Rosochatius/Smorodinsky-Winternitz system on complex projective space

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We construct the superintegrable generalization of Rosochatius/Smorodinsky-Winternitz system on complex projective space interacting with constant magnetic field. The model belongs to the class of the so-called "Kähler oscillators" and admits "weak $\mathcal{N}=4$" (or $su(2|1)$) supersymmetric extension.

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Superintegrable Smorodinsky-Winternitz System

- The N-dimensional oscillator harmonic oscillator is maximally superintegrable system, with $su(N)$ symmetry algebra.
- Its deformation with potential $\sum_i \frac{g_i^2}{x_i^2}$ preserves superintegrability property but yield the highly nonlinear algebra. This singular oscillator sometimes called Smorodinsky-Winternitz system.
- Smorodinsky-Winternitz system possesses superintegrable generalization to (pseudo)sphere (e.g. Groshe, Pogosyan, Sissakian'1995; Harland Yermolayeva'2004; Galajinsky,A.N., Saghatelian'2013).
  It was suggested by Rosochatius in 1877 (without noticing superintegrability)

Is it possible to construct the superintegrable counterparts of Smorodinsky-Winternitz system on complex projective spaces?

YES!

$\mathbb{CP}^N$-**oscillator** (Bellucci, A.N.'2003)

$$\mathcal{H} = g^{a\bar{b}}\pi_a\bar{\pi}_b + \omega^2 z\bar{z}, \qquad \Omega_0 = dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a + \imath B g_{a\bar{b}} dz^a \wedge d\bar{z}^b,$$

Symmetry generators

$$J_{a\bar{b}} = \imath(z^b\pi_a - \bar{\pi}_b\bar{z}^a) - B\frac{\bar{z}^a z^b}{1+z\bar{z}}, \qquad I_{a\bar{b}} = \frac{J_a\bar{J}_b}{r_0^2} + \omega^2\bar{z}^a z^b \ .$$

with $J_a = \pi_a + \bar{z}^a(\bar{z}\bar{\pi}) + \imath B\frac{\bar{z}^a}{1+z\bar{z}}$.

- Symmetry generators form quadratic algebra
- It admits "weak $\mathcal{N} = 4$ " supersymmetric extension
- It is not covariant under transition from one chart to other

$\mathbb{CP}^N$-oscillator potential in homogeneous coordinates

$$V_{osc} = \omega^2 z\bar{z} = \frac{\omega_0^2}{u_0\bar{u}_0} - \omega_0^2,$$

with $z^a = u_a/u_0,\ u_0\bar{u}_0 + \sum_{a=1}^N u_0\bar{u}_0 = 1.$

"Forminvariantisation"

$$V_{SWR} = \sum_{i=0}^N \frac{\omega_i^2}{u_i\bar{u}_i} - \omega_i^2 = (1 + z\bar{z})(\omega_0^2 + \sum_{a=1}^N \frac{\omega_a^2}{z^a\bar{z}^a}) - \sum_{i=0}^N \omega_i^2$$

- In flat limit it results in $\mathbb{C}^N$-Smorodinsky-Winternitz system
- Can be viewed as the analog of Rosochatius system

$\mathbb{CP}^N$-Rosochatius/Smorodinsky-Winternitz model

$\mathbb{C}^N$-**Smorodinsky-Winternitz system**(Shmavonian' 2018)

$$\mathcal{H}_{SW} = \pi\bar{\pi} + \omega_0^2 z\bar{z} + \sum_{a=1}^{N} \frac{g_a^2}{z^a \bar{z}^a},$$

$$\Omega = dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a + \imath B dz^a \wedge d\bar{z}^a.$$

