The component $N = 2$, $d = 6$ Born-Infeld theory

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Spontaneous breaking of supersymmetry

The supersymmetric Born-Infeld theories arise as theories with spontaneously broken supersymmetry with vector multiplets as corresponding Goldstone superfields. Particular interest in this theories is related to the fact that they describe low energy effective actions of various D-branes. One of the best known and simplest examples is the $N = 2$, $d = 4$ Born-Infeld theory, which is a direct supersymmetrization of the original Born-Infeld action. It provides the effective description of the space-filling $D3$-brane. Its superfield action can be constructed as one of the components of the composite $N = 2$, $d = 4$ multiplet, constructed from $N = 1$ superfields.

Another example with higher supersymmetry is $N = 4$, $d = 4$ Born-Infeld theory, which, in addition to the electromagnetic field strength, contains also two scalar fields. Usual superfield methods allow construction of the action only as iterative procedure, because the corresponding $N = 4$, $d = 4$ multiplet it belongs to, is composed of infinite amount of $N = 2$ fields. An attempt to fix the theory from the requirement of self-duality also leads to iterative solution. Also this theory could be obtained as a reduction from $N = 2$, $d = 6$ Born-Infeld theory, but its formulation in superfields is also problematic.

The component actions

In a few previous works, we advocated the component approach to the actions with broken supersymmetry, which is based on the fact that the dependence of the component action on the Goldstone fermions is fixed by the invariance with respect to the broken supersymmetry, and then the whole action is determined by the unbroken half of the supersymmetry. The necessary invariants, covariant differential forms and transformation laws can be obtained from the nonlinear realizations formalism.

In this talk the component version of the $N=2$, $d=6$ Born-Infeld theory is considered. It realizes spontaneous breaking of $N=(2,0)$, $d=6$ supersymmetry to the $N=(1,0)$, $d=6$. Advantage of discussing this theory instead of $N=4$, $d=4$ is the fact that it is formulated in terms of spinor, rather than scalar superfields: as experience with $N=4$, $d=3$ theory shows, this greatly simplifies the task of deriving the Bianchi identities. Also, the physical bosonic components of the theory combine into a single antisymmetric tensor, while in the $N=4$, $d=4$ case one deals with one antisymmetric tensor and two gradients of scalars.

This talk is composed of four parts. In the first one, the $N=(2,0)$ Poincare superalgebra is introduced and the necessary coset machinery is implemented. In the second one, the proper irreducibility conditions of the multiplet are found. In the third, the Bianchi identities are analyzed. In the last one, the ansatz for the action is introduced and its invariance is proven in the first order in fermions.

Algebra and transformation laws

The $N=(2,0)$, $d=6$ superalgebra is composed of two copies of $N=(1,0)$ superalgebra. In spinor notations, it reads

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon^{ij}P_{\alpha\beta}, \quad \{S^i_\alpha, S^j_\beta\} = 2\epsilon^{ij}P_{\alpha\beta},$$

where $i=1,2$, $\alpha=1\ldots4$ and $P_{\alpha\beta}=-P_{\beta\alpha}$. Lorentz $so(1,5)$ and automorphism $so(5)$ complete the algebra.
The transformations, that form this superalgebra, can be realized by left multiplication $g_0 g = g' h$ of the following coset element as

$$g = e^{\mathrm{i}x^{\alpha\beta}P_{\alpha\beta}}\, e^{\mathrm{i}\theta^\alpha_i Q^i_\alpha}\, e^{\mathrm{i}\psi^\alpha_i(x,\theta)S^i_\alpha}.$$

For example, the supersymmetry transformations are

$$g_Q = e^{\mathrm{i}\epsilon^\alpha_i Q^i_\alpha} : \quad \delta_Q x^{\alpha\beta} = -\mathrm{i}\epsilon_i^{[\alpha}\theta^{\beta]i},\ \delta_Q\theta^\alpha_i = \epsilon^\alpha_i,\ \delta_Q\psi^\alpha_i = 0,$$
$$g_S = e^{\mathrm{i}\varepsilon^\alpha_i S^i_\alpha} : \quad \delta_S x^{\alpha\beta} = -\mathrm{i}\epsilon_i^{[\alpha}\psi^{\beta]i},\ \delta_S\psi^\alpha_i = \varepsilon^\alpha_i,\ \delta_S\theta^\alpha_i = 0.$$

