EXTENDED GALILEAN SUSY WITH ALL CENTRAL CHARGES AND N=4 GALILEAN SUPERPARTICLES

1. From general D=4 N-extended Poincaré to d=3 N-extended Galilean superalgebras (D=d+1)
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3. d=3 N=4 Galilean superparticle models: action, phase space formulation and first quantization
4. Final remarks

based on

– J.L. "D=4 extended Galilei superalgebras with central charges", Phys. Lett. B694(2011), 478; arXiv:1009.0182

– S. Fedoruk, E. Ivanov, J.L., "From N=4 Galilean superparticle to three-dimensional nonrelativistic N=4 superfields", JHEP05 (2018)019; arXiv:1803.03159

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D = 4 extended Galilei superalgebras with central charges

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abstract

We perform a nonrelativistic contraction of N-extended Poincaré superalgebra with internal symmetry U(N) and general set of $N(N-1)$ real central charges. We show that for even $N = 2k$ and particular choice of the dependence of Z_{ij} on light velocity c one gets the N-extended Galilei superalgebra with unchanged number of central charges and compact internal symmetry algebra $U(k; H) = USp(2k)$. The Hamiltonian positivity condition is modified only by one central charge. If we put all the central charges equal to zero one gets the 2k-extended Galilei superalgebra as the subalgebra of recently introduced extended Galilei conformal superalgebra (de Azcárraga, Lukierski (2009) [1] and Sakaguchi [2]).

1. Introduction

contraction $c \to \infty$ [7] if we perform the following c-dependent rescaling (P_i, M_i remain unchanged)

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From $\mathcal{N} = 4$ Galilean superparticle to three-dimensional non-relativistic $\mathcal{N} = 4$ superfields

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Abstract: We consider the general $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra with arbitrary central charges and study its dynamical realizations. Using the nonlinear realization techniques, we introduce a class of actions for $\mathcal{N} = 4$ three-dimensional non-relativistic superparticle, such that they are linear in the central charge Maurer-Cartan one-forms. As a prerequisite to the quantization, we analyze the phase space constraints structure of our model for various choices of the central charges. The first class constraints generate gauge transformations, involving fermionic κ-gauge transformations. The quantization of the model gives rise to the collection of free $\mathcal{N} = 4$, $d = 3$ Galilean superfields, which can be further employed, e.g., for description of three-dimensional non-relativistic $\mathcal{N} = 4$ supersymmetric theories.

Keywords: Extended Supersymmetry, Space-Time Symmetries, Superspaces

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1. From general D=4 N-extended Poincaré to d=3 N-extended Galilean superalgebras

N=extended Poincaré superalgebra with central charges generators

$$\underbrace{P_\mu, M_{\mu\nu}}_{\text{Poincaré}}, \underbrace{T_A{}^B, A}_{\text{U(N)}}, \underbrace{Z^{AB}, \bar{Z}^{AB}}_{\text{central charges}}, \underbrace{Q^A_\alpha, \bar{Q}^A_{\dot{\alpha}}}_{\text{supercharges}} :$$

$$\begin{aligned}
\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \, \delta^A_B \\
\left\{Q^A_\alpha, Q^B_\beta\right\} &= 2\varepsilon_{\alpha\beta} Z^{AB} = 2\varepsilon_{\alpha\beta}\left(X^{AB} + iY^{AB}\right) \\
\left\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\right\} &= 2\varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{AB} = 2\varepsilon_{\dot{\alpha}\dot{\beta}}\left(X^{AB} - iY^{AB}\right)
\end{aligned}$$

$\frac{N(N-1)}{2}$ complex $Z^{AB} \leftrightarrow N(N-1)$ real central charges (X^{AB}, Y^{AB}).

Covariance relations:

$$\left[M_{\mu\nu}, Q_\alpha{}^A\right] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta \qquad \left[P_\mu, Q^A_\alpha\right] = 0$$

$$\left[T^A{}_B, Q_\alpha{}^A\right] = \delta^C_B Q^A_\alpha - \tfrac{1}{N}\delta^A_B Q^C_\alpha \qquad \left[A, Q^A_\alpha\right] = \alpha Q^A_\alpha$$

$$\left(\alpha = 1 \;\rightarrow\; A = U(1)\right)$$

plus complex – conjugate relations for $\bar{Q}_{\dot{\alpha}A}$.

D=4 Poincaré superalgebra → $d = 3$ Galilean superalgebra (N=2k even)

i) Bosonic space-time sector (N=0)

$$M_{\mu\nu} = (M_i, N_i) \qquad P_\mu = (P_i, P_0) \quad \xrightarrow[\substack{N_i = cB_i \\ P_0 = m_0 c + \frac{H}{c}}]{c \to \infty}$$

d=3 Galilean algebra

$$[M_i, B_j] = \varepsilon_{ijk} B_k$$
$$[H, B_i] = -P_i$$
$$[P_i, B_j] = -M\,\delta_{ij}$$

$M = m_0$ – Bargmann central charge

ii) Internal sector $U(N) = SU(N) \oplus A$ (R-symmetries)

One splits generators of $SU(N)$ into symmetric Riemannian pair

$$h = T^{+A}_{\;\;B} \in USp(4) \qquad k = T^{-A}_{\;\;B} \in \frac{SU(N)}{USp(4)}$$

$$T^{\pm A}_{\;\;B} = \mp\Omega^{AC} T^{\pm D}_{\;\;C} \Omega_{DB} \longrightarrow \begin{matrix} \Omega^{AB} = -\Omega^{BA} \\ \Omega^{AC}\Omega_{CB} = \delta^A_{\;B} \end{matrix} \Rightarrow \text{symplectic metric}$$

