Deformed oscillators with so(3,4) symmetry

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5-graded decomposition

2Every simple Lie algebra $\mathcal{F}$ (except for $su(1,1)$) admits 5-graded decomposition with respect to a suitable generator $L_0 \in \mathcal{F}$:

$$\mathcal{F} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{-\frac{1}{2}} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{\frac{1}{2}} \oplus \mathfrak{f}_1$$

$$[\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \ for\ i,j \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\},\ |i+j| \leq 1$$

$$[\mathfrak{f}_i, \mathfrak{f}_j] = 0 \ for\ |i+j| > 1$$

We can choose

$$\mathfrak{f}_{\pm 1} = \mathbb{C}L_{\pm 1}, \quad \mathfrak{f}_0 = \mathcal{H} \oplus \mathbb{C}L_0$$

where $\mathcal{H} \subset \mathcal{F}$ is a Lie subalgebra and L_0 commutes with $\mathcal{H}$. A basis for $\mathfrak{f}_{\pm\frac{1}{2}}$ is given by generators $G^A_{\pm\frac{1}{2}}$ which carry an irreducible representation of $\mathcal{H}$.

Here only real Lie algebras and groups are considered so a real forms of $\mathcal{F}$ and $\mathcal{H}$ have to be picked.

Compatibility with the 5-grading requires this real form to be non-compact hence (L_{-1}, L_0, L_1) generate the $su(1,1)$ subalgebra of $\mathcal{F}$.

Let F, H, $SU(1,1)$ be generated by $\mathcal{F}$, $\mathcal{H}$, $su(1,1)$, then a symmetric space can be build as follows:

$$W = \frac{F}{H \times SU(1,1)}$$

The main idea of the construction is to enlarge the coset by reducing the stability subgroup:

$$H \times SU(1,1) \to H \times \mathfrak{B}_{SU(1,1)}$$

where $\mathcal{B}_{SU(1,1)}$ is generated by (L_0, L_1). That gives

$$\mathcal{W} = \frac{F}{H \times \mathfrak{B}_{SU(1,1)}}$$

$\mathcal{W}$ can be parametrised as follows:

$$g = e^{t(L_{-1}+\omega^2 L_1)} e^{u(t)\cdot G_{-\frac{1}{2}}} e^{v(t)\cdot G_{\frac{1}{2}}}$$

where $\cdot$ implies summation over A. The standard definition of the Cartan forms ($\{h_s\}$ form a basis of $\mathcal{H}$):

$$g^{-1}dg = \sum_{i=-1,0,1} \omega_i L_i + \sum_{j=\pm\frac{1}{2}} \omega_j \cdot G_j + \sum_s \omega_h^s h_s$$

One can see, that the equations

$$\omega^A_{-\frac{1}{2}} = 0, \quad A = 1, \ldots, d \tag{1}$$

are invariant under the transformations of the coset, realised by left multiplication by the elements of F. Using this equation we can express the Goldstone fields $v(t)$ via the Goldstone fields $u(t)$ in a covariant fashion (inverse Higgs phenomenon).
After that one can impose additional constraints:

$$\omega^A_{\frac{1}{2}} = 0 \quad A = 1, \ldots, d \tag{2}$$

which are invariant only when (1) are satisfied. Thus the equations (1) and (2) are invariant equations of motion.

The algebra

The algebra $SO(3,4)$ may be defined by the following commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad n,m = -1,0,1$$
$$[M_n, M_m] = (n-m)M_{n+m}, \qquad n,m = -1,0,1$$
$$[P_n, P_m] = (n-m)P_{n+m}, \qquad n,m = -1,0,1$$