The Cartan forms, invariant with respect to these transformations , are:

$$g^{-1}dg = \mathrm{i}\triangle x^{\alpha\beta}P_{\alpha\beta} + \mathrm{i}d\theta^\alpha_i\, Q^i_\alpha + \mathrm{i}d\psi^\alpha_i\, S^i_\alpha, \quad \triangle x^{\alpha\beta} = dx^{\alpha\beta} - \mathrm{i}d\theta_i^{[\alpha}\,\theta^{\beta]i} - \mathrm{i}d\psi_i^{[\alpha}\,\psi^{\beta]i}.$$

Covariant derivatives

If one expands the definition of the the differential in terms of the covariant forms $\triangle x^{\alpha\beta}$, $d\theta^{\alpha}_{i}$, one may obtain the derivatives, which are covariant with respect to both supersymmetries:

$$\nabla_{\alpha\beta} = \left(E^{-1}\right)_{\alpha\beta}{}^{\mu\nu}\partial_{\mu\nu}, \;\; \left(E\right)_{\alpha\beta}{}^{\mu\nu} = \delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta} - \mathrm{i}\partial_{\alpha\beta}\psi^{[\mu}_{i}\psi^{\nu]i},$$

$$\nabla^{i}_{\alpha} = D^{i}_{\alpha} - \mathrm{i}\nabla^{i}_{\alpha}\psi^{\rho}_{m}\psi^{m\sigma}\partial_{\rho\sigma}, \;\; D^{i}_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}_{i}} + \mathrm{i}\theta^{i\beta}\partial_{\alpha\beta}.$$

As $\left\{D^{i}_{\alpha}, D^{j}_{\beta}\right\} = 2\mathrm{i}\epsilon^{ij}\partial_{\alpha\beta}$, their (anti)commutation relations are

$$\left\{\nabla^{i}_{\alpha}, \nabla^{j}_{\beta}\right\} = 2\mathrm{i}\epsilon^{ij}\nabla_{\alpha\beta} + 2\mathrm{i}\nabla^{i}_{\alpha}\psi^{\rho}_{k}\nabla^{j}_{\beta}\psi^{\sigma k}\nabla_{\rho\sigma},$$

$$\left[\nabla_{\alpha\beta}, \nabla^{i}_{\gamma}\right] = 2\mathrm{i}\nabla_{\alpha\beta}\psi^{\rho}_{m}\nabla^{i}_{\gamma}\psi^{\sigma m}\nabla_{\rho\sigma}, \;\; \left[\nabla_{\alpha\beta}, \nabla_{\mu\nu}\right] = -2\mathrm{i}\nabla_{\alpha\beta}\psi^{\rho}_{k}\nabla_{\mu\nu}\psi^{\sigma k}\nabla_{\rho\sigma}.$$

The irreducibility conditions of the vector supermultiplet should be formulated in terms of these derivatives.

Modified constraints

It could be learned from the construction of $N=2$, $d=4$ and $N=4$, $d=3$ Born-Infeld theories, that the proper constraints on the multiplet are fixed by the invariance with respect to one of automorphism groups of the respective superalgebra. In the case of $N=2$, $d=6$ Born-Infeld theory, the relevant automorphism group is $SO(4)$ subgroup of the full $SO(5)\sim Sp(2)$ automorphism group. Its generators are

$$\left[T^{ij},T^{kl}\right]=\mathrm{i}\left(\epsilon^{ik}T^{jl}+\epsilon^{jl}T^{ik}\right),\;\;\left[T^{ij},R^{kl}\right]=\mathrm{i}\left(\epsilon^{ik}R^{jl}+\epsilon^{jl}R^{ik}\right),$$
$$\left[R^{ij},R^{kl}\right]=2\mathrm{i}\left(\epsilon^{ik}T^{jl}+\epsilon^{jl}T^{ik}\right).$$