NR contraction $c \to \infty$ after rescaling of internal sector

$$T^{+A}_{\;\;B} = \mathbb{T}^{+A}_{\;\;B} \qquad T^{-A}_{\;\;B} = c\,\mathbb{T}^{-A}_{\;\;B} \qquad A = c\mathbb{A}$$

$$[h', h] \subset h \quad [h, k] \subset k \quad [k, k] \subset h \quad \xrightarrow{c \to \infty} \quad \begin{matrix} [h, h] \subset h \\ [h, \mathbb{k}] \subset \mathbb{k} \\ [\mathbb{k}, \mathbb{k}] = 0 \end{matrix}$$

One gets inhomogeneous $USp(N)$ as maximal Galilean R-symmetry

iii) Fermionic sector and central charges

We introduce new symplectic–covariant basis of supercharges

$$Q^{\pm A}_{\alpha} = \frac{1}{\sqrt{2}}\left(Q^{A}_{\alpha} \pm \varepsilon_{\alpha\beta}\Omega^{AB}\bar{Q}_{\dot{\beta}B}\right) \qquad \bar{Q}^{\pm}_{\dot{\alpha}A} = \frac{1}{\sqrt{2}}\left(\bar{Q}_{\dot{\alpha}A} \mp \varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}Q^{B}_{\beta}\right)$$

Such supercharges satisfy symplectic ($USp(N)$) Majorana condition

$$(Q^{\pm A}_{\alpha})^{+} \equiv \bar{Q}^{\pm}_{\dot{\alpha}A} = \mp\varepsilon_{\dot{\alpha}\dot{\beta}}\Omega_{AB}Q^{\pm B}_{\beta} \qquad\qquad A, B = 1\ldots N = 2k$$

Due to this constraint it is enough to consider only holomorphic sector described by $Q^{\pm A}_{\alpha}$ (or antiholomorphic $\bar{Q}^{\pm A}_{\dot{\alpha}}$) ← 2N unconstrained

N-extended D=4 Poincaré superalgebra in holomorphic basis

$$\begin{aligned}
\left\{Q^{\pm A}_{\alpha}, Q^{\pm B}_{\beta}\right\} &= \pm 2\Omega^{AB}\varepsilon_{\alpha\beta}P_0 + \varepsilon_{\alpha\beta}(Z^{AB} - \Omega^{AC}\bar{Z}_{CD}\Omega^{DB}) \\
&= \pm 2\Omega^{AB}\varepsilon_{\alpha\beta}P_0 + 2\varepsilon_{\alpha\beta}(X_{-}^{AB} + iY_{+}^{AB})
\end{aligned}$$

$$\begin{aligned}
\left\{Q^{+A}_{\alpha}, Q^{-B}_{\beta}\right\} &= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + \varepsilon_{\alpha\beta}(Z^{AB} + \Omega^{AC}\bar{Z}_{CD}\Omega^{DB}) \\
&= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(X_{+}^{AB} + iY_{-}^{AB})
\end{aligned}$$

where

$$\Omega^{AC}X_{\pm}^{CD}\Omega^{DB} = \pm X_{\pm}^{AB} \qquad\qquad \Omega^{AC}Y_{\pm}^{CD}\Omega^{DB} = \pm Y_{\pm}^{AB}$$

Rescalings of supercharges:

$$Q_\alpha^{+A} = c^{-\frac{1}{2}}\mathbb{Q}_\alpha^A \qquad Q_\alpha^{-A} = c^{\frac{1}{2}}\mathbb{S}_\alpha^A \qquad \text{(scaled differently!)}$$

Rescalings of central charges $Z^{AB} = X^{AB} + iY^{AB}$ $(X^{AB} = \tilde{X}^{AB} + \Omega^{AB}X)$

$$X_-^{AB} = -m_0 c\Omega^{AB} + \frac{1}{c}\mathbb{X}_-^{AB} \qquad Y_+^{AB} = \frac{1}{c}\mathbb{Y}_+^{AB} \qquad \leftarrow \quad \text{physical meaning of } \Omega$$

$$X_+^{AB} = \mathbb{X}_+^{AB} \qquad Y_-^{AB} = \mathbb{Y}_-^{AB} \qquad \leftarrow \quad \text{not rescaled!}$$

where $\Omega^{AC}X_\pm^{CD}\Omega^{DB} = \pm X_\pm^{AB}$ $\quad \Omega^{AC}Y_\pm^{CD}\Omega^{DB} = \pm Y_\pm^{AB}$

Contraction limits in holomorphic basis: d=3 Galilean superalgebra

$$\begin{aligned}
\left\{\mathbb{Q}_\alpha^A, \mathbb{Q}_\beta^B\right\} &= \lim_{c\to\infty}[c\cdot\{2\Omega^{AB}\varepsilon_{\alpha\beta}(P_0 + X)\} + 2\varepsilon_{\alpha\beta}(\tilde{X}_-^{AB} + iY_+^{AB})] \\
&= 2\Omega^{AB}\varepsilon_{\alpha\beta}(H + \mathbb{X}) + 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}_-^{AB} + i\mathbb{Y}_+^{AB}) \qquad X = -m_0 c + \frac{\mathbb{X}}{c} \\
\left\{\mathbb{Q}_\alpha^A, \mathbb{S}_\beta^B\right\} &= 2\Omega^{AB}(\sigma_i P_i)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(\mathbb{X}_+^{AB} + i\mathbb{Y}_-^{AB}) \qquad P_0 = m_0 c + \frac{H}{c} \\
\left\{\mathbb{S}_\alpha^A, \mathbb{S}_\beta^B\right\} &= \lim_{c\to\infty}[\frac{1}{c}2\Omega^{AB}\varepsilon_{\alpha\beta}(-P_0 + X)] + 2\varepsilon_{\alpha\beta}(\tilde{X}_-^{AB} + iY_+^{AB}) = 4m_0\varepsilon_{\alpha\beta}\Omega^{AB}
\end{aligned}$$