$$[L_n, G_{r,a,b}] = (n-r)G_{n+r,a,b}, \qquad n,r = -1,0,1, \qquad a,b = -\frac{1}{2}, \frac{1}{2}$$
$$[M_n, G_{r,a,b}] = (\frac{n}{2}-a)G_{r,n+a,b}, \qquad n,r = -1,0,1, \qquad a,b = -\frac{1}{2}, \frac{1}{2}$$
$$[P_n, G_{r,a,b}] = (\frac{n}{2}-b)G_{n,a,n+b}, \qquad n,r = -1,0,1, \qquad a,b = -\frac{1}{2}, \frac{1}{2}$$

$$[G_{r,a,b}, G_{r',a',b'}] = \delta_{r+r',0}\delta_{b+b',0}b(-1+3r^2)M_{a+a'} +$$
$$+\delta_{r+r',0}\delta_{a+a',0}a(-1+3r^2)P_{b+b'} + 2\delta_{a+a',0}\delta_{b+b',0}ab(r-r')L_{r+r'}$$
$$r, r' = -1,0,1, a, a', b, b' = -\frac{1}{2}, \frac{1}{2}$$

The reason this algebra has been chosen for consideration is that it has 3 $su(1,1)$ subalgebras (generated by L_n, M_n, and P_n) which allows construction of different actions. Every $su(1,1)$ generator rotates $G_{r,a,b}$ by the corresponding index.
In case of $[G_{r,a,b}, G_{r',a',b'}] = 0$ we get the Galilei algebra.

The coset is built as follows:

$$\mathcal{W} = \frac{SO(3,4)}{SU(1,1)\times SU(1,1)\times \mathfrak{B}_{SU(1,1)}}$$

One of the possible parametrisations is

$$g = e^{t(M_{-1}+\omega^2 M_1)}$$
$$e^{u_1 G_{-1,-\frac{1}{2},-\frac{1}{2}} + u_2 G_{-1,-\frac{1}{2},\frac{1}{2}} + u_3 G_{0,-\frac{1}{2},-\frac{1}{2}} + u_4 G_{0,-\frac{1}{2},\frac{1}{2}} + u_5 G_{1,-\frac{1}{2},-\frac{1}{2}} + u_6 G_{1,-\frac{1}{2},\frac{1}{2}}}$$
$$e^{v_1 G_{-1,\frac{1}{2},-\frac{1}{2}} + v_2 G_{-1,\frac{1}{2},\frac{1}{2}} + v_3 G_{0,\frac{1}{2},-\frac{1}{2}} + v_4 G_{0,\frac{1}{2},\frac{1}{2}} + v_5 G_{1,\frac{1}{2},-\frac{1}{2}} + v_6 G_{1,\frac{1}{2},\frac{1}{2}}}$$

The Cartan forms:

$$g^{-1}dg = \sum_n \omega_{L_n} L_n + \sum_a \omega_{M_a} M_a + \sum_b \omega_{P_b} P_b + \sum_{r,a,b} \omega_{r,a,b} G_{r,a,b}$$

Linearised Cartan forms

The linearised Cartan forms can be written as follows:

$$
\begin{aligned}
(g^{-1}dg)_{\text{lin}} = &(du_1 - dtv_1)G_{-1,-\frac{1}{2},-\frac{1}{2}} + (du_2 - dtv_2)G_{-1,-\frac{1}{2},\frac{1}{2}} + \\
&+ (du_3 - dtv_3)G_{0,-\frac{1}{2},-\frac{1}{2}} + (du_4 - dtv_4)G_{0,-\frac{1}{2},\frac{1}{2}} + \\
&+ (du_5 - dtv_5)G_{1,-\frac{1}{2},-\frac{1}{2}} + (du_6 - dtv_6)G_{1,-\frac{1}{2},\frac{1}{2}} + \\
&+ (dv_1 + dt\omega^2 u_1)G_{-1,\frac{1}{2},-\frac{1}{2}} + (dv_2 + dt\omega^2 u_2)G_{-1,\frac{1}{2},\frac{1}{2}} + \\
&+ (dv_3 + dt\omega^2 u_3)G_{0,\frac{1}{2},-\frac{1}{2}} + (dv_4 + dt\omega^2 u_4)G_{0,\frac{1}{2},\frac{1}{2}} + \\
&+ (dv_5 + dt\omega^2 u_5)G_{1,\frac{1}{2},-\frac{1}{2}} + (dv_6 + dt\omega^2 u_6)G_{1,\frac{1}{2},\frac{1}{2}}
\end{aligned}
$$

