Low-energy spectrum of SU(3) Yang-Mills QM from the Integrable System closest to it

Hans-Peter Pavel (BLTP, JINR)

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- Introduction: Hamiltonian fomulation of SU(3) Yang-Mills QM constrained by the non-Abelian Gauss-laws
- Unonstrained Hamiltonian fomulation via exact resolution of the Gauss-laws using a new algebraic gauge (flux-tube gauge) with a simple but non-trivial Jacobian (Faddeev-Popov det)
- Exact solution of the 'corresponding harmonic oscillator problem', the integrable system obtained by replacing the magnetic potential by the multiple harm. oscillator potential $B_{ai}^2(A) \longrightarrow \omega^2 A_{ai}^2$, but keeping the non-trivial measure and the kinetic term unchanged $\longrightarrow$ ONB (orthonormal polynomials)
- Calculation of the low-energy spectrum of SU(3) YM QM: using ONB truncated at increas. polyn.degree $\rightarrow$ convergence
- Comparison of the results with those of Weisz and Zieman (1986) obtained in the constrained approach, and comparison with the low glueball spectrum obtained using lattice QCD (Morningstar and Peardon (1997) and Chen et. al. (2006))
- Conclusions.

The action of $SU(3)$ Yang-Mills theory of the gluon fields $V_\mu(x) \equiv V_{a\mu}(x)\lambda_a/2$

$$\mathcal{S}[V] := \int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right] \qquad F^a_{\mu\nu} := \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g f_{abc} V^b_\mu V^c_\nu \ , \quad a = 1, .., 8$$

is invariant under the $SU(3)$ gauge transformations

$$V^\omega_{a\mu}(x)\lambda_a/2 \quad = \quad U[\omega(x)] \left(V_{a\mu}(x)\lambda_a/2 + \frac{i}{g}\partial_\mu \right) U^{-1}[\omega(x)]$$

chromoelectric : $\quad E^a_i \equiv F^a_{i0}$ and chromomagnetic $\quad B^a_i \equiv \frac{1}{2}\epsilon_{ijk} F^a_{jk}$

$\Pi_{ai} = -E_{ai}$ momenta can. conj. to the spatial $V_{ai} \to$ canonical Hamiltonian

$$H_C \quad = \quad \int d^3x \left[\frac{1}{2} E^2_{ai} + \frac{1}{2} B^2_{ai}(V) - V_{a0} \left(D_i(V)_{ab} E_{bi} \right) \right]$$

with the covariant derivative $D_i(V)_{ab} \equiv \delta_{ab}\partial_i - g f_{abc} V_{ci}$
Exploit the time dependence of the gauge transformations to put

$$V_{a0} = 0 \ , \qquad a = 1, .., 8 \qquad \text{(Weyl gauge)}$$

The dynam. vaiables V_{ai}, $-E_{ai}$ are quantized in the Schrödinger functional approach imposing the equal-time CR, $-E_{ai} = -i\partial/\partial V_{ai}$. The physical states Φ satisfy

$$H_0\,\Phi \equiv \int d^3x \left[\frac{1}{2} E^2_{ai} + \frac{1}{2} \left(B_{ai}(V) \right)^2 \right] \Phi = E\,\Phi \ , \qquad G_a(x)\,\Phi \equiv D_i(V)_{ab} E_{bi}\ \Phi = 0$$

The Gauss law operators G_a are the generators of the residual time independent gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = i f_{abc} G_c$.

matrix elements given in the Cartesian form $\quad \langle \Phi' | O | \Phi \rangle \propto \int dV \ \Phi'^*(V)\, O\, \Phi(V)$.

Constrained Quantisation of $SU(3)$ YM QM of spatially constant fields

For the case of $SU(3)$ YM QM of spatially constant fields, the physical states Φ satisfy

$$H_0\,\Phi \equiv \left[\frac{1}{2}E_{ai}^2 + \frac{1}{2}\left(B_{ai}^{\rm hom}(V)\right)^2\right]\Phi = E\left[\frac{g^{2/3}}{\mathrm{Vol}^{1/3}}\right]\Phi\ , \qquad B_{ai}^{\rm hom} := (1/2)g\epsilon_{ijk}\,f_{abc}\,V_{b\,j}V_{c\,k}$$

$$G_a\,\Phi \equiv f_{abc}\;V_{ai}E_{bi}\;\Phi = 0\ , \qquad a = 1,..,8\ . \qquad V_{ai}^{\omega}\lambda_a/2 = U[\omega]\;V_{ai}\lambda_a/2\;U^{-1}[\omega]$$

The Gauss law operators G_a are the generators of the residual time independent global gauge transformations, satisfying $[G_a, H] = 0$ and $[G_a, G_b] = if_{abc}G_c$.
H_0 is invariant under spatial rotations $[H_0, J_i] = 0$ with

$$J_i = -\epsilon_{ijk}V_{aj}E_{ak} \quad i = 1,2,3\ , \quad [J_i, J_j] = i\epsilon_{ijk}J_k$$

and invariant under parity $[H_0, P] = 0$ and charge conjugation $[H_0, C] = 0$ $\qquad J^{PC}$

$$P:\quad V_{ai}\lambda_a \to -V_{ai}\lambda_a \qquad\qquad C:\quad V_{ai}\lambda_a \to -(V_{ai}\lambda_a)^*$$