Rescalings: $\tilde{X}_-^{AB} = \frac{1}{c}\tilde{\mathbb{X}}_-^{AB}$ $\quad Y_+^{AB} = \frac{1}{c}\mathbb{Y}_+^{AB}$; $\quad$ not rescaled: $\quad X_+^{AB} = \mathbb{X}_+^{AB}$ $\quad Y_+^{AB} = \mathbb{Y}_+^{AB}$

If $c \to \infty$ one obtains identical in form Galilean symplectic Majorana condition (A,B=1,2…N; N=2k) for $\mathbb{Q}$ and $\mathbb{S}$ supercharges

$$\bar{\mathbb{Q}}_{\dot\alpha A} = -\varepsilon_{\dot\alpha\dot\beta}\Omega_{AB}\mathbb{Q}^B_\beta \qquad \bar{\mathbb{S}}_{\dot\alpha A} = \varepsilon_{\dot\alpha\dot\beta}\Omega_{AB}\mathbb{S}^B_\beta$$

Second algebraic option: resolving Galilean symplectic Majorana condition by choosing as independent supercharges Hermitean–conjugate

$$(\mathbb{Q}^A_\alpha, \mathbb{S}^A_\alpha) \;\to\; (\mathbb{Q}^i_\alpha, \bar{\mathbb{Q}}^i_{\dot\alpha i}), (\mathbb{S}^i_\alpha, \bar{\mathbb{S}}^i_{\dot\alpha i}) \qquad i = 1,2,\ldots\tfrac{N}{2} = k \qquad \left(\text{for } \Omega = \begin{pmatrix} 0 & -\mathbb{1}_k \\ \mathbb{1}_k & 0 \end{pmatrix}\right)$$

One passes from holomorphic superalgebra to Hermitean superalgebra:

$$\left\{\mathbb{Q}^i_\alpha, \bar{\mathbb{Q}}^j_\beta\right\} = 2\delta^{ij}(H + \mathbb{X}) + 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}_-^{i\,k+j} + i\mathbb{Y}_+^{i\,k+j})$$

$$\left\{\mathbb{Q}^i_\alpha, \mathbb{Q}^j_\beta\right\} = 2\varepsilon_{\alpha\beta}(\tilde{\mathbb{X}}_-^{ij} + i\mathbb{Y}_+^{ij})$$

$$\left\{\mathbb{Q}^i_\alpha, \bar{\mathbb{S}}^j_\beta\right\} = 2\delta^{ij}(\sigma_r P_r)_{\alpha\beta} + 2\varepsilon_{\alpha\beta}(\mathbb{X}_+^{i\,k+j} + i\mathbb{Y}_-^{i\,k+j}) \qquad r = 1,2,3$$

$$\left\{\mathbb{Q}^i_\alpha, \mathbb{S}^j_\beta\right\} = 2\varepsilon_{\alpha\beta}(\mathbb{X}_+^{ij} + i\mathbb{Y}_-^{ij})$$

$$\left\{\mathbb{S}^i_\alpha, \bar{\mathbb{S}}^j_{\dot\beta}\right\} = 4m_0\delta_{\alpha\dot\beta}\delta^{ij} \qquad \left\{\mathbb{S}^i_\alpha, \mathbb{S}^j_\beta\right\} = 0$$

Proper basis for describing N-extended Galilean SUSY in QM!

Remarks:

1. One central charge is distinguished, implying the choice of Ω^{AB}

$$Z^{AB} = Z\Omega^{AB} \qquad Z = X + iY = -m_0 c + \frac{\mathbb{X}}{c} + i\mathbb{Y} - \text{unique for N=2}$$

We denote by $\tilde{X}^{AB}$ the central charges X^{AB} without X.
The role of $-m_0 c$ term in Z : to compensate the term $m_0 c$ from P_0.

2. If only $Z \neq 0$ (one complex central charge), the R-symmetry $SU(N)$ is reduced to $USp(N) \equiv U(\frac{N}{2};\mathbb{H})$ (quaternionic unitary group)

3. For $N = 2\mathbb{k}$ one can introduce k complex quasi-triangular central charges $Z_1 \dots Z_k$

$$Z^{AB} = \left(\begin{array}{cc|c|cc} 0 & Z_1 & 0 & 0 & \\ -Z_1 & 0 & & & \\ \hline 0 & & \ddots & 0 & \\ \hline 0 & & 0 & 0 & Z_k \\ & & & -Z_k & 0 \end{array}\right) \Rightarrow \begin{array}{c}\text{one gets R-symmetry:} \\ \underbrace{USp(2) \otimes \dots \otimes USp(2)}_{\text{k times}}\end{array}$$

For $N \geq 4$ necessary to consider also off-diagonal central charges.