They act on the supercharges as

$$\left[T^{ij},Q^{k}_{\alpha}\right]=\frac{\mathrm{i}}{2}\left(\epsilon^{ik}Q^{j}_{\alpha}+\epsilon^{jk}Q^{i}_{\alpha}\right),\;\;\left[T^{ij},S^{k}_{\alpha}\right]=\frac{\mathrm{i}}{2}\left(\epsilon^{ik}S^{j}_{\alpha}+\epsilon^{jk}S^{i}_{\alpha}\right),$$
$$\left[R^{ij},Q^{k}_{\alpha}\right]=\mathrm{i}\left(\epsilon^{ik}S^{j}_{\alpha}+\epsilon^{jk}S^{i}_{\alpha}\right),\;\;\left[R^{ij},S^{k}_{\alpha}\right]=\mathrm{i}\left(\epsilon^{ik}Q^{j}_{\alpha}+\epsilon^{jk}Q^{i}_{\alpha}\right).$$

Acting by $g_0=\exp\left(\mathrm{i}a_{ij}R^{ij}\right)$ on the coset element g, one may find that the superspace coordinates,the basic superfield ψ^{α}_{i} and its covariant derivative transform as

$$\delta x^{\alpha\beta}=0,\;\;\delta\theta^{\alpha}_{i}=a^{k}_{i}\psi^{\alpha}_{k},\;\;\delta\psi^{\alpha}_{k}=a^{k}_{i}\theta^{\alpha}_{k},$$
$$\delta\nabla^{i}_{\alpha}\psi^{\beta}_{j}=a^{i}_{j}\delta^{\beta}_{\alpha}-a^{k}_{m}\nabla^{i}_{\alpha}\psi^{\gamma}_{k}\,\nabla^{m}_{\gamma}\psi^{\beta}_{j}.$$

Standard irreducibility conditions

The standard $N=(1,0)$, $d=6$ vector multiplet is described by the superfield ψ_i^α, subjected to the conditions

$$D^i_\alpha \psi^\alpha_i = 0,\, D^i_\alpha \psi^{j\beta} + D^j_\alpha \psi^{i\beta} = \frac{1}{2}\delta^\beta_\alpha D^i_\gamma \psi^{j\gamma}.$$

This leaves in the multiplet the components

$$\psi^\alpha_i = \psi^\alpha_i|,\quad F_\alpha{}^\beta = D^i_\alpha \psi^\beta_i|,\quad B^{ij} = B^{ji} = D^i_\alpha \psi^{\alpha j}|.$$

In addition, the component $F_\alpha{}^\beta$ is subjected to two differential (Bianchi) identities

$$\partial_{\alpha\gamma} F_\beta{}^\gamma + \partial_{\beta\gamma} F_\alpha{}^\gamma = 0,\quad \partial^{\alpha\gamma} F_\gamma{}^\beta + \partial^{\beta\gamma} F_\gamma{}^\alpha = 0,\quad \partial^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\partial_{\mu\nu}.$$

The first one could be derived by acting by two covariant derivatives on one of the irreducibility conditions, $D^i_\alpha D^j_\beta \left(D^k_\gamma \psi^\gamma_k\right) = 0$. The second one results form the analysis of the expression $\epsilon^{\alpha\mu\nu\lambda} D^i_\mu D^j_\nu D^k_\lambda \psi^\beta_k$:

$$\epsilon^{\alpha\mu\nu\lambda} D^i_\mu D^j_\nu D^k_\lambda \psi^\beta_k + (\alpha \leftrightarrow \beta) = 4\mathrm{i}\left(\partial^{\alpha\gamma} D^k_\gamma \psi^\beta_k + \partial^{\beta\gamma} D^k_\gamma \psi^\alpha_k\right)\epsilon^{ij}$$

Multiplying this by ϵ_{ij} and using the fact that $\epsilon^{\alpha\beta\mu\nu}\epsilon_{ij} D^i_\mu D^j_\nu = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon_{ij}\left\{D^i_\mu, D^j_\nu\right\}$, one finds the expected identity.

In vector notations, they are just self-dual and anti-self-dual parts of $\partial_{[A}F_{BC]} = 0$, $A, B, C = 0, \ldots, 5$, and ensure that $F_\alpha{}^\beta$ is the electromagnetic field strength.