It corresponds to the Galilei algebra where $[G_{r,a,b}, G_{r',a',b'}] = 0$ for $r, r' = -1, 0, 1$, $a, a', b, b' = -\frac{1}{2}, \frac{1}{2}$.
The system

$$
\begin{cases} du_i - dtv_i = 0 \\ dv_i + dt\omega^2 u_i = 0 \end{cases}
$$

transforms to

$$
\ddot{u}_i + \omega^2 u_i = 0
$$

where $i = 1, \ldots, 6$.

Shortcuts

In order to shorten the following formulae, let us introduce the new variables

$$U^{111} = u_1\,, \quad U^{112} = u_2\,, \quad U^{121} = u_3\,, \quad U^{122} = u_4\,, \quad U^{221} = u_5\,, \quad U^{222} = u_6$$
$$V^{111} = v_1\,, \quad V^{112} = v_2\,, \quad V^{121} = v_3\,, \quad V^{122} = v_4\,, \quad V^{221} = v_5\,, \quad V^{222} = v_6$$

Also, the following shortcuts are used:

$$(\mathcal{AB}) = \mathcal{A}^{ij\alpha}\mathcal{B}_{ij\alpha}$$
$$(\mathcal{ABC})^{ij\alpha} = \mathcal{A}^{ij\beta}\mathcal{B}_{kl\beta}\mathcal{C}^{kl\alpha}$$
$$(\mathcal{ABCD}) = \mathcal{A}^{ij\alpha}\mathcal{B}_{ij\beta}\mathcal{C}^{kl\beta}\mathcal{D}_{kl\alpha}$$

By nullifying Cartan forms $\omega_{r,a,b}$ we gain equations of motion:

$$\dot{V}^{ij\alpha} - \frac{1}{8}(\dot{U}VV)^{ij\alpha} - \frac{1}{4}(V\dot{U}V)^{ij\alpha} +$$
$$+\frac{1}{8}\left(1 + \frac{1}{8}(U\dot{U}) - \frac{1}{128}\omega^2(UUUU)\right)(VVV)^{ij\alpha} +$$
$$+\omega^2\left(U^{ij\alpha} + \frac{1}{8}(VUU)^{ij\alpha} + \frac{1}{4}(UUV) - \frac{1}{64}((UUU)VV)^{ij\alpha} - \frac{1}{32}(V(UUU)V)^{ij\alpha}\right) = 0$$

where

$$V^{ij\alpha} = \frac{\dot{U}^{ij\alpha} + \frac{1}{8}\omega^2(UUU)^{ij\alpha}}{1 + \frac{1}{8}(U\dot{U}) - \frac{1}{128}\omega^2(UUUU)}$$

In the $\omega = 0$ the equations simplify to

$$\ddot{U}^{ij\alpha}\left(1 + \frac{1}{8}\left(U\dot{U}\right)\right) - \frac{1}{32}\left(U\left(\dot{U}\dot{U}\dot{U}\right)\right)\dot{U}^{ij\alpha} - \frac{1}{4}\left(\dot{U}\dot{U}\dot{U}\right)^{ij\alpha} = 0 \qquad (3)$$