2. Example: d=3 N=4 Galilean SUSY with 12+1 real central charges (cc).

a) d=3 N= 1 Galilean superalgebra with 1cc (Puzalowski 1978)

Hermitean basis

$$\left\{\mathbb{S}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 4m_0\delta_\alpha{}^{\dot\beta} \qquad [\mathbb{B}_i, \mathbb{S}_\alpha] = 0$$

$$[J_i, \mathbb{S}_\alpha] = (\sigma_i)_\alpha{}^\beta \mathbb{S}_\beta \qquad \left[J, \bar{\mathbb{S}}^{\dot\alpha}\right] = -\bar{\mathbb{S}}^{\dot\beta}(\sigma_i)_{\dot\beta}{}^{\dot\alpha}$$

b) d=3 N=2 Galilean superalgebra (Bergman, Thorn 1995) with 3cc

Hermitean basis

$$\left\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}^\beta\right\} = 2\delta_\alpha{}^{\dot\beta}(H + \mathbb{X})$$

$$\left\{\mathbb{Q}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 2(\sigma_i P_i)_\alpha{}^{\dot\beta} + 2i\mathbb{Y}\delta_\alpha{}^{\dot\beta} \qquad i = 1, 2, 3$$

$$\left\{\mathbb{S}_\alpha, \bar{\mathbb{S}}^{\dot\beta}\right\} = 4m_0\delta_\alpha{}^{\dot\beta}$$

$$[B_i, \mathbb{Q}_\alpha] = (\sigma_i)_\alpha{}^\beta \mathbb{S}_\beta \qquad [B_i, \mathbb{S}_\alpha] = 0$$

Hermitean basis ($\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha, \mathbb{S}_\alpha, \bar{\mathbb{S}}_{\dot\beta}$) is related with holomorphic one ($\mathbb{Q}^A_\alpha, \mathbb{S}^A_\alpha$) (A=1,2) by the use of $USp(N)$ subsidiary condition

$$\mathbb{Q}_\alpha = \mathbb{Q}^1_\alpha \qquad \bar{\mathbb{Q}}_{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\mathbb{Q}^2_\beta \qquad \mathbb{S}_\alpha = \mathbb{S}^1_\alpha \qquad \bar{\mathbb{S}}_{\dot\alpha} = \varepsilon_{\alpha\dot\beta}\mathbb{S}^2_\alpha$$

c) d=3 N= 4 Galilean SUSY with 12+1=13 cc

The relativistic N=4 central charges matrix can be introduced as representation of $USp(2) \otimes USp(2) = SU(2) \otimes SU(2) \simeq O(4)$ internal R-symmetries

$$Z^{AB} = \left(\begin{array}{c|c} \mathbb{Z}_1 \varepsilon_{ab} & \mathbb{Z}_{a\tilde{b}} \\ \hline -\mathbb{Z}_{\tilde{a}b} & \mathbb{Z}_2 \varepsilon_{\tilde{a}\tilde{b}} \end{array} \right) \qquad \begin{array}{l} A = (a, \tilde{a}) \\ B = (b, \tilde{b}) \end{array}$$

where $a = (1,2)$ and $\tilde{a} = (\tilde{1}, \tilde{2})$ describe two independent $USp(2) \simeq SU(2)$ spinorial indices. The supercharges are

$$\mathbb{Q}^A_\alpha = (\mathbb{Q}^a_\alpha, \mathbb{Q}^{\tilde{a}}_\alpha)\,, \quad \mathbb{S}^A_\alpha = (\mathbb{S}^a_\alpha, \mathbb{S}^{\tilde{a}}_\alpha) \qquad \longrightarrow$$

holomorphic basis:

Hermitean basis:

$$(\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha, \tilde{\mathbb{Q}}_\alpha, \bar{\tilde{\mathbb{Q}}}_\alpha; \mathbb{S}_\alpha, \bar{\boldsymbol{S}}_\alpha, \tilde{\mathbb{S}}_\alpha, \bar{\tilde{\boldsymbol{S}}}_\alpha)$$

$$\mathbb{Q}_\alpha = \mathbb{Q}^1_\alpha \quad \bar{\mathbb{Q}}_\alpha = -\varepsilon_{\alpha\beta}\mathbb{Q}^{\beta 2}$$

$$\tilde{\mathbb{Q}}_\alpha = \mathbb{Q}^{\tilde{1}}_\alpha \quad \bar{\tilde{\mathbb{Q}}}_\alpha = -\varepsilon_{\alpha\beta}\mathbb{Q}^{\beta \tilde{2}}$$

etc.

i) N=4 NR superalgebra in holomorphic basis

$$\left\{Q^a_\alpha, Q^b_\beta\right\} = 2\varepsilon^{ab}\varepsilon_{\alpha\beta}(H + \mathbb{X}_1)$$

$$\left\{Q^{\tilde{a}}_\alpha, Q^{\tilde{b}}_\beta\right\} = 2\varepsilon^{ab}\varepsilon_{\alpha\beta}(H + \mathbb{X}_2) \qquad \longleftarrow$$

quasidiagonal two central charges

$$\left\{Q^a_\alpha, Q^{\tilde{b}}_\beta\right\} = 2\varepsilon_{\alpha\beta} W^{a\tilde{b}} \quad W^{a\tilde{b}} = \mathbb{X}^{a\tilde{b}}_- + i\mathbb{Y}^{a\tilde{b}}_+ \qquad \longleftarrow$$

