





# Rational Maxwell knots from de Sitter space

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- Conformal equivalence of  $dS_4$  to  $\mathcal{I} \times S^3$  and two copies of  $\mathbb{R}_+^{1,3}$
- The correspondence
- Construction of electromagnetic solutions
- Some properties of the solutions
- Examples
- Summary and discussion

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## Conformal equivalence of $dS_4$ to $\mathcal{I} \times S^3$ and two copies of $\mathbb{R}_+^{1,3}$

Four-dimensional de Sitter space is a one-sheeted hyperboloid in  $\mathbb{R}^{1,4}$  given by

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = \ell^2$$

metric:  $ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2$

Constant  $Z_0$  slices are 3-spheres of varying radius, parametrization:

$$Z_0 = -\ell \cot \tau \quad \text{and} \quad Z_A = \frac{\ell}{\sin \tau} \omega_A \quad \text{for } A = 1, \dots, 4$$

with  $\omega_A \omega_A = 1$  and  $\tau \in \mathcal{I} := (0, \pi)$

metric:  $ds^2 = \frac{\ell^2}{\sin^2 \tau} (-d\tau^2 + d\Omega_3^2) \quad d\Omega_3^2 \quad \text{for } S^3$

Hence,  $dS_4$  is conformally equivalent to a finite cylinder  $\mathcal{I} \times S^3$

The  $Z_0 + Z_4 < 0$  half of  $dS_4$  is also conformally equivalent to future Minkowski space:

$$Z_0 = \frac{t^2 - r^2 - \ell^2}{2t}, \quad Z_1 = \ell \frac{x}{t}, \quad Z_2 = \ell \frac{y}{t}, \quad Z_3 = \ell \frac{z}{t}, \quad Z_4 = \frac{r^2 - t^2 - \ell^2}{2t}$$

with  $x, y, z \in \mathbb{R}$  and  $r^2 = x^2 + y^2 + z^2$  but  $t \in \mathbb{R}_+$

since  $t \in [0, \infty]$  corresponds to  $Z_0 \in [-\infty, \infty]$  but  $Z_0 + Z_4 < 0$

metric:  $ds^2 = \frac{\ell^2}{t^2} (-dt^2 + dx^2 + dy^2 + dz^2)$

Can cover whole  $\mathbb{R}^{1,3}$  by gluing a second  $dS_4$  copy and using patch  $Z_0 + Z_4 > 0$



Direct relation between cylinder and Minkowski coordinates:

$$\cot \tau = \frac{r^2 - t^2 + \ell^2}{2 \ell t}, \quad \omega_1 = \gamma \frac{x}{\ell}, \quad \omega_2 = \gamma \frac{y}{\ell}, \quad \omega_3 = \gamma \frac{z}{\ell}, \quad \omega_4 = \gamma \frac{r^2 - t^2 - \ell^2}{2 \ell^2}$$

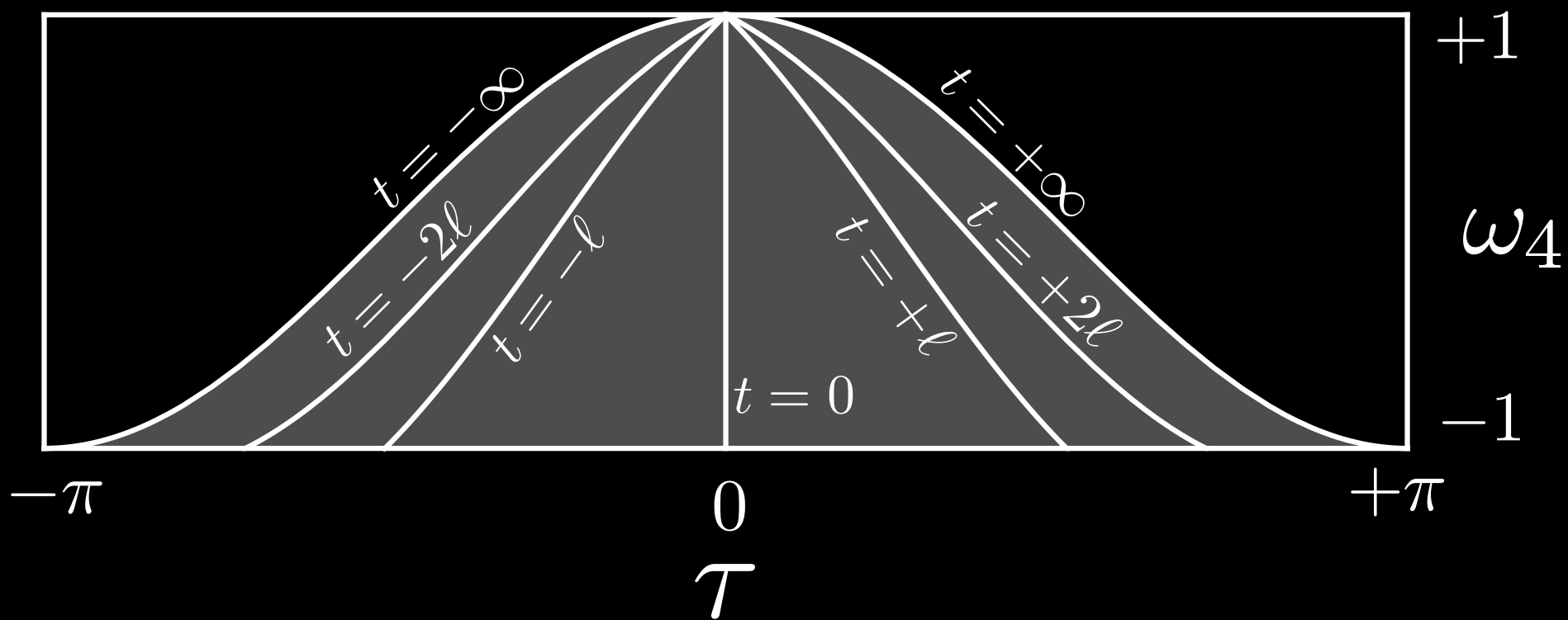
with the convenient abbreviation 
$$\gamma = \frac{2 \ell^2}{\sqrt{4 \ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}}$$