Symmetry generators

$$J_{a\bar{a}} = \imath\pi_a z^b - \imath\bar{\pi}_b \bar{z}^a - B z^a \bar{z}^b, \quad I_a = \pi_a \bar{\pi}_{\bar{a}} + \omega_0^2 z^a \bar{z}^a + \frac{\omega_a^2}{\bar{z}^a z^a},$$

$$I_{ab} = J_{a\bar{b}} J_{b\bar{a}} + \left(\omega_a^2 \frac{z^b \bar{z}^b}{z^a \bar{z}^a} + \omega_b^2 \frac{z^a \bar{z}^a}{z^b \bar{z}^b}\right)$$

where $J_{a\bar{b}} = \imath(z^b \pi_a - \bar{\pi}_b \bar{z}^a) - B\bar{z}^a z^b$

$\mathbb{CP}^N$-Rosochatius/Smorodinsky-Winternitz model

$$\mathcal{H}_{SWR} = g^{\bar{a}b}\bar{\pi}_a\pi_b + (1+z\bar{z})(\omega_0^2 + \sum_{a=1}^{N}\frac{\omega_a^2}{z^a\bar{z}^a}) - \sum_{i=0}^{N}\omega_i^2$$

$$\Omega = dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a + \imath B g_{a\bar{b}} dz^a \wedge d\bar{z}^b.$$

Constants of Motion

$$J_{a\bar{a}} = \imath\pi_a z^a - \imath\bar{\pi}_a\bar{z}^a - B\frac{z^a\bar{z}^a}{1+z\bar{z}},$$

$$I_{0a} = J_{0a}\bar{J}_{0\bar{a}} + \omega_0^2 z^a\bar{z}^a + \frac{\omega_a^2}{\bar{z}^a z^a}, \quad I_{ab} = J_{a\bar{b}}J_{b\bar{a}} + \omega_a^2\frac{z^b\bar{z}^b}{z^a\bar{z}^a} + \omega_b^2\frac{z^a\bar{z}^a}{z^b\bar{z}^b}$$

where

$J_{a\bar{b}} = i(z^b\pi_a - \bar{\pi}_b\bar{z}^a) - B\frac{\bar{z}^a z^b}{1+z\bar{z}}$, $J_{0a} = \pi_a + \bar{z}^a(\bar{z}\bar{\pi}) + \imath B\frac{\bar{z}^a}{1+z\bar{z}}$

are $su(N+1)$ generators

Symmetry algebra of $\mathbb{CP}^N$-R/SW model

$$\{J_{a\bar a}, I_{ij}\} = 0, \qquad \{I_{ij}, I_{kl}\} = \delta_{jk} T_{ijl} + \delta_{ik} T_{jkl} - \delta_{jl} T_{ikl} - \delta_{il} T_{ijk},$$

where

$$T_{ijk}^2 = 2(I_{ij} - J_{i\bar i}J_{j\bar j})(I_{jk} - J_{j\bar j}J_{k\bar k})(I_{ik} - J_{i\bar i}J_{k\bar k}) +$$

$$+2I_{ij}I_{ik}I_{jk} + J^2_{i\bar i}J^2_{j\bar j}J^2_{k\bar k} - (I^2_{jk}J^2_{i\bar i} + I^2_{ij}J^2_{k\bar k} + I^2_{ik}J^2_{j\bar j}) - 4(\omega_k^2 I_{ij}(I_{ij} - J_{i\bar i}J_{j\bar j}) +$$

$$+\omega_i^2 I_{jk}(I_{jk} - J_{j\bar j}J_{k\bar k}) + \omega_j^2 I_{ik}(I_{ik} - J_{i\bar i}J_{k\bar k})) + 4\omega_j^2\omega_k^2 J^2_{i\bar i} + 4\omega_i^2\omega_k^2 J^2_{j\bar j} +$$

$$+4\omega_i^2\omega_j^2 J^2_{k\bar k} + 16\omega_i^2\omega_j^2\omega_k^2.$$

with $I_{ij} = (I_{0a}, I_{ab}),\ i, j, k, l = 0, \ldots, N.$

Reduction to (spherical) Rosochatius system

Coordinate transformation

$$z^a = y_a e^{\imath\varphi_a}, \quad \pi_a = \frac{1}{2}\left(p_a - \imath(\frac{p_{\varphi_a}}{y_a} + \frac{By_a}{1+y^2})\right) e^{-\imath\varphi_a}$$