off-diagonal four central charges

$$\left\{Q^a_\alpha, S^b_\beta\right\} = 2\varepsilon^{ab}((\sigma_i)_{\alpha\beta}P_i + i\varepsilon_{\alpha\beta}\mathbb{Y}_1)$$

quasi-diagonal two cc

$$\left\{Q^{\tilde a}_\alpha, S^{\tilde b}_\beta\right\} = 2\varepsilon^{\tilde a\tilde b}((\sigma_i)_{\alpha\beta}P_i + i\varepsilon_{\alpha\beta}\mathbb{Y}_2)$$

off-diagonal four cc

$$\left\{Q^a_\alpha, S^{\tilde b}_\beta\right\} = \left\{\mathbb{S}^a_\alpha, Q^{\tilde b}_\beta\right\} = 2i\varepsilon_{\alpha\beta}V^{a\tilde b} \qquad V^{a\tilde b} = \mathbb{X}^{a\tilde b}_+ + i\mathbb{Y}^{a\tilde b}_-$$

$$\left\{S^a_\alpha, S^b_\beta\right\} = -4m_0\varepsilon^{ab}\varepsilon_{\alpha\beta} \qquad \left\{S^{\tilde a}_\alpha, S^{\tilde b}_\beta\right\} = -4m_0\varepsilon^{\tilde a b}\varepsilon_{\alpha\beta} \qquad \leftarrow 13^{\text{th}}\text{ cc}$$

$$\left\{S^a_\alpha, S^{\tilde b}_\beta\right\} = 0$$

ii) Hermitean basis: $(Q_\alpha, \bar Q_\alpha, \tilde Q_\alpha, \bar{\tilde Q}_\alpha)$; $\mathbb{X}_1, \mathbb{X}_2$, $W^{a\bar b}$ − c.c:

$$\begin{pmatrix} Q_\alpha \equiv Q^1_\alpha & \bar Q_\alpha = -\varepsilon_{\alpha\beta}Q^2_\alpha \\ \tilde Q_\alpha = Q^{\tilde 1}_\alpha & \bar{\tilde Q}_\alpha = -\varepsilon_{\alpha\beta}Q^{\tilde 2}_\alpha \end{pmatrix}$$

$(S_\alpha, \tilde S_\alpha, \bar S_\alpha, \bar{\tilde S}_\alpha)$; $Y_1, Y_2, V^{a\tilde b} - c.c$

$$\begin{pmatrix} \tilde S_\alpha = S^1_\alpha & \bar S_\alpha = -\epsilon_{\alpha\beta}S^2_\alpha \\ \tilde S_\alpha = S^{\tilde 1}_\alpha & \tilde S_\alpha = -\epsilon_{\alpha\beta}\tilde S^{\tilde 2}_\alpha \end{pmatrix}$$

General N=2 SUSY QM relation as subsuperalgebra:

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\delta_{\alpha\beta}(H + \mathbb{X}_1)$$

$$\left\{\tilde{Q}_\alpha, \bar{\tilde{Q}}_{\dot\beta}\right\} = 2\delta_{\alpha\beta}(H + \mathbb{X}_2)$$

$$\left\{Q_\alpha, \bar{\tilde{Q}}_\beta\right\} = -2\delta_{\alpha\beta}W^{1\tilde{2}} \qquad \left\{\tilde{Q}_\alpha, \bar{Q}_\alpha\right\} = 2\delta_{\alpha\dot\beta}W^{2\tilde{1}}$$

$$\left\{Q_\alpha, \tilde{Q}_\beta\right\} = 2\varepsilon_{\alpha\beta}W^{1\tilde{1}} \qquad \left\{\bar{Q}_\alpha, \bar{\tilde{Q}}_{\bar\beta}\right\} = 2\varepsilon_{\dot\alpha\dot\beta}W^{2\tilde{2}} \qquad \text{etc.}$$

$$\{Q_\alpha, \bar{S}_{\dot\beta}\} = 2(\sigma_i)_{\alpha\dot\beta}P_i - i\delta_{\alpha\dot\beta}Y_1$$

$$\{\tilde{S}_\alpha, \bar{\tilde{Q}}_{\dot\beta}\} = -2(\sigma_i)_{\alpha\dot\beta}P_i - i\delta_{\alpha\dot\beta}Y_2$$

$$\{Q_\alpha, S_\beta\} = 2i\epsilon_{\alpha\beta}Y^{1\tilde{1}} \qquad \{\bar{Q}_\alpha, \bar{\tilde{S}}_\beta\} = 2i\epsilon_{\alpha\beta}Y^{2\tilde{2}}$$

$$\{Q_\alpha, \bar{S}_{\dot\beta}\} = 2i\delta_{\alpha\dot\beta}Y^{1\tilde{2}} \qquad \{\bar{\tilde{S}}_{\dot\alpha}, Q_\beta\} = 2i\delta_{\alpha\beta}Y^{21}$$

and

$$\{S_\alpha, \bar{S}_{\dot\beta}\} = 4\delta_{\alpha\dot\beta}m_0$$

$$\{\tilde{S}_\alpha, \bar{\tilde{S}}_{\dot\beta}\} = 4\delta_{\alpha\dot\beta}m_0$$

$$\{S_\alpha, \bar{\tilde{S}}_\beta\} = \{\tilde{S}_\alpha, \bar{S}_\beta\} = 0$$

Hermitean form of Galilean superalgebra permits to obtain two generalized positivity conditions for H (any Ψ belongs to Hilbert space.)