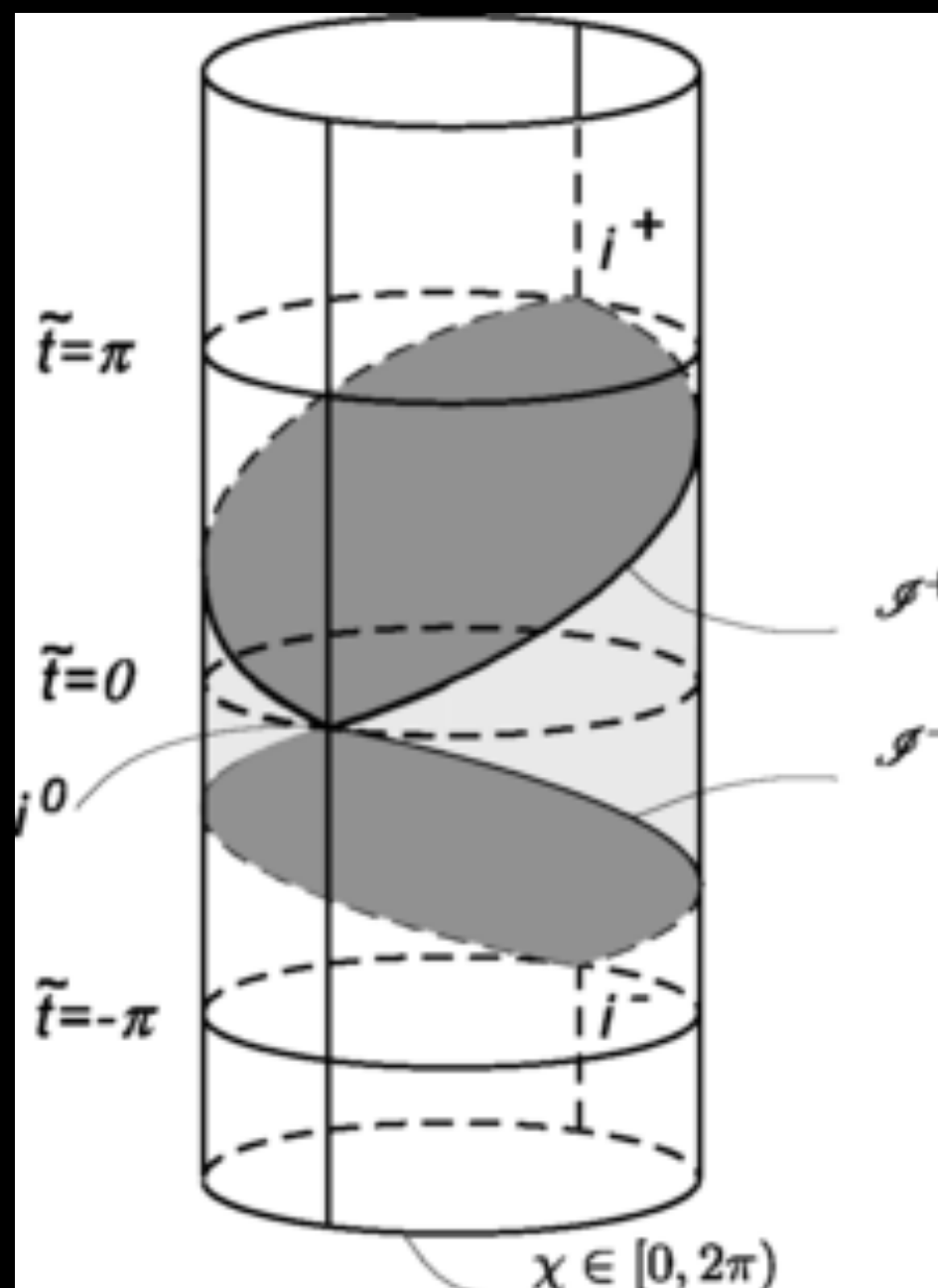
$t = -\infty, 0, \infty$  corresponds to  $\tau = -\pi, 0, \pi$  so the cylinder is doubled to  $2\mathcal{I} \times S^3$