The transformations are defined by the following expression:

$$g_0 \cdot g = \tilde{g} \cdot h$$

where h is from the stability subgroup, g and $\tilde{g} \in \mathcal{W}$.
In the case of $SO(3,4)$ under consideration the following transformations have been calculated

$$g^{aM_{-1}+bM_0+cM_1} e^{t(M_{-1}+\omega^2 M_1)} e^{u\cdot G_{-\frac{1}{2}}} e^{v\cdot G_{\frac{1}{2}}} =$$

$$e^{(t+\delta t)(M_{-1}+\omega^2 M_1)} e^{(u+\delta u)\cdot G_{-\frac{1}{2}}} e^{(v+\delta v)\cdot G_{\frac{1}{2}}} h$$

Transformations

We will use the function f:

$$f(t) = \frac{1+\cos(2\omega t)}{2}\cdot a + \frac{\sin(2\omega t)}{2\omega}\cdot b + \frac{1-\cos(2\omega t)}{2\omega^2}\cdot c$$

which satisfies the equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{f} + 4\omega^2 f\right) = 0$.
Then

$$\delta t = \frac{1}{64}\ddot{f}\frac{(UUUU)}{-4+\frac{1}{32}\omega^2(UUUU)} + f$$

$$\delta U^{ij\alpha} = -\frac{4\dot{f}U^{ij\alpha} + \ddot{f}(UUU)^{ij\alpha} - \frac{1}{16}\omega^2\dot{f}(UU(UUU))^{ij\alpha}}{2(-4+\frac{1}{32}\omega^2(UUUU))}$$

$$\delta V^{ij\alpha} = -\frac{1}{2(-4+\frac{1}{32}\omega^2(UUUU))}\cdot(4\ddot{f}U^{ij\alpha} - 4\dot{f}V^{ij\alpha} + \ddot{f}(UUV)^{ij\alpha} + \frac{1}{2}\ddot{f}(VUU)^{ij\alpha} - \frac{1}{16}\omega^2\dot{f}(V(UUU)U)^{ij\alpha})$$

In order to construct an action let us expand the coset:

$$\mathcal{W} = \frac{SO(3,4)}{SU(1,1) \times SU(1,1) \times \mathfrak{B}_{SU(1,1)}} \rightarrow \mathcal{W}_{\text{imp}} = \frac{SO(3,4)}{SU(1,1) \times U(1) \times \mathfrak{B}_{SU(1,1)}}$$

then a coset element changes to:

$$g_{\text{imp}} = g e^{\Lambda_{-1} P_{-1} + \Lambda_1 P_1}$$

The new Cartan forms are:

$$g_{\text{imp}}^{-1} dg_{\text{imp}} = \sum_n \Omega_{L_n} L_n + \sum_a \Omega_{M_a} M_a + \sum_b \Omega_{P_b} P_b + \sum_{ij\alpha} \Omega_u^{ij\alpha} G_{-\frac{1}{2},ij\alpha} + \sum_{ij\alpha} \Omega_v^{ij\alpha} G_{\frac{1}{2},ij\alpha}$$

$$\Omega_u^{ij\alpha} = \Omega_u^{ji\alpha}, \quad \Omega_v^{ij\alpha} = \Omega_v^{ji\alpha}$$

and $\Omega_{M_n} = \omega_{M_n}$.

As follows from this equation

$$\Omega_u^{ij\alpha} G_{-\frac{1}{2},ij\alpha} = e^{-\Lambda_{-1}P_{-1}-\Lambda_1 P_1} \omega_u^{ij\alpha} G_{-\frac{1}{2},ij\alpha} e^{\Lambda_{-1}P_{-1}+\Lambda_1 P_1}$$
$$\Omega_v^{ij\alpha} G_{\frac{1}{2},ij\alpha} = e^{-\Lambda_{-1}P_{-1}-\Lambda_1 P_1} \omega_v^{ij\alpha} G_{\frac{1}{2},ij\alpha} e^{\Lambda_{-1}P_{-1}+\Lambda_1 P_1}$$

we have

$$\omega_u^{ij\alpha} = \omega_v^{ij\alpha} = 0 \quad \Rightarrow \quad \Omega_u^{ij\alpha} = \Omega_v^{ij\alpha} = 0$$