The matrix element of an operator O is given in the Cartesian form

$$\langle\Phi'|O|\Phi\rangle \;\propto \int dV\;\Phi'^*(V)\,O\,\Phi(V)\ .$$

Weisz and Ziemann (1986): Variat. calc. with trial functions

$$\Phi^{(J)PC}(V) = P^{(J)PC}_{\rm gauge\ inv.}(V)\;\exp[-(\omega/2)\,(V_{ai})^2]$$

I shall show that the above constrained system becomes integrable if one replaces

$$\left(B_{ai}^{\rm hom}(V)\right)^2 \;\longrightarrow\; \omega^2\,(V_{ai})^2\ , \qquad \omega > 0\ \ \text{free parameter}$$

Using an exact gauge reduction the energy-eigensystem can be found and used as a Hilbert-basis for the YM QM. Truncating at higher and higher numbers of nodes, a converging low-energy eigensystem of YM QM is obtained.

Lambda-Matrices

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Unconstrained formulation using the flux-tube gauge

Point trafo to new set of adapt coord., 24 $V_{ai} \to$ 8 gauge angles q_j, 16 reduced A_{ai}

$V_{ai}\,(q,S) = O_{ab}\,(q)\,A_{bi}$, $\qquad O_{ab}(q)$ orth. 8×8 matrix adjoint to $U(q)$

$$O_{ab}(q) = (1/8)\mathrm{Tr}\left[U^{-1}(q)\lambda_a U(q)\lambda_b\right]$$

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ 0 & A_{42} & A_{43} \\ 0 & 0 & A_{53} \\ 0 & A_{62} & A_{63} \\ 0 & 0 & A_{73} \\ A_{81} & A_{82} & A_{83} \end{pmatrix} \equiv (X\ Y\ Z)\ .$$

$$\chi_a(A) = (\Gamma_i)_{ab}\,A_{bi} = 0\ : \qquad A_{a1} = 0 \quad \forall a = 1,2,4,5,6,7 \quad \wedge \quad A_{a2} = 0 \quad \forall a = 5,7\,.$$

Preserving the CCR $\to$ old canonical momenta in terms of the new variables

$$-E_{ai}(q,A,p,P) = O_{ab}\,(q)\left[P_{bi} - (\Gamma_i)_{bl}\,\gamma^{-1}_{ls}(A)\left(\frac{1}{g}\Omega^{-1}_{st}(q)p_t + T_s(A,P)\right)\right]$$

with the FP operator $\gamma_{ab}(A) \equiv (\Gamma_i)_{ad}\, f_{dbc}A_{ci}$ and the $T_a(A,P) \equiv f_{abc}A_{bi}P_{ci}$.

$$\Rightarrow \qquad G_a\Phi \equiv O_{ak}(q)\Omega^{-1}_{ki}(q)p_i\Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i}\Phi = 0 \quad \text{(Abelianisation)}$$

(for the case of unconstrained $SU(2)$ YM: Khvedelidze and Pavel 1999, 2000)

The correctly ordered unconstrained Hamiltonian (e.g. Christ and Lee 1980) of SU(3) YM-QM in the flux-tube gauge takes the form

$$H = \frac{1}{2}\left[\frac{1}{|\gamma(A)|}P_{ai}\ |\gamma(A)|\ P_{ai} + \frac{1}{|\gamma(A)|}T_a\ |\gamma(A)|\left(\gamma^{-1}\gamma^{-1T}\right)_{ac}T_c + \left(B^{\rm hom}_{ai}(A)\right)^2\right],$$

using the homogeneus part $B^{\rm hom}_{a\,i} := (1/2)g\epsilon_{ijk}\,f_{abc}\,A_{b\,j}A_{c\,k}$ of the chromomagnetic field and the operators

$$T_a(A,P) := f_{abc}A_{bi}P_{ci} \equiv T^Y_a(Y,P_Y) + T^Z_a(Z,P_Z)\ .$$

Note, that the components of the (non-reduced) $T^Z_a = -if_{abc}\,Z_b\,\partial/\partial Z_c$ satisfy the $su(3)$ algebra

$$[T^Z_a, T^Z_b] = i\,f_{abc}\,T^Z_c\ ,$$

whereas the reduced $T^Y_a = -if_{abc}\,Y_b\,\partial/\partial Y_c$ do not.
The matrix element of a physical operator O is given by

$$\langle\Psi'|O|\Psi\rangle\ \propto \int dA\ |\gamma(A)|\ \Psi'^*[A]\ O\ \Psi[A]$$