New irreducibility conditions

New irreducibility conditions should be explicitly covariant with respect to the broken supersymmetry and should be preserved by the automorphism transformations $\delta\nabla^i_\alpha\psi^\beta_j = a^i_j\delta^\beta_\alpha - a^k_m\nabla^i_\alpha\psi^\gamma_k\,\nabla^m_\gamma\psi^\beta_j$.

It is simple to generalize the relation $D^i_\alpha\psi^\alpha_i = 0$. Observing that

$$\delta\nabla^i_\alpha\psi^\alpha_i = -a^k_m\nabla^m_\gamma\psi^\alpha_i\,\nabla^i_\alpha\psi^\gamma_k \equiv -a^k_m\left(\nabla\psi^2\right)_{\gamma k}{}^{m\gamma},$$

$$\delta\left(\nabla\psi^3\right)_{\gamma k}{}^{k\gamma} = 3a^k_m\left(\nabla\psi^2\right)_{\gamma k}{}^{m\gamma} - 3a^k_m\left(\nabla\psi^4\right)_{\gamma k}{}^{m\gamma}, \text{ e.t.c,}$$

one may note that

$$\left(\nabla\psi + \frac{1}{3}(\nabla\psi)^3 + \frac{1}{5}(\nabla\psi)^5 + \ldots\right)_{\gamma m}{}^{m\gamma} = \mathrm{tr}\left[\operatorname{arctanh}\left(\nabla^i_\alpha\psi^\beta_j\right)\right]$$

is covariant and could be set to zero.

New irreducibility conditions

Another irreducibility condition may be generalized as

$$\nabla_\alpha^{(i}\psi^{j)\beta} = \frac{1}{4}\boldsymbol{Y}_\alpha{}^\beta \nabla_\gamma^{(i}\psi^{j)\gamma}, \quad \mathrm{tr}\,\boldsymbol{Y} = 4.$$

Matrix $\boldsymbol{Y}_\alpha{}^\beta$ here should depend on $\boldsymbol{V}_\alpha{}^\beta = \nabla^i_\alpha \psi^\beta_i$ and $\boldsymbol{B}^2 = \boldsymbol{B}^{ij}\boldsymbol{B}_{ij}$, $\boldsymbol{B}^{ij} = \nabla_\gamma^{(i}\psi^{j)\gamma}$. Straightforward analysis shows that $\boldsymbol{Y}_\alpha{}^\beta$ should satisfy the relation

$$\delta_\alpha^\beta - \frac{1}{4}\left(\boldsymbol{V}^2\right)_\alpha{}^\beta - \frac{1}{32}\boldsymbol{B}^2\left(\boldsymbol{Y}^2\right)_\alpha{}^\beta = \boldsymbol{Y}_\alpha{}^\beta\left(1 - \frac{1}{16}\mathrm{tr}(\boldsymbol{V}^2) - \frac{1}{128}\mathrm{tr}\left(\boldsymbol{Y}^2\right)\boldsymbol{B}^2\right).$$

As we want to find the on-shell identity for the field strength, it is sufficient to know the irreducibility conditions in the first approximation in $\boldsymbol{B}^{ij}$, or the $\boldsymbol{Y}_\alpha{}^\beta$ in the limit $\boldsymbol{B} \to 0$. Then it reads

$$\nabla_\alpha^{(i}\psi^{j)\beta} = \frac{\boldsymbol{Z}_\alpha{}^\beta}{\mathrm{tr}\boldsymbol{Z}} \nabla_\gamma^{(i}\psi^{j)\gamma}, \quad \boldsymbol{Z}_\alpha{}^\beta = \delta_\alpha{}^\beta - \frac{1}{4}\left(\boldsymbol{V}^2\right)_\alpha{}^\beta.$$

