0710.4789]. This shows that passing from one branch of the square root to another introduces additional sine factors in the normalization of the wave functions in a way to keep unchanged the two-point functions.