Full Minkowski space is covered by the cylinder patch  $\omega_4 \leq \cos \tau$

Cylinder time  $\tau$  becomes a regular smooth function of  $(t, x, y, z)$ , but more useful is

$$\exp(i \tau) = \frac{(\ell + it)^2 + r^2}{\sqrt{4 \ell^2 t^2 + (r^2 - t^2 + \ell^2)^2}}$$







## The correspondence

Yang–Mills and Maxwell are conformally invariant in four spacetime dimensions

$\Rightarrow$  may solve on cylinder  $2\mathcal{I} \times S^3$  rather than directly on Minkowski space  $\mathbb{R}^{1,3}$

Why?  $S^3$  enables manifestly  $SO(4)$ -covariant formalism!

Geometric ansatz for the gauge potential taking values in a Lie algebra  $\mathfrak{g}$

$$A = \sum_{a=1}^3 X_a(\tau, \omega) e^a \quad \text{on } 2\mathcal{I} \times S^3$$

where  $X_a \in \mathfrak{g}$  and  $\{e^a\}$  are the left-invariant one-forms on  $S^3$ , so  $A_\tau = 0$

Translate YM or Maxwell solutions from  $2\mathcal{I} \times S^3$  to  $\mathbb{R}^{1,3}$  simply by coordinate change

Behavior at the boundary  $\cos \tau = \omega_4$  yields fall-off properties at  $t \rightarrow \pm\infty$

To become explicit, need the one-forms  $e^0 \equiv d\tau$  and  $e^a$ , subject to

$$de^a + \varepsilon^a_{bc} e^b \wedge e^c = 0 \quad \text{and} \quad e^a e^a = d\Omega_3^2$$

In terms of  $S^3$  coordinates ( $a, i, j, k = 1, 2, 3$ ):

$$e^a = -\eta^a_{BC} \omega_B d\omega_C \quad \text{where} \quad \eta^i_{jk} = \varepsilon^i_{jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j$$

In terms of Minkowski coordinates (calculation!):

$$e^0 = \frac{\gamma^2}{\ell^3} \left( \frac{1}{2}(t^2 + r^2 + \ell^2) dt - t x^k dx^k \right)$$

$$e^a = \frac{\gamma^2}{\ell^3} \left( t x^a dt - \left( \frac{1}{2}(t^2 - r^2 + \ell^2) \delta^a_k + x^a x^k + \ell \varepsilon^a_{jk} x^j \right) dx^k \right)$$

with notation  $(x^i) = (x, y, z)$  and (for later)  $(x^\mu) = (x^0, x^i) = (t, x, y, z)$

Simplest YM solutions: impose  $SO(4)$  symmetry  $\Rightarrow X_a(\tau, \omega) = X_a(\tau)$

YM equations become ordinary matrix differential equations:

$$\frac{d^2}{d\tau^2} X_a = -4 X_a + 3 \varepsilon_{abc} [X_b, X_c] - [X_b, [X_a, X_b]] \quad \text{and} \quad \left[ \frac{d}{d\tau} X_a, X_a \right] = 0$$

Known analytic non-Abelian solutions, for  $\mathfrak{g} = su(2)$ :  $[T_a, T_b] = 2\varepsilon_{abc} T_c$

$$X_a(\tau) = q(\tau) T_a \quad \text{with} \quad \frac{d^2 q}{d\tau^2} = -\frac{\partial V}{\partial q} \quad \text{for} \quad V(q) = 2q^2(q-1)^2$$

Double-well potential: vacua  $q = 0$  or  $1$ , sphaleron  $q = \frac{1}{2}$ , bounce, ...

$$\Rightarrow A = q(\tau) g^{-1} dg \quad \text{for} \quad g : S^3 \rightarrow SU(2)$$

Nontrivial static homogeneous solution (on cylinder) is  $A = \frac{1}{2} T_a e^a = \frac{1}{2} g^{-1} dg$

Translates to finite-action homogeneous color-magnetic YM solution on  $dS_4$

Known analytic Abelian solutions,  $\mathfrak{g} = \mathbb{R}$ :

$$X_a(\tau) \Rightarrow X_a(\tau) T_3 \quad \text{with} \quad \frac{d^2 X_a}{d\tau^2} = -4 X_a$$

Harmonic motion: oscillations  $X_a(\tau) = c_a \cos(2(\tau - \tau_a))$

From now on focus on Maxwell's equations  $\Leftrightarrow X_a(\tau, \omega)$  are real-valued functions

Task: transfer oscillatory cylinder solutions to Minkowski space

Helpful:  $\exp(2i\tau)$  is a rational function of  $t$  and  $r$   $x \equiv \{x^\mu\}$

$$A = X_a(\tau(x)) e^a(x) = A_\mu(x) dx^\mu \quad \text{yields } A_\mu(x) \quad A_t \neq 0$$

$$dA = \frac{d}{d\tau} X_a e^0 \wedge e^a - \varepsilon^a_{bc} X_a e^b \wedge e^c = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{yields } F_{\mu\nu}(x)$$

and hence electric and magnetic fields  $E_i = F_{i0}$  and  $B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$