The opposite statement, however, is not always true. Next let us introduce the new variables:

$$\lambda_{-1} = \frac{\tan\sqrt{\Lambda_{-1}\Lambda_1}}{\sqrt{\Lambda_{-1}\Lambda_1}}\Lambda_{-1} \quad \lambda_1 = \frac{\tan\sqrt{\Lambda_{-1}\Lambda_1}}{\sqrt{\Lambda_{-1}\Lambda_1}}\Lambda_1$$

Construction of the action

Then equations for $\lambda_{\pm 1}$ are:

$$\Omega_{P_{-1}} = \frac{1}{1+\lambda_{-1}\lambda_1}(\omega_{P_{-1}} + \lambda_{-1}\omega_{P_0} + \lambda_{-1}^2\omega_{P_1} + d\lambda_{-1}) = 0$$

$$\Omega_{P_1} = \frac{1}{1+\lambda_{-1}\lambda_1}(\omega_{P_1} - \lambda_1\omega_{P_0} + \lambda_1^2\omega_{P_{-1}} + d\lambda_1) = 0$$

They are needed to get rid of the λ terms in the previous equations to acquire the required ones.
One can build an invariant action using Ω_{P_0}:

$$S = -\int \Omega_{P_0} =$$

$$-\int \frac{1}{1+\lambda_{-1}\lambda_1}(\lambda_{-1}\omega^{22} - \lambda_1\omega^{11} + (1-\lambda_{-1}\lambda_1)\omega^{12} + (\lambda_{-1}d\lambda_1 - \lambda_1 d\lambda_{-1}))$$

where

$$\omega^{11} = 2\omega_{P_{-1}}, \quad \omega^{12} = \omega_{P_0}, \quad \omega^{22} = 2\omega_{P_1}$$

Another possible coset parametrisation is built as follows:

$$e^{t(M_{-1}+\omega^2 M_1)} e^{u_1 G_{-1,-\frac{1}{2},-\frac{1}{2}} + u_2 G_{-1,\frac{1}{2},-\frac{1}{2}}} e^{v_1 G_{-1,-\frac{1}{2},\frac{1}{2}} + v_2 G_{-1,\frac{1}{2},\frac{1}{2}}}$$
$$e^{r_1 G_{0,-\frac{1}{2},-\frac{1}{2}} + r_2 G_{0,\frac{1}{2},-\frac{1}{2}}} e^{s_1 G_{0,-\frac{1}{2},\frac{1}{2}} + s_2 G_{0,\frac{1}{2},\frac{1}{2}}} e^{x_1 G_{1,-\frac{1}{2},-\frac{1}{2}} + x_2 G_{1,\frac{1}{2},-\frac{1}{2}}}$$
$$e^{y_1 G_{1,-\frac{1}{2},\frac{1}{2}} + y_2 G_{1,\frac{1}{2},\frac{1}{2}}}$$

The Cartan forms:

$$g^{-1}dg = \sum_n \hat{\omega}_{L_n} L_n + \sum_a \hat{\omega}_{M_a} M_a + \sum_b \hat{\omega}_{P_b} P_b + \sum_{r,a,b} \hat{\omega}_{r,a,b} G_{r,a,b}$$

Nullifying all the forms $\hat{\omega}_{r,a,b}$ yields these equations for 6 non-interacting harmonic oscillators:

$$\ddot{w} + \omega^2 w = 0$$

where $w = u_1, v_1, r_1, s_1, x_1, y_1$.

Conclusion

The method of nonlinear realisations in application to the Lie algebra $SO(3,4)$ has been considered and equations of motion have been obtained. Also the transformation leaving the equations invariant have been gained and a corresponding action has been built. As it has been mentioned, the chosen algebra admits building of a second action which is left for the further research. It has also been acquired, that the system of 6 harmonic oscillators admits symmetry under the action of the $so(3,4)$ group.