The Bianchi identities

As in the case without broken supersymmetry, the tensor component of the multiplet is subjected to the differential and algebraic constraints as a consequence of the irreducibility conditions, from which the physical field strength should be derived. To construct the action in the lowest approximation in fermions, it is sufficient to know the bosonic limit of these identities. Also, one may calculate the identity on-shell, eliminating the auxiliary field $B^{ij} = \boldsymbol{B}^{ij}|$ by the equation of motion $B^{ij} = 0$ where $\boldsymbol{B}^{ij}$ appears without a covariant derivative applied to it. This way, one may obtain identities

$$\nabla^i_\alpha \nabla^j_\beta \left(\nabla^k_\gamma \psi^\gamma_k\right) = 0 \quad \Rightarrow \quad \left(\partial_{\alpha\rho} V_\beta{}^\sigma + \frac{1}{4} V_\alpha{}^\mu V_\rho{}^\nu \partial_{\mu\nu} V_\beta{}^\sigma\right) \left(Z^{-1}\right)_\sigma{}^\rho + (\alpha \leftrightarrow \beta) = 0,$$

$$\epsilon^{\mu\nu\lambda\alpha} \nabla^i_\mu \nabla^j_\nu V_\lambda{}^\beta \quad \Rightarrow \quad \epsilon^{\alpha\mu\nu\lambda} \left(\partial_{\mu\nu} V_\lambda{}^\gamma + \frac{1}{4} V_\mu{}^\rho V_\nu{}^\sigma \partial_{\rho\sigma} V_\lambda{}^\gamma\right) \left(Z^{-1}\right)_\gamma{}^\beta + (\alpha \leftrightarrow \beta) = 0.$$

Just like in the linear system bispinor $V_\alpha{}^\beta$ satisfies an algebraic relation, which eliminates one component. This relation could be simplified using identity $\det e^A = e^{\mathrm{tr} A}$:

$$\mathrm{tr}\left(\mathrm{arctanh} \frac{1}{2} V_\alpha{}^\beta\right) = 0, \; d\,\mathrm{tr}\left(\mathrm{arctanh} \frac{1}{2} V_\alpha{}^\beta\right) = 0 \;\Rightarrow\; \det\left(\frac{1 + \frac{1}{2} V}{1 - \frac{1}{2} V}\right) = 1$$

$$\Rightarrow\; 24\mathrm{tr}(V) + \left(\mathrm{tr}(V)\right)^3 - 3\mathrm{tr}(V)\mathrm{tr}(V^2) + 2\mathrm{tr}(V^3) = 0, \quad dV_\alpha{}^\beta \left(Z^{-1}\right)_\beta{}^\alpha = 0.$$

The physical field strength

The found bosonic Bianchi identities should be equivalent to the standard ones $\partial_{\alpha\gamma}F_{\beta}{}^{\gamma}+\partial_{\beta\gamma}F_{\alpha}{}^{\gamma}=0$, $\partial^{\alpha\gamma}F_{\gamma}{}^{\beta}+\partial^{\beta\gamma}F_{\gamma}{}^{\alpha}=0$. A way to prove this is to combine the both identities with some matrix coefficients, which are polynomials in $V_{\alpha}{}^{\beta}$, and introduce an ansatz for $F_{\alpha}{}^{\beta}$ also as a polynomial:

$$(BI)_{\alpha\beta}=\left(\partial_{\alpha\rho}V_{\beta}{}^{\sigma}+\frac{1}{4}V_{\alpha}{}^{\mu}V_{\rho}{}^{\nu}\partial_{\mu\nu}V_{\beta}{}^{\sigma}\right)\left(Z^{-1}\right)_{\sigma}{}^{\rho}+(\alpha\leftrightarrow\beta)=0,$$

$$(\widetilde{BI})^{\alpha\beta}=\epsilon^{\alpha\mu\nu\lambda}\left(\partial_{\mu\nu}V_{\lambda}{}^{\gamma}+\frac{1}{4}V_{\mu}{}^{\rho}V_{\nu}{}^{\sigma}\partial_{\rho\sigma}V_{\lambda}{}^{\gamma}\right)\left(Z^{-1}\right)_{\gamma}{}^{\beta}+(\alpha\leftrightarrow\beta)=0,$$

$$\partial_{\alpha\gamma}\widehat{F}_{\beta}{}^{\gamma}+\partial_{\beta\gamma}\widehat{F}_{\alpha}{}^{\gamma}=M^{\mu\nu}_{\alpha\beta}(BI)_{\mu\nu}+N_{\alpha\beta\mu\nu}(\widetilde{BI})^{\mu\nu},$$