An example (putting  $\ell = 1$ ):

$$X_1(\tau) = -\frac{1}{8}\sin 2\tau, \quad X_2(\tau) = -\frac{1}{8}\cos 2\tau, \quad X_3(\tau) = 0$$

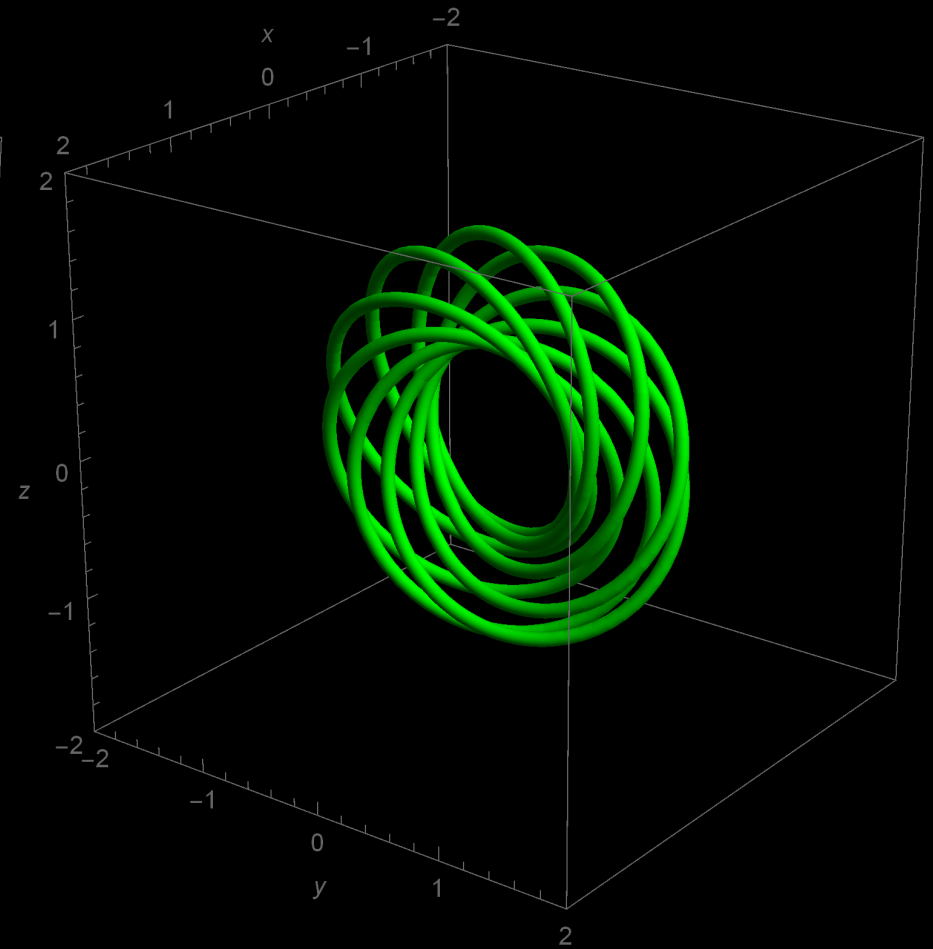
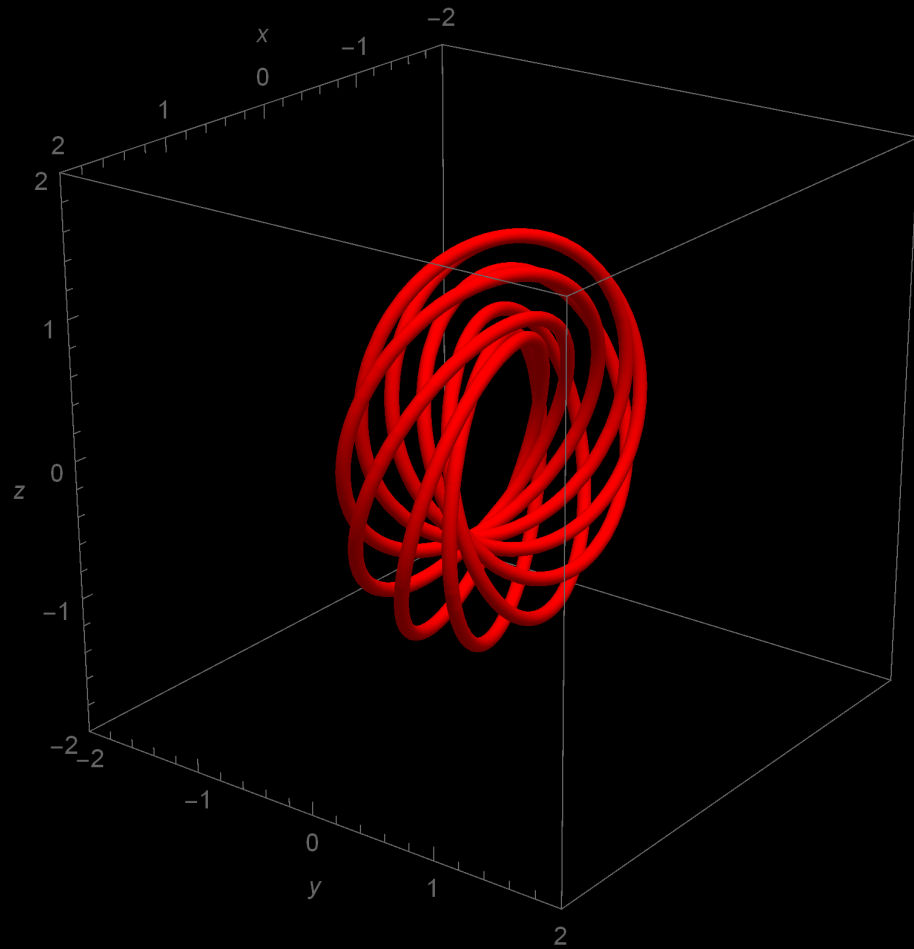
Result of short computation:

$$\vec{E} + i\vec{B} = \frac{1}{\left((t-i)^2 - r^2\right)^3} \begin{pmatrix} (x-iy)^2 - (t-i-z)^2 \\ i(x-iy)^2 + i(t-i-z)^2 \\ -2(x-iy)(t-i-z) \end{pmatrix}$$

This is the celebrated Hopf–Rañada electromagnetic knot.

Our approach also yields its gauge potential.





Some magnetic (red) and electric (green) field lines at  $t=0$

Energy density in  $y=0$  plane, changing with time

## Construction of electromagnetic solutions

Admit arbitrary SO(4)-non-symmetric solutions  $\Rightarrow X_a = X_a(\tau, \omega)$

But capture the  $\omega$ -dependence in an SO(4)-covariant fashion!

Ingredients:

Left-invariant right multiplication:  $R_a = -\eta_{BC}^a \omega_B \frac{\partial}{\partial \omega_C} \Rightarrow [R_a, R_b] = 2 \varepsilon_{abc} R_c$

Right-invariant left multiplication:  $L_a = -\tilde{\eta}_{BC}^a \omega_B \frac{\partial}{\partial \omega_C} \Rightarrow [L_a, L_b] = 2 \varepsilon_{abc} L_c$

Dual to the corresponding one-forms, e.g.  $e^a(R_b) = \delta_b^a$ , and commute,  $[R_a, L_b] = 0$

An arbitrary function  $\Phi$  on  $S^3$  obeys  $d\Phi = e^a R_a \Phi$

Space of functions on  $S^3$  decomposes into irreps of  $su(2)_L \oplus su(2)_R$   
 labelled by a common spin  $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

Define hermitian “angular momenta”  $I_a := \frac{i}{2} L_a$  and  $J_a := \frac{i}{2} R_a$

A basis of hyperspherical harmonics

$Y_{j;m,n}(\omega)$  with  $m, n = -j, -j+1, \dots, +j$  and  $2j = 0, 1, 2, \dots$

is specified by the relations

$$I^2 Y_{j;m,n} = J^2 Y_{j;m,n} = j(j+1) Y_{j;m,n} \quad ,$$

$$I_3 Y_{j;m,n} = m Y_{j;m,n} \quad \text{and} \quad J_3 Y_{j;m,n} = n Y_{j;m,n}$$

For an explicit construction, introduce two complex coordinates

$$\alpha = \omega_1 + i\omega_2 \quad \text{and} \quad \beta = \omega_3 + i\omega_4 \quad \text{subject to} \quad \bar{\alpha}\alpha + \bar{\beta}\beta = 1$$

Angular momenta generators read

$$\begin{aligned} I_+ &= (\bar{\beta}\partial_{\bar{\alpha}} - \alpha\partial_{\beta})/\sqrt{2}, & J_+ &= (\beta\partial_{\bar{\alpha}} - \alpha\partial_{\bar{\beta}})/\sqrt{2} \\ I_3 &= (\alpha\partial_{\alpha} + \bar{\beta}\partial_{\bar{\beta}} - \bar{\alpha}\partial_{\bar{\alpha}} - \beta\partial_{\beta})/2, & J_3 &= (\alpha\partial_{\alpha} + \beta\partial_{\beta} - \bar{\alpha}\partial_{\bar{\alpha}} - \bar{\beta}\partial_{\bar{\beta}})/2 \\ I_- &= (\bar{\alpha}\partial_{\bar{\beta}} - \beta\partial_{\alpha})/\sqrt{2}, & J_- &= (\bar{\alpha}\partial_{\beta} - \bar{\beta}\partial_{\alpha})/\sqrt{2} \end{aligned}$$