The inverse γ^{-1} of the homogeneous part of the Faddeev-Popov operator exists in the regions of non-vanishing determinant

$$|\gamma(A)| = X_3^2\left(X_3^2 - 3\,X_8^2\right)^2\ Y_4Y_6$$

with the homogeneous part of the FP operator

$$\gamma_{ab} \equiv (\Gamma_i)_{ad}\; f_{dbc} A_{ci} \; . \tag{1}$$

Its explicit expression is

$$\gamma = \begin{pmatrix} 0 & -X_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -Y_6/2 & 0 & -Y_4/2 & -Y_+ & 0 & Y_1/2 & -Y_2/2 & -\sqrt{3}\,Y_4/2 \\ 0 & 0 & 0 & 0 & X_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & -X_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -X_- & 0 \\ 0 & 0 & 0 & 0 & 0 & X_- & 0 & 0 \\ -Y_4/2 & 0 & Y_6/2 & Y_1/2 & Y_2/2 & Y_- & 0 & -\sqrt{3}\,Y_6/2 \end{pmatrix} , \tag{2}$$

using the abbreviations $X_\pm := -(X_3 \pm \sqrt{3}X_8)/2$ and $Y_\pm := -(Y_3 \pm \sqrt{3}Y_8)/2$. The inverse γ^{-1} of the homogeneous part of the Faddeev-Popov operator exists in the regions of non-vanishing determinant

$$\det \gamma = X_3^2 \left(X_3^2 - 3\,X_8^2\right)^2 Y_4 Y_6$$

Analytical invertibility of the FP-operator in the flux-tube gauge

The inverse γ^{-1} of the homogeneous part of the Faddeev-Popov operator exists in the regions of non-vanishing determinant

$$\det\gamma = X_3^2\left(X_3^2 - 3\,X_8^2\right)^2 Y_4 Y_6$$

and the non-vanishing matrix elements of the inverse γ^{-1} are rather simple,

$$(\gamma^{-1})_{12} = -\,(\gamma^{-1})_{21} = X_3^{-1}\;,$$

$$(\gamma^{-1})_{32} = \frac{1}{2}X_3^{-1}\,(Y_4/Y_6 - Y_6/Y_4)\;,\quad (\gamma^{-1})_{82} = -\frac{1}{2\sqrt{3}}X_3^{-1}\,(Y_4/Y_6 + Y_6/Y_4)\;,$$

$$(\gamma^{-1})_{45} = -\,(\gamma^{-1})_{54} = -X_+^{-1}\;,\qquad (\gamma^{-1})_{34} = -\sqrt{3}\,(\gamma^{-1})_{84} = -\frac{1}{2}X_+^{-1}(Y_2/Y_6)\;,$$

$$(\gamma^{-1})_{35} = -\frac{1}{2}X_+^{-1}\,(-Y_1/Y_6 - 2\,Y_+/Y_4)\;,\quad (\gamma^{-1})_{85} = -\frac{1}{2\sqrt{3}}X_+^{-1}\,(Y_1/Y_6 - 2\,Y_+/Y_4)\;,$$

$$(\gamma^{-1})_{67} = -\,(\gamma^{-1})_{76} = X_-^{-1}\;,\qquad (\gamma^{-1})_{36} = \sqrt{3}\,(\gamma^{-1})_{86} = \frac{1}{2}X_-^{-1}(Y_2/Y_4)\;,$$

$$(\gamma^{-1})_{37} = \frac{1}{2}X_-^{-1}\,(Y_1/Y_4 - 2\,Y_-/Y_6)\;,\qquad (\gamma^{-1})_{87} = \frac{1}{2\sqrt{3}}X_-^{-1}\,(Y_1/Y_4 + 2\,Y_-/Y_6)\;,$$

$$(\gamma^{-1})_{33} = \sqrt{3}\,(\gamma^{-1})_{83} = Y_4^{-1}\;,\qquad (\gamma^{-1})_{38} = -\sqrt{3}\,(\gamma^{-1})_{88} = Y_6^{-1}\;,$$

grouped into those prop. to X_3^{-1}, X_+^{-1}, and X_-^{-1}, and those indep. of X.
Such a "Weyl-decomposition" $\rightarrow$ considerable simplification of the non-local potential.

$$\begin{aligned}
H[A,P] &= K_X + K_Y + K_Z + \\
&+ \frac{1}{2\,r^2}\left[\frac{(I_1^{YZ}+I_2^{YZ})}{\cos^2\psi} + \frac{(I_4^{YZ}+I_5^{YZ})}{\cos^2[\psi+2\pi/3]} + \frac{(I_6^{YZ}+I_7^{YZ})}{\cos^2[\psi+4\pi/3]}\right] \\
&+ \frac{1}{2\,Y_4^2} I_+^{YZ} + \frac{1}{2\,Y_6^2} I_-^{YZ} + \frac{1}{2}\left(B_{ai}^{\rm hom}[X,Y,Z]\right)^2 \;,
\end{aligned}$$

$$\langle\Psi_1|O|\Psi_2\rangle = \int d\mu_X \int d\mu_Y \int d\mu_Z \;\Psi_1^\dagger O\;\Psi_2 \;, \qquad X_3 = r\cos\psi,\;\; X_8 = r\sin\psi$$