$$M^{\mu\nu}_{\alpha\beta}=\sum_{m,n=0,\ldots,3}M_{m,n}\left(V^{m}\right)^{\mu}_{(\alpha}\left(V^{n}\right)^{\nu}_{\beta)},\quad N_{\alpha\beta\mu\nu}=\sum_{m,n=1,\ldots,3}N_{m,n}\epsilon_{\alpha\mu\rho\sigma}\left(V^{m}\right)_{\beta}{}^{\rho}\left(V^{n}\right)_{\nu}{}^{\sigma},$$

the latter symmetrized with respect to α,β and μ,ν independently. Then, if $\widehat{F}_{\alpha}{}^{\beta}=\sum_{k=1,2,3}\Lambda_{k}\left(V^{k}\right)_{\alpha}{}^{\beta}$, the both sides can be equated (by analytical computer calculation) and functions, entering the $\widehat{F}_{\alpha}{}^{\beta}$, determined. The actual physical bosonic field strength is $F_{\alpha}{}^{\beta}=\widehat{F}_{\alpha}{}^{\beta}-\frac{1}{4}\delta^{\beta}_{\alpha}\widehat{F}_{\gamma}{}^{\gamma}$:

$$F_{\alpha}{}^{\beta}=\frac{\frac{1}{2}\mathrm{tr}(V)\delta^{\beta}_{\alpha}+\left(1+\frac{1}{8}\left(\mathrm{tr}(V)\right)^{2}-\frac{1}{8}\mathrm{tr}(V^{2})\right)V^{\beta}_{\alpha}-\frac{1}{4}\mathrm{tr}(V)\left(V^{2}\right)_{\alpha}{}^{\beta}+\frac{1}{4}\left(V^{3}\right)_{\alpha}{}^{\beta}}{1+\frac{1}{4}\left(\mathrm{tr}(V)\right)^{2}+\frac{1}{128}\left(\mathrm{tr}(V)\right)^{4}-\frac{1}{128}\left(\mathrm{tr}(V^{2})\right)^{2}-\frac{1}{64}\left(\mathrm{tr}(V)\right)^{2}\mathrm{tr}(V^{2})+\frac{1}{64}\mathrm{tr}(V^{4})}.$$

The ansatz for the action

When considering the component actions, one should, at first, consider an ansatz, invariant with respect to the broken supersymmetry. It involves replacing the integration measure with covariant one,

$$d^6x \;\rightarrow\; d^6x \det\mathcal{E}, \quad \left(\mathcal{E}\right)_{\alpha\beta}{}^{\mu\nu} = \left(E\right)_{\alpha\beta}{}^{\mu\nu}|_{\theta\rightarrow 0} = \delta_\alpha^{[\mu}\delta_\beta^{\nu]} - \mathrm{i}\partial_{\alpha\beta}\psi_i^{[\mu}\,\psi^{\nu]i}.$$

As the bosonic component of multiplet is already invariant, it should be used as it is (the true physical field strength, which takes into account fermions, is not invariant), and any derivative of the fermionic fields should be covariantized: $\partial_{\alpha\beta} \rightarrow \mathcal{D}_{\alpha\beta} = \left(\mathcal{E}^{-1}\right)_{\alpha\beta}{}^{\rho\sigma}\partial_{\rho\sigma}$. Then the main part of the action, which generalizes bosonic one, reads

$$\mathcal{L}_0 = -\det\mathcal{E}\left(C_1 + \sqrt{-\det\left(\eta_{AB} + F_{AB}\right)}\right).$$

the constant C_1 could by fixed by considering supersymmetry in linearized limit as $C_1 = 1$. Therefore, in terms of $V_\alpha{}^\beta$ the main part of the Lagrangian reads

$$\mathcal{L}_0 = -\frac{2\det\mathcal{E}\left(1 + \frac{1}{16}\left(\mathrm{tr}\left(V\right)\right)^2 - \frac{1}{16}\mathrm{tr}\left(V^2\right)\right)}{1 + \frac{1}{4}\left(\mathrm{tr}\left(V\right)\right)^2 + \frac{1}{128}\left(\mathrm{tr}\left(V\right)\right)^4 - \frac{1}{128}\left(\mathrm{tr}\left(V^2\right)\right)^2 - \frac{1}{64}\left(\mathrm{tr}\left(V\right)\right)^2\mathrm{tr}\left(V^2\right) + \frac{1}{64}\mathrm{tr}\left(V^4\right)}.$$

The Wess-Zumino term also should be added.