Normalized hyperspherical harmonics are represented as

$$Y_{j;m,n} = \sqrt{\frac{2j+1}{2\pi^2}} \sqrt{\frac{2^{j-m}(j+m)!}{(2j)!(j-m)!} \frac{2^{j-n}(j+n)!}{(2j)!(j-n)!}} (I_-)^{j-m} (J_-)^{j-n} \alpha^{2j}$$

$\Rightarrow$  homogenous polynomials of degree  $2j$  in  $\{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$

Set up a left-invariant formulation

General Maxwell gauge potential on  $2\mathcal{I} \times S^3$ :

$$A = X_0(\tau, \omega) d\tau + X_a(\tau, \omega) e^a$$

Coulomb gauge:  $X_0(\tau, \omega) = 0$  and  $J_a X_a(\tau, \omega) = 0$

Maxwell's equations are coupled wave equations:

$$-\frac{1}{4} \partial_\tau^2 X_a = (J^2 + 1) X_a + i \varepsilon_{abc} J_b X_c$$

More transparent rewriting:

$$X_\pm = (X_1 \pm iX_2)/\sqrt{2}$$

$$-\frac{1}{4} \partial_\tau^2 X_+ = (J^2 - J_3 + 1) X_+ + J_+ X_3$$

$$-\frac{1}{4} \partial_\tau^2 X_3 = (J^2 + 1) X_3 - J_+ X_- + J_- X_+$$

$$-\frac{1}{4} \partial_\tau^2 X_- = (J^2 + J_3 + 1) X_- - J_- X_3$$

$$0 = J_3 X_3 + J_+ X_- + J_- X_+$$

Expand in our basis of hyperspherical harmonics:

$$X_a(\tau, \omega) = \sum_{jmn} X_a^{j;m,n}(\tau) Y_{j;m,n}(\alpha, \beta)$$

From the form of the equations it is obvious that

- the equations are diagonal in  $j$  and  $m$ , so these may be kept fixed
- they only couple triplets  $(X_3^{j;m,n}, X_+^{j;m,n+1}, X_-^{j;m,n-1})$ ,  
so  $X_\pm \propto J_\pm X_3$  for  $X_3 \propto Y_{j;m,n}$
- the ansatz  $X_a^{j;m,n}(\tau) \propto e^{i\Omega_a^{j;n}\tau} c_a^{j;n}$  gives a linear system for  $\Omega_a^{j;n}$  and  $c_a^{j;n}$

The frequencies turn out to be integral:  $\Omega_a^{j;n} = \pm 2(j+1)$  or  $\pm 2j$

Two types of solutions:

- type I :  $j \geq 0$  ,  $m = -j, \dots, +j$  ,  $n = -j-1, \dots, j+1$  ,  $\Omega^j = \pm 2(j+1)$

$$X_+ = \sqrt{(j-n)(j-n+1)/2} e^{\pm 2(j+1)i\tau} Y_{j;m,n+1}$$

$$X_3 = \sqrt{(j+1)^2 - n^2} e^{\pm 2(j+1)i\tau} Y_{j;m,n}$$

$$X_- = -\sqrt{(j+n)(j+n+1)/2} e^{\pm 2(j+1)i\tau} Y_{j;m,n-1}$$

- type II :  $j \geq 1$  ,  $m = -j, \dots, +j$  ,  $n = -j+1, \dots, j-1$  ,  $\Omega^j = \pm 2j$

$$X_+ = -\sqrt{(j+n)(j+n+1)/2} e^{\pm 2j i\tau} Y_{j;m,n+1}$$

$$X_3 = \sqrt{j^2 - n^2} e^{\pm 2j i\tau} Y_{j;m,n}$$

$$X_- = \sqrt{(j-n)(j-n+1)/2} e^{\pm 2j i\tau} Y_{j;m,n-1}$$



## Some properties of the solutions

Each complex solution yields two real ones (real part and imaginary part)

Count for  $j$  fixed:

$2(2j+1)(2j+3)$  type-I solutions    and     $2(2j+1)(2j-1)$  type-II solutions ( $j>0$ )  
together:     $4(2j+1)^2$  solutions for  $j>0$     and    6 solutions for  $j=0$

Constant solutions ( $\Omega = 0$ ) are not allowed; simplest are  $j=0$  type I (Hopf–Rañada)

Spin  $j$  type I     $\longleftrightarrow$     parity ( $L \leftrightarrow R, m \leftrightarrow n$ )     $\longrightarrow$     spin  $j+1$  type II

Electromagnetic duality:    shifting  $|\Omega^j|_\tau$  by  $\pm\frac{\pi}{2}$  yields a dual solution  $A_D$

Main technical task:

transform a chosen solution on  $2\mathcal{I} \times S^3$  to Minkowski coordinates  $(t, x, y, z)$ ,

straightforward due to explicit formulæ for all ingredients     $\Rightarrow$     only rational functions

Helicity  $h = \frac{1}{2} \int_{\mathbb{R}^3} (A \wedge F + A_D \wedge F_D)$  is conserved

Energy  $E = \frac{1}{2} \int_{\mathbb{R}^3} d^3x (\vec{E}^2 + \vec{B}^2)$  is conserved and related to  $h$