$$\int d\mu_X \propto \int_0^\infty dr\, r^{\,7} \int_0^{2\pi} d\psi\;\cos^2[3\,\psi]\;,$$

$$\int d\mu_Y \propto \int_{-\infty}^\infty dY_1 \int_{-\infty}^\infty dY_2 \int_{-\infty}^\infty dY_3 \int_{-\infty}^\infty dY_8 \int_0^\infty dY_4 Y_4 \int_0^\infty dY_6 Y_6\;,$$

$$\int d\mu_Z \propto \prod_{a=1}^{8}\int_{-\infty}^\infty dZ_a \;.$$

The single-direction kinetic terms read

$$K_X = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{7}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(-6\tan[3\psi]\frac{\partial}{\partial\psi} + \frac{\partial^2}{\partial\psi^2}\right)\right]\;, \qquad K_Z = -\frac{1}{2}\sum_{a=1}^{8}\frac{\partial^2}{\partial Z_a^2},$$

$$K_Y = -\frac{1}{2}\left[\sum_{a=1,2,3,8}\frac{\partial^2}{\partial Y_a^2} + \sum_{a=4,6}\left(\frac{\partial^2}{\partial Y_a^2} + \frac{1}{Y_a}\frac{\partial}{\partial Y_a} - \frac{1}{Y_a^2}\left(Y_1\frac{\partial}{\partial Y_2} - Y_2\frac{\partial}{\partial Y_1}\right)^2\right)\right]\,,$$

and the interations

$$
\begin{aligned}
I_m^{YZ} &:= \left(\frac{1}{Y_4 Y_6}\widetilde{T}_m^{Y\dagger} Y_4\, Y_6 + \widetilde{T}_m^Z\right)\left(\widetilde{T}_m^Y + \widetilde{T}_m^Z\right) \\
I_\pm^{YZ} &:= \left[\left(T_3^Z \pm \frac{1}{\sqrt{3}} T_8^Z\right) + 2T_3^Y\right]\left(T_3^Z \pm \frac{1}{\sqrt{3}} T_8^Z\right)
\end{aligned}
$$

with the shifted non-Hermitean $\widetilde{T}_a^Y$ and Hermitean $\widetilde{T}_a^Z$, defined as

$$
\begin{aligned}
\widetilde{T}_1 &:= T_1 - \frac{Y_6}{2Y_4}\left(T_3 + \frac{1}{\sqrt{3}}T_8\right) + \frac{Y_4}{2Y_6}\left(T_3 - \frac{1}{\sqrt{3}}T_8\right) , \\
\widetilde{T}_2 &:= T_2 , \\
\widetilde{T}_4 &:= T_4 - \frac{Y_1}{2Y_6}\left(T_3 - \frac{1}{\sqrt{3}}T_8\right) - \frac{Y_+}{Y_4}\left(T_3 + \frac{1}{\sqrt{3}}T_8\right) , \\
\widetilde{T}_5 &:= T_5 - \frac{Y_2}{2Y_6}\left(T_3 - \frac{1}{\sqrt{3}}T_8\right) , \\
\widetilde{T}_6 &:= T_6 + \frac{Y_1}{2Y_4}\left(T_3 + \frac{1}{\sqrt{3}}T_8\right) - \frac{Y_-}{Y_6}\left(T_3 - \frac{1}{\sqrt{3}}T_8\right) , \\
\widetilde{T}_7 &:= T_7 - \frac{Y_2}{2Y_4}\left(T_3 + \frac{1}{\sqrt{3}}T_8\right) ,
\end{aligned}
$$

The corresponding harmonic oscillator problem

Replacing in $H(A,P)$ the magnetic potential by the separable harmonic oscillator potential with free parameter $\omega > 0$

$$\frac{1}{2}\left(B^{\text{hom}}_{ai}(A)\right)^2 \longrightarrow \frac{1}{2}\,\omega^2\,(A_{ai})^2 \equiv \frac{1}{2}\,\omega^2\left[r^2+Y_1^2+Y_2^2+Y_3^2+Y_4^2+Y_6^2+Y_8^2+Z_a^2\right]$$

we obtain the corresponding harmonic oscillator problem (with the same measure !!!)

$$\begin{aligned} H_{h.o.}[A,P] \quad &= \quad H_X + H_Y + H_Z + \frac{1}{2\,Y_4^2} I_+^{YZ} + \frac{1}{2\,Y_6^2} I_-^{YZ} + \\ &\quad + \frac{1}{2\,r^2}\left[\frac{(I_1^{YZ}+I_2^{YZ})}{\cos^2\psi} + \frac{(I_4^{YZ}+I_5^{YZ})}{\cos^2[\psi+2\pi/3]} + \frac{(I_6^{YZ}+I_7^{YZ})}{\cos^2[\psi+4\pi/3]}\right] \end{aligned}$$