$$\langle\Psi|(H + X_r)|\Psi\rangle \ \geq 0 \qquad r = 1, 2$$

General features of d=3 N-extended Galilean SUSY:

i) We derived NR superalgebra which is d=3 NR counterpart of Haag–Łopuszański–Sohnius classification of D=4 Poincaré superalgebras

ii) The central charges $Z^{AB} = X^{AB} + iY^{AB}$ after contraction enter as

$$\{Q,Q\} \simeq \mathbb{H} + \frac{N(N-1)}{2} \quad \text{real central charges} \quad W^{AB} = \mathbb{X}_-^{AB} + i\mathbb{Y}_+^{AB}$$

$$\{Q,S\} \simeq \sigma_i P_i + \frac{N(N-1)}{2} \quad \text{real central charges} \quad V^{AB} = X_+^{AB} + iY_-^{AB}$$

$$\{S,S\} \simeq m_0 \leftarrow \quad \text{additional Bargmann mass central charge}$$

We have k^2 central charges $\mathbb{X}_-^{AB}(Y_-^{AB})$ and $k(k-1)$ of $X_+^{AB}(\mathbb{Y}_+^{AB})$

$(2k^2 + 2k(k-1) = N(N-1) \qquad N = 2k)$

iii) Special feature of N=4 (6 complex c.c.)

We have 2 complex quasi-diagonal central charges Z_1, Z_2 and 4 complex off-diagonal ones described by complex fourvector Z_A $(A = 1, 2, \ldots 4)$

$$X^{a\tilde{b}} = (\sigma_A^E)^{a\tilde{b}} Z_A \qquad \sigma_\mu^E = (\sigma_i, -iI_2) \qquad Z_A = n_A + iv_A$$

where σ_μ^E are D=4 Euclidean σ-matrices and n_A, v_A are two real O(4) internal fourvectors $\Rightarrow$ new type of KK theory?

3. N=4 d=3 Galilean superparticle models: action, phase space formulation, first quantization

One performs the following steps:

i) Maurer-Cartan (MC) one-form $\omega(g_r)$ for the coset G

$$G = \frac{\mathcal{G}}{H}$$

$\mathcal{G}$ – $N=4$ $d=3$ Galilean supergroup with generators $\hat{g}_r$

$H - USp(4) \times O(3) \times A$ – stability group (standard choice)

All central charges are located in G. One gets

$$G^{-1}dG = i\sum_r \omega(g_r)\cdot \hat{g}_r$$

$\omega(g_r)$ – linear representation of stability group H

$$\hat{g}_r = (\underbrace{H, B_i, P_i, M,}_{\text{Galilei algebra}/O(3)}\ \underset{\uparrow\ SU(4)/USp(4)}{\mathbb{T}^{-A}{}_{B}}, \underbrace{\mathbb{X}_1, \mathbb{X}_2, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{X}^{a\tilde{b}}, \mathbb{Y}^{a\tilde{b}}}_{\text{12 central charges; M – 13th cc}}, \underbrace{Q^a_\alpha, Q^{\tilde{a}}_\alpha, S^a_\alpha, S^{\tilde{a}}_\alpha}_{\text{all supercharges}})$$

One gets the model of classical mechanics if all group parameters g_r of coset G are promoted to $d=1$ fields: $g_r \rightarrow g_r(\tau)$; τ – evolution parameter

Important: one can in H-covariant way eliminate some of $d=1$ fields by imposing algebraic inverse Higgs constraints.

The coset element G in our d=3 N=4 NR model can be written as

$$G = G_{(1)}\,G_{(2)}\,G_{(3)}\,G_{(4)}\,G_{(5)}\,G_{(6)} \equiv \hat{G}\,G_{(6)}\,,$$

where explicitly

$$\begin{aligned}
G_{(1)} &= \exp i\{tH + x^i P^i\}\,,\\
G_{(2)} &= \exp i\{\xi^\alpha_a \mathbb{Q}^a_\alpha + \xi^\alpha_{\tilde{a}} \mathbb{Q}^{\tilde{a}}_\alpha\}\,,\\
G_{(3)} &= \exp i\{\theta^\alpha_a \mathbb{S}^a_\alpha + \theta^\alpha_{\tilde{a}} \mathbb{S}^{\tilde{a}}_\alpha\}\,,\\
G_{(4)} &= \exp i\{k^i B^i\}\,,\\
G_{(5)} &= \exp i\{s\mathbb{M} + h_1\mathbb{X}_1 + h_2\mathbb{X}_2 + h_{a\tilde{b}}\mathbb{X}^{a\tilde{b}} + f_1\mathbb{Y}_1 + f_2\mathbb{Y}_2 + f_{a\tilde{b}}\mathbb{Y}^{a\tilde{b}}\}\,,\\
G_{(6)} &= \exp i\{u^b_a \mathbb{T}^{-a}{}_b + u^{\tilde{b}}_{\tilde{a}} \mathbb{T}^{-\tilde{a}}{}_{\tilde{b}} + u^{\tilde{b}}_a \mathbb{T}^{-a}{}_{\tilde{b}}\}.
\end{aligned}$$

The factors $G_{(1)}$, $G_{(4)}$ are parametrized by d=3 Galilei group parameters, $G_{(5)}$ by the central charge parameters dual to central charges, $G_{(6)}$ represents the Abelian 5-dimensional coset IUSp(4)/USp(4) and $G_{(2)}$, $G_{(3)}$ collect parameters of the fermionic (odd) sector.