Best computed in “sphere frame” at  $t = \tau = 0$ :  $F = \mathcal{E}_a e^a \wedge e^0 + \frac{1}{2} \mathcal{B}_a \varepsilon^a_{bc} e^b \wedge e^c$

$$\int_{\mathbb{R}^3} d^3x \vec{E}^2 = \frac{1}{\ell} \int_{S^3} d^3\Omega_3 (1 - \omega_4) \mathcal{E}_a \mathcal{E}_a \quad \text{and} \quad \int_{\mathbb{R}^3} d^3x \vec{B}^2 = \frac{1}{\ell} \int_{S^3} d^3\Omega_3 (1 - \omega_4) \mathcal{B}_a \mathcal{B}_a$$

exploiting orthogonality properties of the hyperspherical harmonics

## Examples

Example 1:  $(j; m, n) = (1, 0, 0)$ , type I, combine  $e^{4i\tau} + e^{-4i\tau} = 2 \cos 4\tau$

$$X_+ = -\frac{\sqrt{3}}{\pi} \alpha \beta \cos 4\tau, \quad X_3 = \frac{\sqrt{6}}{\pi} (\beta \bar{\beta} - \alpha \bar{\alpha}) \cos 4\tau, \quad X_- = -\frac{\sqrt{3}}{\pi} \bar{\alpha} \bar{\beta} \cos 4\tau$$

$$\Rightarrow \quad h = 12 \quad \text{and} \quad E = 48/\ell$$

Example 2:  $(j; m, n) = (2; 1, -1)$ , type I  $E = 6 h/\ell$

Example 3:  $(j; m, n) = (\frac{5}{2}; \frac{3}{2}, \frac{1}{2})$ , type I  $E = 7 h/\ell$

Example 4:  $(j; m, n) = (2; 1, 2)$ , type I  $E = 6 h/\ell$

Example 1

$$(E+iB)_x = \frac{-2i}{((t-i)^2 - x^2 - y^2 - z^2)^5} \times$$

$$\times \left\{ 2y + 3ity - xz + 2t^2y + 2itxz - 8x^2y - 8y^3 + 4yz^2 \right.$$

$$+ 4it^3y - 6t^2xz - 8itx^2y - 8ity^3 + 4ityz^2 + 10x^3z + 10xy^2z - 2xz^3$$

$$+ 2(itxz + x^2y + y^3 + yz^2)(-t^2 + x^2 + y^2 + z^2) + (ity - xz)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$

$$(E+iB)_y = \frac{2i}{((t-i)^2 - x^2 - y^2 - z^2)^5} \times$$

$$\times \left\{ 2x + 3itx + yz + 2t^2x - 2ityz - 8x^3 - 8xy^2 + 4xz^2 \right.$$

$$+ 4it^3x + 6t^2yz - 8itx^3 - 8itxy^2 + 4itxz^2 - 10x^2yz - 10y^3z + 2yz^3$$

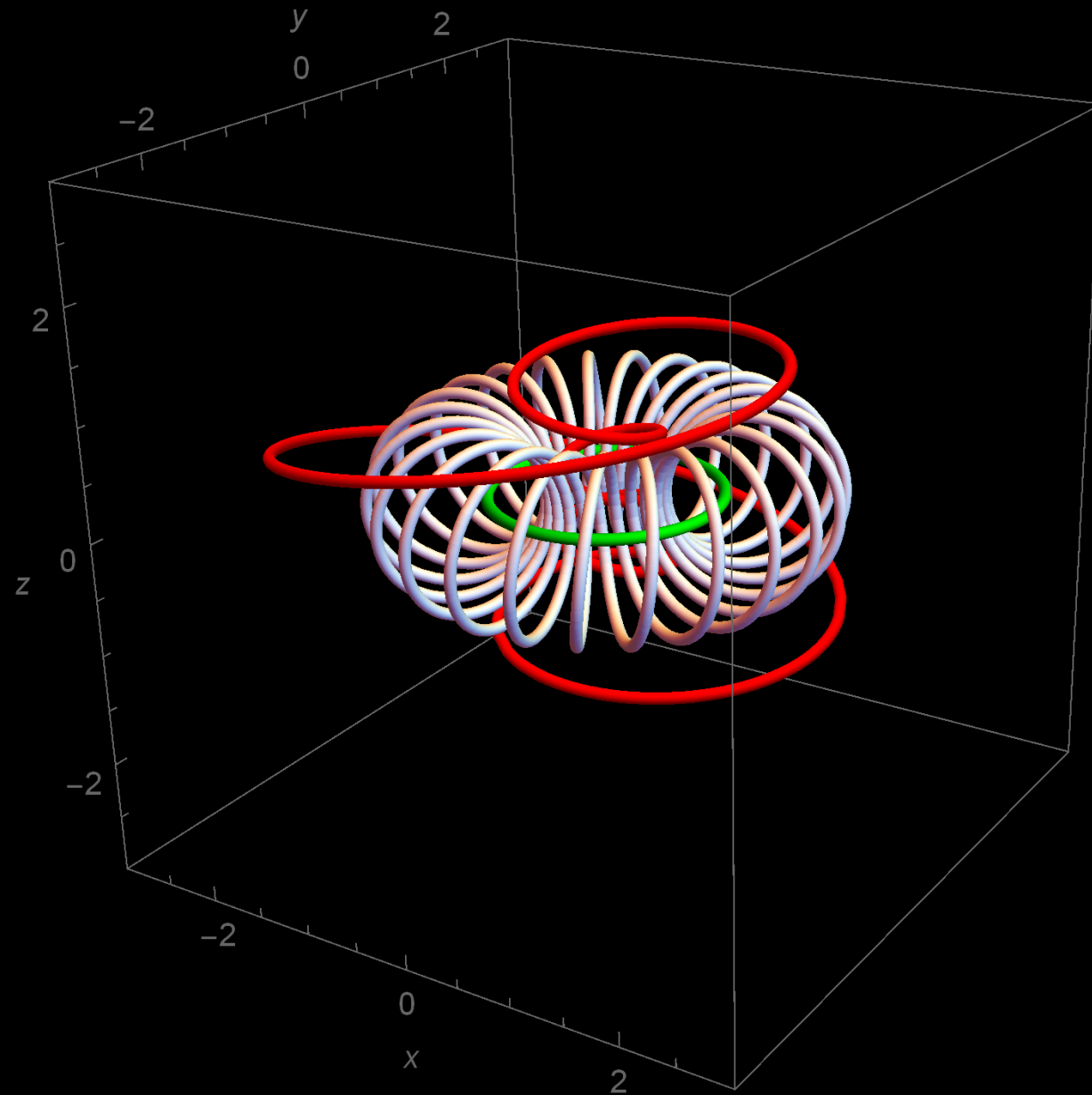
$$+ 2(-ityz + x^3 + xy^2 + xz^2)(-t^2 + x^2 + y^2 + z^2) + (itx + yz)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$