The single-direction Hamiltonions read

$$\begin{aligned} H_X &= \frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{7}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(6\tan[3\psi]\frac{\partial}{\partial\psi} - \frac{\partial^2}{\partial\psi^2}\right) + \omega^2 r^2\right] , \\ H_Y &= \frac{1}{2}\Bigg[\sum_{a=1,2,3,8}\left(-\frac{\partial^2}{\partial Y_a^2} + \omega^2 Y_a^2\right) \\ &\qquad + \sum_{a=4,6}\left(-\frac{\partial^2}{\partial Y_a^2} - \frac{1}{Y_a}\frac{\partial}{\partial Y_a} + \frac{1}{Y_a^2}\left(Y_1\frac{\partial}{\partial Y_2} - Y_2\frac{\partial}{\partial Y_1}\right)^2 + \omega^2 Y_a^2\right)\Bigg] , \\ H_Z &= \frac{1}{2}\sum_{a=1}^{8}\left[\frac{\partial^2}{\partial Z_a^2} + \omega^2 Z_a^2\right], \end{aligned}$$

The corresponding harmonic oscillator problem (with the same measure !!!)

$$H_{h.o.}(A,P)|\Phi^{(J)PC}_{i,M}\rangle = \epsilon^{(J)PC}_{h.o.}\left(\frac{g^{2/3}}{a}\right)|\Phi^{(J)PC}_{i,M}\rangle\ ,$$

turns out to be trigonal in the space of the monomial functionals

$$M[\omega s_{[2]}, \omega^{3/2} s_{[3]}, \omega^2 b_{[4]}, \omega^{5/2} b_{[5]}, \omega^{3/2} a_{[3]}, \omega^2 a_{[4]}, \omega^{5/2} a_{[5]}]\ \exp[-\frac{1}{2}\omega\,(A_{ai})^2]\ ,$$

The M are monomials in the 35 components of seven elementary $SU(3)$-invariant spatial tensors in reduced A-space shown in Table 1. Note $(A_{ai})^2 \equiv (s_{11}+s_{22}+s_{33})$

sym. $s^{++}_{[2]ij}[A] := A_{ai}A_{aj}\ ,\quad (i\le j)$	$0^{++}, 2^{++}$
sym. $s^{--}_{[3]ijk}[A] := d_{abc}\,A_{ai}A_{bj}A_{ck}\ ,\quad (i\le j\le k)$	$1^{--}, 3^{--}$
sym. $b^{++}_{[4]ij}[A] := B^{\rm hom}_{ai}B^{\rm hom}_{aj}\ ,\quad (i\le j)\ ,\quad B^{\rm hom}_{ai} := (1/2)\epsilon_{ijk}\,f_{abc}\,A_{bj}A_{ck}$	$0^{++}, 2^{++}$
$b^{--}_{[5]i}[A] := d_{abc}\,B^{\rm hom}_{ai}B^{\rm hom}_{bi}A_{ci} + \frac{1}{4}\left(2s_{jk}s_{123} - s_{jj}s_{ikk} - s_{kk}s_{ijj}\right),\ (i\neq j\neq k)$	1^{--}
$a^{-+}_{[3]}[A] := f_{abc}\,A_{a1}A_{b2}A_{c3} = B^{\rm hom}_{a1}A_{a1} = B^{\rm hom}_{a2}A_{a2} = B^{\rm hom}_{a3}A_{a3}$	0^{-+}
$a^{+-}_{[4]i}[A] := d_{abc}\,B^{\rm hom}_{ai}A_{bi}A_{ci}\ ,\quad (i=1,2,3)$	1^{+-}
sym. $a^{-+}_{[5]ij}[A] := d_{abc}\,B^{\rm hom}_{ak}A_{bk}(d_{cde}\,A_{di}A_{ej})\ ,\quad (i\le j\wedge k\neq i,j)$	$0^{-+}, 2^{-+}$

Our result is in agreement with theorem by Dittner (1972):
"maximal number of primitive $SU(3)$-invariants in original constrained V-space is 35".

e.g. in constrained V-space, the $SU(3)$-invariant

$$d_{abc} C^a_{12}[V] C^b_{12}[V] C^c_{12}[V] \qquad \text{with} \qquad C^a_{12}[V] := d_{abc}\, V^b_1\, V^c_2 \ ,$$

is independent of $s_{11}[V]$, $s_{12}[V]$, $s_{12}[V]$, $s_{111}[V]$, $s_{112}[V]$, $s_{122}[V]$, $s_{222}[V]$, $b_{33}[V]$, in the sense, that it cannot be represented as a sum of products of them. It is, however, not primitive because it is related to them via outer products.