Hamiltonian and symplectic structure

$$\mathcal{H} = \frac{1}{4}(1+y^2)\left[\sum_{a,b=1}^{N}(\delta_{ab} + y_a y_b)p_a p_b + \widetilde{\omega}_0^2 + \sum_{a=1}^{N}\frac{\widetilde{\omega}_a^2}{y_a^2}\right] - E_0$$

$$\Omega = dp_a \wedge dy_a + dp_{\varphi_a} \wedge d\varphi_a$$

with

$$\widetilde{\omega}_a^2 = 4\omega_a^2 + p_{\varphi_a}^2, \quad \widetilde{\omega}_0^2 = 4\omega_0^2 + \left(B + \sum_a p_{\varphi_a}\right)^2, \quad E_0 = \frac{B^2}{4} + \sum_{i=0}^{N}\omega_i^2.$$

Reducing by p_{φ}^a, we arrive the (spherical) Rosochatius systemwith $y_a = x_a/x_0$, with (x_0, x_a) be Cartesian coordinates of $\mathbb{R}^{N+1}$, $sum_{i=0}^N x_i^2 = 1$

Supersymmetrization

The Hamiltonian of $\mathbb{CP}^N$-SW/R model can be represented in the form of "Kähler oscillator" (shifted by constant),

$$\mathcal{H} = g^{a\bar{b}}\left(\pi_a\bar{\pi}_b + \omega^2\partial_a K\partial_{\bar{a}}K\right) - E_0, \quad E_0 \equiv |\sum_{i=0}^{N}\omega_i|^2 - \sum_{i=0}^{N}|\omega_i|^2$$

with

$$K = \log(1 + z\bar{z}) - \frac{1}{\omega}\sum_{a=1}^{N}(\omega_a \log z^a + \bar{\omega}_a \log \bar{z}^a), \quad \omega = |\sum_{i=0}^{N}\omega_i|.$$

- $\omega = 0, B = 0$: Admits $\mathcal{N} = 4$ supersymmetric extension.
- $\omega \neq 0$: Admits "Weak $\mathcal{N} = 4$" (or $SU(2|1)$) supersymmetric extension

Weak $\mathcal{N}=4$ supesymmetrization of Kähler oscillator

Supersymplectic Structure

$$\Omega = d\pi_a \wedge dz^a + d\bar\pi_a \wedge d\bar z^a + i(Bg_{a\bar b} + iR_{a\bar b c\bar d}\eta^c_\alpha\bar\eta^d_\alpha)dz^a\wedge d\bar z^b + g_{a\bar b}D\eta^a_\alpha \wedge D\bar\eta^b_\alpha$$

where $D\eta^a_\alpha = d\eta^a_\alpha + \Gamma^a_{bc}\eta^b_\alpha dz^c, \alpha = 1,2,$

Supercharges

$$\Theta^+_\alpha = \pi_a\eta^a_\alpha + i\omega\varepsilon_{\alpha\beta}\bar\partial_a K\bar\eta^a_\beta, \quad \Theta^-_\alpha = \bar\pi_a\bar\eta^a_\alpha - i\omega\varepsilon_{\alpha\beta}\partial_a K\eta^a_\beta$$

Hamiltonian and R-charges

$$\mathcal{H}_{SUSY} = g^{\bar a b}(\pi_a\bar\pi_b + \omega^2\partial_a K\partial_b K) - R_{a\bar b c\bar d}\eta^a_1\bar\eta^b_1\eta^c_2\bar\eta^d_2$$

$$-\imath\omega K_{a;b}\eta^a_1\eta^b_2 + \imath\omega K_{\bar a;\bar b}\bar\eta^a_1\bar\eta^b_2 + \frac{B}{2}\imath g_{a\bar b}\eta^a_\alpha\bar\eta^b_\alpha$$

$$\mathcal{R}_i = \imath g_{a\bar b}\eta^a_\alpha\sigma_i^{\alpha\bar\beta}\bar\eta^b_\beta$$