We can write the MC one-forms in the following way

$$\hat{G}^{-1}d\hat{G} := i\sum_K \hat{\omega}_{(K)}T_{(K)}\,,$$

where $T_{(K)}$ stand for all coset G generators, and $\hat{\omega}_{(K)}$ denote the corresponding MC one-forms:

$$\hat{\omega}^{\alpha}_{(Q)\,a} = d\xi^{\alpha}_{a}\,, \hat{\omega}^{\alpha}_{(Q)\,\tilde{a}} = d\xi^{\alpha}_{\tilde{a}}\,,$$

$$\hat{\omega}^{\alpha}_{(S)\,a} = d\theta^{\alpha}_{a} + \frac{1}{2}k_i(\sigma_i)_{\beta}{}^{\alpha}d\xi^{\beta}_{a}\,,$$

$$\hat{\omega}^{\alpha}_{(S)\,\tilde{a}} = d\theta^{\alpha}_{\tilde{a}} + \frac{1}{2}k_i(\sigma_i)_{\beta}{}^{\alpha}d\xi^{\beta}_{\tilde{a}},$$

$$\hat{\omega}_{(H)} = dt + i\left(\xi^{\alpha}_{a}d\xi^{a}_{\alpha} + \xi^{\alpha}_{\tilde{a}}d\xi^{\tilde{a}}_{\alpha}\right),$$

$$\hat{\omega}_{(B)\,i} = dk_i\,,$$

$$\hat{\omega}_{(P)\,i} = \left[dx_i + 2i(\sigma_i)_{\alpha\beta}\left(\theta^{b\alpha}d\xi^{\beta}_{b} + \theta^{\tilde{b}\alpha}d\xi^{\beta}_{\tilde{b}}\right)\right] + k_i\,\hat{\omega}_{(H)}\,,$$

$$\hat{\omega}_{(M)} = ds + k_i\,\hat{\omega}_{(P)\,i} - \frac{1}{2}k^2\hat{\omega}_{(H)} - 2i\left(\theta^{\alpha}_{a}d\theta^{a}_{\alpha} + \theta^{\alpha}_{\tilde{a}}d\theta^{\tilde{a}}_{\alpha}\right), \qquad \longleftarrow S_0$$

$$\hat{\omega}_{(X)\,1} = dh_1 + i\xi^{\alpha}_{a}d\xi^{a}_{\alpha}\,,$$

$$\hat{\omega}_{(X)\,2} = dh_2 + i\xi^{\alpha}_{\tilde{a}}d\xi^{\tilde{a}}_{\alpha}\,,$$

$$\hat{\omega}_{(X)\,a\tilde{b}} = dh_{a\tilde{b}} + i\left(\xi^{\alpha}_{a}d\xi_{\alpha\tilde{b}} - \xi^{\alpha}_{\tilde{b}}d\xi_{\alpha a}\right),$$

$$\hat{\omega}_{(Y)\,1} = df_1 + 2\theta^{\alpha a}d\xi_{a\alpha}\,,$$

$$\hat{\omega}_{(Y)\,2} = df_2 + 2\theta^{\alpha\tilde{a}}d\xi_{\alpha\tilde{a}}\,,$$

$$\hat{\omega}_{(Y)\,a\tilde{b}} = df_{a\tilde{b}} - 2\left(\theta^{\alpha}_{a}d\xi_{\alpha\tilde{b}} - \theta^{\alpha}_{\tilde{b}}d\xi_{\alpha a}\right),$$

where $k^2 := k_ik_i$. Using as action $S_0 = \int\hat{\omega}_(M)$ one gets d=3 N=4 NR superparticle model.

Simple example: geometric model of Galilean particle with mass m_0:

$$G = \frac{\text{d=3 Galilei group}}{O(3)}$$

8 generators: H, P_i, B_i, M
8 group parameters: t, x_i, k_i, s

MC one-forms:

$$\omega_{(H)} = dt \quad \omega_{(B_i)} = dk_i \quad \omega_{(P_i)} = dx_i + k_i dt \qquad t = t(\tau) \quad x_i = x_i(\tau)$$

$$\omega_{(M)} = ds + k_i \omega_{(P_i)} - \tfrac{1}{2}k^2 dt \qquad k_i = k_i(\tau) \quad s = s(\tau)$$

Inverse Higgs mechanism: $\omega_{(P_i)} = 0 \ \rightarrow \ k_i = -\frac{dx_i}{dt} = -v_i$

Action: $\mathcal{S}_0 = -m_0 \int \omega_{(M)} = -m_0 \int d\tau (\dot{s} - \frac{1}{2}k^2 \dot{t}) \qquad \dot{a} = \frac{da}{d\tau} \qquad k_i = \frac{dx_i}{d\tau} \cdot \frac{d\tau}{dt}$

$$\frac{dx_i}{dt} = \frac{dx_i}{d\tau}\frac{d\tau}{dt} = \frac{\dot{x}_i}{\dot{t}} \qquad = \frac{m_0}{2}\int d\tau \left(\frac{\dot{x}_i}{\dot{t}}\right)^2 \cdot \dot{t} = \frac{m_0}{2}\int d\tau \frac{\dot{x}_i^2}{\dot{t}}$$

τ – evolution parameter

$$\int d\tau \dot{s} = 0 \qquad p_i = \frac{\partial L}{\partial \dot{x}_i} = m_0 \frac{\dot{x}_i}{\dot{t}} \qquad p_0 = \frac{\partial L}{\partial \dot{t}} = -\frac{m_0}{2}\frac{\dot{x}_i^2}{\dot{t}^2} \quad \Rightarrow p_0 = -\frac{p_i^2}{2m_0}$$

p_0 – energy E

One gets NR energy-momentum dispersion relation for Schrödinger free particle!