In reduced A-space, however, where outer products of invariant tensors are absent, the corresponding polynomial is indeed reducible

$$d_{abc} C^a_{12}[A] C^b_{12}[A] C^c_{12}[A] \quad = \quad \frac{1}{18} s^3_{12}[A] - \frac{1}{6} s_{12}[A]\, s_{11}[A]\, s_{22}[A] - \frac{1}{12} s_{111}[A]\, s_{222}[A] + \frac{3}{4} s_{112}[A]\, s_{122}[A] + \frac{1}{6} s_{12}[A]\, b_{33}[A] \ .$$

All solutions of the corresponding harmonic oscillator problem

Organising the monomial functionals according to the degree n (as a polynomial in the A) and the conserved quantum numbers J,M,P,C and applying a Gram-Schmidt orthogonalisation with respect to the measure, we obtain all exact solutions

$$\Phi^{(J)PC}_{[n]\,i,M}[A] = P^{(J)PC}_{[n]\,i,M}[\omega s_{[2]}, \omega^{3/2} s_{[3]}, \omega^2 b_{[4]}, \omega^{5/2} b_{[5]}, \omega^{3/2} a_{[3]}, \omega^2 a_{[4]}, \omega^{5/2} a_{[5]}] \times \exp[-\omega\,(A_{ai})^2/2]\ ,$$

of the corresponding harmonic oscillator problem with energies

$$\epsilon^{(J)PC}_{h.o.} = (12+n)\,\omega\ ,$$

where n is the degree of $P_{[n]}$ as a polynom in the A.

The lowest 0^{++} eigenstates e.g., are $\quad (s^{(0)++}_{[2]} \equiv (s_{11}+s_{22}+s_{33})/\sqrt{3})$

$$\epsilon^{(0)++}_{h.o.} = 12\,\omega\ : \quad P^{(0)++}_{[0]} \propto 1\ ,$$

$$\epsilon^{(0)++}_{h.o.} = 14\,\omega\ : \quad P^{(0)++}_{[2]} \propto -2\sqrt{3} + \frac{1}{2}\,\omega\, s^{(0)++}_{[2]}\ ,$$

$$\epsilon^{(0)++}_{h.o.} = 16\,\omega\ : \quad P^{(0)++}_{[4]1} \propto \sqrt{78} - \sqrt{\frac{13}{2}}\,\omega\, s^{(0)++}_{[2]} + \frac{1}{2}\sqrt{\frac{3}{26}}\,\omega^2 \left(s^{(0)++}_{[2]}\right)^2\ ,$$

$$P^{(0)++}_{[4]2} \propto -\frac{1}{2\sqrt{273}}\omega^2 \left(s^{(0)++}_{[2]}\right)^2 + \frac{1}{2}\sqrt{\frac{13}{105}}\omega^2 \langle s^{(2)}_{[2]} s^{(2)}_{[2]} \rangle^{(0)++}\ ,$$

$$P^{(0)++}_{[4]3} \propto -\frac{1}{3}\sqrt{\frac{2}{35}}\omega^2 \left(s^{(0)++}_{[2]}\right)^2 + \frac{1}{3\sqrt{14}}\omega^2 \langle s^{(2)}_{[2]} s^{(2)}_{[2]} \rangle^{(0)++} + \frac{1}{9}\sqrt{\frac{14}{5}}\,\omega^2\, b^{(0)++}_{[4]}\ , \ldots\ldots$$

Consider the basis of energy eigenstates of the correponding unconstrained harmonic oscillator Schrödinger equation orthonormal with respect to the Yang-Mills measure

$$H_{\text{h.o.}}\Phi_n[A,\omega] \equiv \left[T_{\text{kin}} + \frac{1}{2}\omega^2 A_{ai}^2\right]\Phi_n[A,\omega] = \epsilon_n^{\text{h.o.}}\Phi_n[A,\omega]\ .$$

Then the matrix elements of the unconstrained Yang-Mills Hamiltonian are given as

$$\begin{aligned}\mathcal{M}_{mn} &:= \langle\Phi_m^\dagger[A,\omega]\left(T_{\text{kin}} + \frac{1}{2}B_{ai}^2[A]\right)\Phi_n[A,\omega]\rangle_A \\ &= \left[\delta_{nm}\epsilon_n^{\text{h.o.}} - \langle\Phi_m^\dagger[A,\omega]\left(\frac{1}{2}\omega^2 A_{ai}^2\right)\Phi_n[A,\omega]\rangle_A\right] + \frac{1}{2}\langle\Phi_m^\dagger[A,\omega]\left(B_{ai}^2[A]\right)\Phi_n[A,\omega]\rangle_A\end{aligned}$$

since the kinetic terms T_{kin} are the same for the Yang-Mills and the corresponding harmonic oscillator problem. We treat ω as a variational parameter, which in each symmetry sector can be choose to minimize the lowest eigenvalue of the matrix $\mathcal{M}$.

The spectrum is purely discrete and the lowest energy is $\epsilon_0^{++} = 12.589$.

The results are in good agreement with the results of Weisz and Zieman (1986) using the constrained Hamiltonian approach in the 0^{++} and 2^{++} sectors, much more accurate values in other sectors considered by them, e.g. in 1^{--} and 3^{--} sectors , and give quite accurate "new results" for the states not considered by them, as e.g. 2^{--}, 3^{++} .