Weak $\mathcal{N}=4$ supersymmetry algebra

$$\{\Theta_\alpha, \overline{\Theta}_\beta\} = \delta_{\alpha\beta}\mathcal{H}_{SUSY} + B\sigma^i_{\alpha\beta}\mathcal{R}_i$$
$$\{\Theta^\pm_1, \mathcal{H}_{SUSY}\} = i\left[(B/2)\,\Theta^\pm_1 - |\omega|\Theta^\pm_2\right],$$
$$\{\mathcal{R}_\pm, \mathcal{R}_3\} = \mp 2\imath\mathcal{R}_\pm, \quad \{\mathcal{R}_+, \mathcal{R}_-\} = \imath\mathcal{R}_3$$
$$\{\Theta^\pm_\alpha, \mathcal{R}_\pm\} = 0, \quad \{\Theta^\pm_\alpha, \mathcal{R}_\mp\} = \pm\imath\epsilon_{\alpha\beta}\Theta^\mp_\beta, \quad \{\Theta^\pm_\alpha, \mathcal{R}_3\} = \pm\imath\Theta^\pm_\alpha,$$
$$\{\mathcal{R}_\pm, \mathcal{H}_{SUSY}\} = 0, \quad \{\mathcal{R}_3, \mathcal{H}_{SUSY}\} = 0,$$

Kinematical symmetries

$$\mathcal{J}_\mu = J_\mu - \imath\frac{\partial^2 h_\mu}{\partial z^c \partial \bar{z}^d}\eta^c_\alpha\bar{\eta}^d_\alpha, \quad \{\mathcal{J}_\mu, \mathcal{H}_{SUSY}\} = \{\mathcal{J}_\mu, \Theta^\pm_\alpha\} = \{\mathcal{J}_\mu, R_i\} = 0$$

with $J_\mu = (J_{a\bar{b}}, J_a, \bar{J}_a)$ be initial $su(N+1)$ symmetry generators and h_μ respective Killing potentials

$$h_{a\bar{b}} = \frac{\bar{z}^a z^b}{1+z\bar{z}}, \qquad h_a = \frac{\bar{z}^a}{1+z\bar{z}}.$$

“Supersymmetric” remarks

- ‘Weak $\mathcal{N}=4$” $\mathbb{CP}^N$-RSW model inherits kinematical $SU(N)$ symmetries. What about hidden symmetries?
- $|\omega|=0, B=0$: $\mathcal{N}=4$ supersymmetry

$$|\omega|=0 \quad \Rightarrow \quad |\omega_0| \leq \sum_{a=1}^{N} |\omega_a|, \quad \text{with} \quad |\omega_0| \leq |\omega_1| \leq \ldots \leq |\omega_N|$$

  CP^N-oscillator does not belong to this set of models.
- $\omega_i=0, B\neq 0$: $SU(N+1)$-symmetric “weak $\mathcal{N}=4$” super-$\mathbb{CP}^N$-Landau problem.
- All these statements remain correct for $\mathbb{C}^N$-Smorodinsky-Winternitz.

Concluding remarks

- Quantum mechanics of $\mathbb{CP}^N$-RSW model? In progress
- We are sure that $\mathbb{HP}^N$-RSW model defined by $\mathbb{CP}^N$-RSW Hamiltonian with z^a be quaternionic oordinates is superintegrable system admitting interaction with BPST instanton field
- Quantum mechanics of "weak $\mathcal{N} = 4$" super-$\mathbb{CP}^N$-RSW will hopefully be done in collaboration with E.Ivanov and S.Sidorov.
- Could hidden symmetries of $\mathbb{CP}^N$-RSW model be extended to "weak $\mathcal{N} = 4$" super $\mathbb{CP}^N$-RSW model? One should clarified in "weak $\mathcal{N} = 4$"$\mathbb{C}^N$-Smorodinsky-Winternitz at first.
- Is it possible to construct the superintegrable $\mathbb{CP}^N$-Calogero model?

Thank you for your attention