Quantization: $p_i = i\frac{\partial}{\partial x_i} \quad p_0 = -i\frac{\partial}{\partial t} \ \Rightarrow \ i\frac{\partial}{\partial t}\psi = -\frac{\Delta}{2m_0}\psi$ free Schrödinger eq.

Our general model: **linear combination of MC forms** associated with all **12+1 central charges**

$$S = S_0 + S_1 + S_2 \qquad S_1 = \sum_{i=1}^{2} \int (m_i \hat{\omega}(X_i) + \mu_i \hat{\omega}(Y_i)) \qquad \text{quasidiagonal c.c.}$$

$$S_2 = \sum_{a,\tilde{b}} \int (n^{a\tilde{b}} \hat{\omega}(X^{a\tilde{b}}) + \nu^{a\tilde{b}} \hat{\omega}(Y^{a\tilde{b}}) \qquad \text{off-diagonal c.c.}$$

Two subclasses of models:

i) $S = S_0 + S_1 \quad m_1 = -\frac{(\mu_1)^2}{2m_0}$ and/or $m_2 = -\frac{(\mu_2)^2}{2m_0} \Rightarrow$
$\Rightarrow$ necessary conditions for getting first class constraints

Odd (fermionic) coordinates: generated by all possible supercharges in G;

16 real Grassmann variables: $\theta_\alpha, \tilde{\theta}_\alpha, \xi_\alpha, \tilde{\xi}_\alpha, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_{\dot{\alpha}}, \bar{\xi}_\alpha, \bar{\tilde{\xi}}_\alpha$

One gets: 8 first class constraints $\longrightarrow$ 8 wave equations for Ψ

8 second class constraints $\longrightarrow$ 4 wave equations for Ψ
(Gupta–Bleuler quantization!)

Superfield solution Ψ of all 12 wave equations depends on arbitrary doubly chiral superfield $\chi^{(2)}(x_i, t; \theta_\alpha, \tilde{\theta}_\alpha)$, which is determined by the initial values of Ψ at $\tau = 0$.

ii) $S = S_0 + S_2$ $\left(\frac{v_A^2}{2m_0}\right)^2 = n_A^2$ $\Leftarrow$ required for first class constraints

off-diagonal central charges

n_A, v_A – two $O(4)$ fourvectors

One gets 16 constraints which follow from the definition of 16 odd (fermionic) momenta – one gets:

4 first class constraints $\Rightarrow$ 4 wave equations

12 second class constraints $\Rightarrow$ 6 wave equations

(Gupta-Bleuler quantization again used)

N=4 SUSY wave function Ψ:

$$\Psi(x_i, t; \underbrace{\theta_\alpha, \tilde{\theta}_\alpha, \zeta_\alpha, \tilde{\zeta}_\alpha, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_\alpha, \bar{\zeta}_\alpha, \bar{\tilde{\zeta}}_\alpha}_{\text{16 odd variables}}) \xrightarrow[\text{10 wave equations}]{16-10=6} \text{solution depends arbitrarily on } \underbrace{\theta_\alpha, \tilde{\theta}_\alpha, \zeta_\alpha}_{\text{6 odd variables}}$$

Form of the solution describing SUSY wave function:

$$\Psi = \hat{R}(\tilde{\chi}, \bar{\theta}_\alpha, \bar{\tilde{\theta}}_\alpha, \bar{\zeta}_\alpha, \bar{\tilde{\zeta}}_\alpha; \frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\tilde{\theta}_\alpha}, \frac{\partial}{\partial\tilde{\zeta}_\alpha})\ \chi^{(3)}(x_i, t; \theta_\alpha, \tilde{\theta}_\alpha, \tilde{\zeta}_\alpha)$$

$\Uparrow$ polynomial dependence

$\chi^{(3)}$– arbitrary triply chiral superfield

All component fields $\psi_A(x_i, t)$ of the wave function Ψ satisfy free Schrödinger wave eq., due to the presence of S_0 in the action.

4. Final remarks

i) One can choose the actions as nonlinear functions of MC one-forms, e.g. as $S_0 + S_2'$, where (by analogy with free relativistic particle)

$$S_2' = \int (k_1 \sqrt{\omega^{a\tilde{b}}(X)\omega_{a\tilde{b}}(X)} + k_2 \sqrt{\omega^{a\tilde{b}}(Y)\omega_{a\tilde{b}}(Y)})$$

with additional dynamical bosonic internal coordinates $h_{a\tilde{b}}$, $f_{a\tilde{b}}$, associated by duality with central charges $X^{a\tilde{b}}, Y^{a\tilde{b}}$. Model after quantization leads to SUSY field KK theory and is under consideration.

ii) Other generalization is to introduce the NR couplings to EM, YM and gravitational (super)backgrounds. Because component fields contain many spins which have different couplings to such backgrounds, one expects to obtain spin-dependent modifications of the Schrödinger eq.

iii) An important task for the future is to study d=3 NR N=4 SUSY YM theory with nonvanishing central charges. It is also interesting to obtain it as consistent NR contraction limit $c \to \infty$ of relativistic D=4 N=4 SUSY YM field theory with central charges.

THANK YOU!