Eigenstates of $H_{\rm h.o.}$ from Gram-Schmidt Orthogonalisation

Enumerate all possible monomials for given J^{PC} in increasing order n and multipl. m

$$M_r^{(J)PC}[A] := M_{\{[n]m\}}^{(J)PC}[s_{[2]}[A], s_{[3]}[A], b_{[4]}[A], b_{[5]}[A], a_{[3]}[A], a_{[4]}[A], a_{[5]}[A]] \ ,$$

$$\text{e.g.} \quad M_1^{(0)++} = M_{[0]}^{(0)++} \ , \quad M_2^{(0)++} = M_{[2]}^{(0)++} , \quad M_{3,4,5}^{(0)++} = M_{[4]1,2,3}^{(0)++} \ , \ldots..$$

Gram matrix

$$(G_{rs})^{(J)PC} := \left(\langle\langle M_r^{(J)PC} M_s^{(J)PC} \rangle\rangle_A \right)$$

$$\langle\langle \ \rangle\rangle_A \quad := \quad \int_0^{2\pi} d\psi \, \cos^2[3\,\psi] \int_0^{\infty} dr\, r^{\,7} \exp[-\omega r^2] \left[\prod_{a=1}^{8} \int_{-\infty}^{\infty} dZ_a \exp[-\omega Z_a^2] \right]$$
$$\left[\prod_{a}^{1,2,3,8} \int_{-\infty}^{\infty} dY_a \ . \exp[-\omega Y_a^2] \right] \int_0^{\infty} dY_4 Y_4 \exp[-\omega Y_4^2] \int_0^{\infty} dY_6 Y_6 \exp[-\omega Y_4^2]$$

Gram-Schmidt orthogonalisation (T lower-triangular) $\rightarrow$ orthog. polynomials

$$M_n^{(J)PC}[A] \quad \rightarrow \quad P_n^{(J)PC}[A] := \sum_{k=1}^{n} T_{nk} M_k^{(J)PC}[A] \qquad T\, G^{(J)PC} T^T = 1$$

ONB of sol.s of corresp. harm. oscill. : $\Phi_n^{(J)PC}[A,\omega] \equiv P_n^{(J)PC}[A] \exp[-\frac{1}{2}\omega\,(A_{ai})^2]$
Calculate finally magnetic matrix elements:

$$\langle \Phi_m^{\dagger}[A,\omega] \left(B_{ai}^2[A]\right) \Phi_n[A,\omega] \rangle_A^{(J)PC} = T_{mr} T_{ns} \left(\langle\langle M_r^{(J)PC} B^2[A] \ M_s^{(J)PC} \rangle\rangle_A \right)$$

Results: Low-energy spectrum of SU(3) YM QM

0^{PC}	0^{++} $(0,2,4,6,8,10)$	0^{-+} $(3,5,7,9,11)$	0^{+-} (10)	0^{--} $(9,11)$
$aE_1[g^{2/3}]$	12.589(12.591) [12.589]	17.74(17.76) [17.8]	28.55 [28]	25.20(25.56)
$aE_2[g^{2/3}]$	15.39(15.45) [15.38]	19.99(20.05)	29.80	26.76(27.44)
$aE_3[g^{2/3}]$	17.24(17.34) [17.23]	21.40(21.66)	−−	27.68(28.35)
dim basis	$1+1+3+8+17+39=69$	$1+2+6+17+40=66$	2	$3+9=12$

1^{PC}	1^{--} $(3,5,7,9)$	1^{+-} $(4,6,8,10)$	1^{-+} $(7,9)$	1^{++} $(6,8,10)$
$aE_1[g^{2/3}]$	16.58(16.74) [17.05]	18.77(18.82) [18]	23.45(23.70) [23]	21.52(21.93)
$aE_2[g^{2/3}]$	18.62(19.18)	21.33(21.84)	24.15(24.53)	22.95(23.64)
$aE_3[g^{2/3}]$	18.92(19.47)	22.14(22.55)	26.51	24.16(24.73)
dim basis	$1+4+12+35=52$	$1+3+13+39=56$	$2+9=11$	$1+7+26=34$

2^{PC}	2^{++} $(2,4,6,8,10)$	2^{-+} $(5,7,9)$	2^{+-} $(6,8,10)$	2^{--} $(5,7,9)$
$aE_1[g^{2/3}]$	14.81(14.86) [14.85]	19.95(20.30) [21]	21.69(22.15) [22.1]	18.53(19.04)
$aE_2[g^{2/3}]$	17.18(17.29) [17.26]	20.87(21.22)	24.36(24.72)	20.14(20.51)
$aE_3[g^{2/3}]$	17.60(17.83)	22.78(23.22)	24.55(25.15)	21.50(22.19)
dim basis	$1+3+9+29+80=122$	$2+8+27=37$	$1+7+33=41$	$2+8+29=39$

3^{PC}	3^{--} $(3,5,7,9)$	3^{+-} $(6,8,10)$	3^{-+} $(7,9)$	3^{++} $(6,8,10)$
$aE_1[g^{2/3}]$	16.09(16.15) [16.5]	21.20(21.68)	23.38(23.71)	19.56(20.14)
$aE_2[g^{2/3}]$	18.73(19.39)	21.79(22.13)	24.03(24.29)	21.36(21.76)
$aE_3[g^{2/3}]$	19.13(19.71)	24.10(24.98)	25.99	22.04(22.46)
dim basis	$1+3+13+44=61$	$2+10+45=57$	$2+13=15$	$3+12+46=61$

The values obtained by Weisz and Ziemann (1986) are shown in []-brackets.