$$(E+iB)_z = \frac{i}{((t-i)^2 - x^2 - y^2 - z^2)^5} \times$$

$$\times \left\{ 1 + 2it + t^2 - 11x^2 - 11y^2 + 3z^2 + 4it^3 - 16itx^2 - 16ity^2 + 4itz^2 \right.$$

$$- t^4 - 2t^2x^2 - 2t^2y^2 - 2t^2z^2 + 11x^4 + 22x^2y^2 + 10x^2z^2 + 11y^4 - 10y^2z^2 + 3z^4$$

$$+ 2it(t^2 - 3x^2 - 3y^2 - z^2)(t^2 - x^2 - y^2 - z^2) - (t^2 + x^2 + y^2 - z^2)(-t^2 + x^2 + y^2 + z^2)^2 \Big\}$$



$(j; m, n) = (1; 0, 0)$ : three particular magnetic field lines at  $t=0$

$(j; m, n) = (1; 0, 0)$ : energy density in  $y=0$  plane, changing with time

$(j; m, n) = (2; 1, -1)$ : energy density in  $y=0$  plane, changing with time

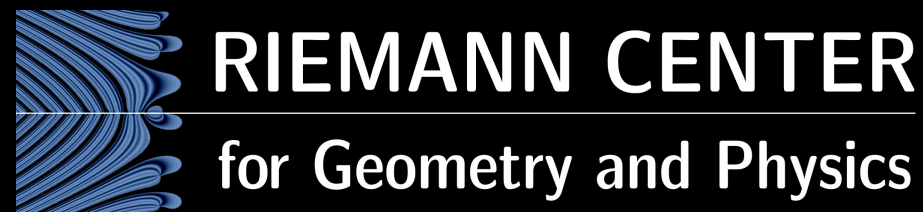
$(j; m, n) = (\frac{5}{2}; \frac{3}{2}, \frac{1}{2})$ : energy density at  $t=0$ , scanning  $z = \text{const}$  planes



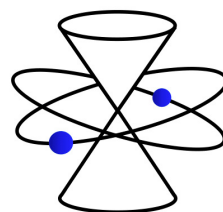
$(j; m, n) = (2; 1, 2)$ : energy density at  $t=0$ , scanning  $z = \text{const}$  planes

## Summary and discussion

- Rational electromagnetic fields with nontrivial topology investigated since 1989
- We introduced a new construction method based on two insights:
  - simplicity of solving Maxwell on a temporal cylinder over a three-sphere
  - conformal equivalence of a cylinder patch to four-dimensional Minkowski space
- $A = X_\nu(\tau, \omega) e^\nu = X_\nu(\tau(x), \omega(x)) e^\nu_\mu(x) dx^\mu$
- Only finite-time  $\tau \in (-\pi, +\pi)$  dynamics is required on the cylinder
- Our solutions have finite energy and action, by construction
- A complete basis for sufficiently fast spatially and temporally decaying fields
- Non-Abelian extension couples different  $j$  components of  $X_a \Rightarrow$  much harder
- Useful tool for numerical study of classical YM dynamics in Minkowski space



THANK YOU !



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