Energy-eigenvalues as a function of polynomial order of truncation

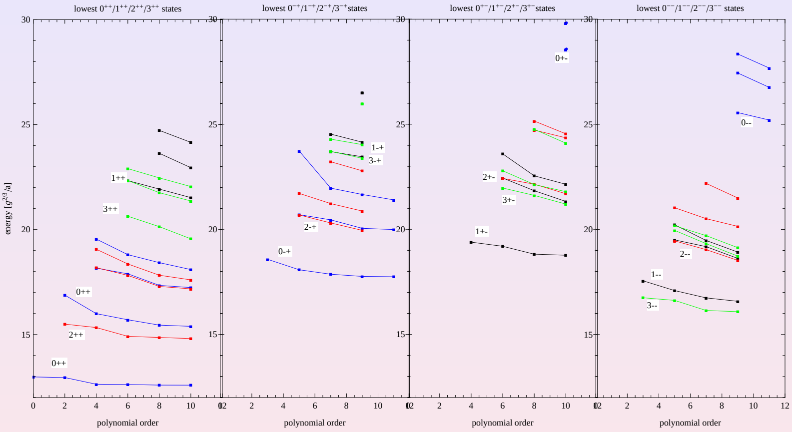
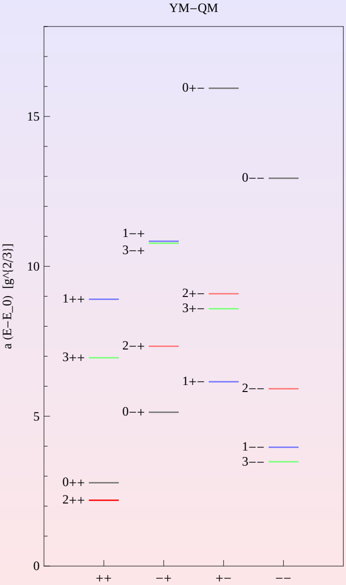
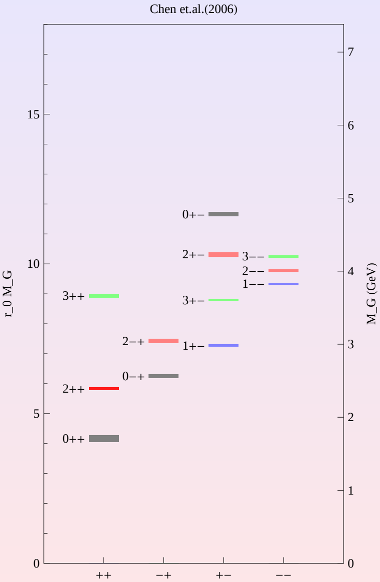


Figure: Energy-eigenvalues as a function of polynomial order of truncation. The blue/black/red/green lines correspond to spin-0/spin-1/spin-2/spin-3

energy-spectrum of YM-QM vs glueball-spectrum of lattice YM



$a > 1$ fm



$a \sim 0.2$ fm $(r_0 \sim 0.5$ fm$)$

Conclusions

- An unconstrained Hamiltonian formulation of SU(3) Yang-Mills QM (lowest order in an strong coupling expansion of YM theory) is possible with a simple, analytically invertible, but non-trivial Faddeev-Popov operator.
- The spectrum of the Hamiltonian of SU(3) Yang-Mills QM of spatially constant fields can be determined in an effective way using the exact solutions of the corresponding harmonic oscillator problem. The results are in good agreement with the results of Weisz and Zieman (1986) using the constrained Hamiltonian approach in the 0^{++} and 2^{++} sectors, much more accurate values in other sectors considered by them, e.g. in 1^{--} and 3^{--} sectors , and give quite accurate "new results" for the states not considered by them, as e.g. 2^{--}, 3^{++} .
- In order to get convergent results polynomials in orders of at least 10 are necessary. Hence we need powerful computers and very effective Computer algorithms to cope with very large numbers of terms.
- A quite accurate knowledge of the eigensystem of SU(3) Yang-Mills QM is a good basis for strong coupling pert.theory in small $\lambda = g^{-2/3}$ analogous to the SU(2) approach (H.-P. P., Phys. Lett. B **685** (2010) 353-364.)
- The calculation can straightforwardly be generalised to the inclusion of quarks analogous to the case of SU(2) Dirac-Yang-Mills QM (H.-P. P., Phys. Lett. B **700** (2011) 265-276.) $\longrightarrow$ meson spectrum ?
- The very effective formulation in terms of the 35 components of 7 spatial symmetric tensors might give some useful information about the decay channels of glueballs, important for their experimental detection.