Physical field strength with the fermions

While the physical field strength was determined only in the bosonic limit, one may fix its dependence on the fermionic fields by requiring the Bianchi identities to be covariant with respect to broken supersymmetry. This may be illustrated using vector notations, which involve antisymmetric matrices $\left(\gamma^A\right)_{\alpha\beta}$, $\left(\gamma^A\right)^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\left(\gamma^A\right)_{\mu\nu}$. In this notations, the spacetime coordinates, the fermion and transform as

$$\delta x^A = -\frac{\mathrm{i}}{2}\varepsilon_i^\mu \psi^{\nu i}\left(\gamma^A\right)_{\mu\nu} \equiv -U^A,\ \delta\psi_i^\alpha = \varepsilon_i^\alpha.$$

One may turn to "active" transformation laws $\delta^\star f = \delta f - \delta x^A \partial_A f$ and that the vielbein $\mathcal{E}_A{}^B$ transforms as

$$\mathcal{E}_A{}^B = \frac{1}{4}\left(\gamma_A\right)^{\alpha\beta}\left(\gamma^B\right)_{\mu\nu}\left(\mathcal{E}\right)_{\alpha\beta}{}^{\mu\nu},\ \ \delta^\star \mathcal{E}_A{}^B = \partial_A U^C \mathcal{E}_C{}^B + U^C \partial_C \mathcal{E}_A{}^B$$

Usually inert fields, such as F_{AB}, transform with respect to the active transformations as $\delta^\star F_{AB} = U^C \partial_C F_{AB}$, while derivatives do not transform. One may, therefore, note, that expression $\partial_{[A} F_{BC]}$ is not covariant with respect to the broken supersymmetry, while the $\partial_{[A}\mathcal{F}_{BC]}$, $\mathcal{F}_{BC} = \mathcal{E}_B{}^M \mathcal{E}_C{}^N F_{MN}$, is:

$$\begin{aligned}\delta^\star \partial_{[A}\mathcal{F}_{BC]} &= -2\partial_{[B} U^K \partial_A \mathcal{F}_{C]K} + \partial_K \mathcal{F}_{[BC}\partial_{A]} U^K + U^K \partial_K\left(\partial_{[A}\mathcal{F}_{BC]}\right) = \\ &= \partial_{[B} U^K \partial_A \mathcal{F}_{KC]} - \partial_K U^K\, \partial_{[A}\mathcal{F}_{BC]} + U^K \partial_K\left(\partial_{[A}\mathcal{F}_{BC]}\right).\end{aligned}$$

The Wess-Zumino term

Unlike the case of $N=2$, $d=4$ Born-Infeld theory, in which the Wess-Zumino term is absent, the $N=2$, $d=6$ theory contains term, quadratic in the field strength. In vector notations, it can be written as

$$\mathcal{L}_{WZ} \sim \det\mathcal{E}\,\epsilon^{ABCDMN}\psi^{\alpha}_{i}\,\mathcal{D}_A\psi^{\beta i}\left(\mathcal{E}^{-1}\right)_B{}^K\left(\gamma_K\right)_{\alpha\beta}F_{CD}F_{MN}.$$

With already known transformation laws, it is easy to write down its variation

$$\begin{aligned}
\delta^{\star}_{S}\mathcal{L}_{WZ} \sim\; & \det\mathcal{E}\,\epsilon^{ABCDMN}\varepsilon^{\alpha}_{i}\,\mathcal{D}_A\psi^{\beta i}\left(\mathcal{E}^{-1}\right)_B{}^K\left(\gamma_K\right)_{\alpha\beta}F_{CD}F_{MN} - \\
& -\det\mathcal{E}\,\epsilon^{ABCDMN}\psi^{\alpha}_{i}\,\mathcal{D}_A\psi^{\beta i}\mathcal{D}_B U^K\left(\gamma_K\right)_{\alpha\beta}F_{CD}F_{MN} + \partial_A\left(U^A\mathcal{L}_{WZ}\right) = \\
\sim\; & \epsilon^{ABCDMN}\varepsilon^{\alpha}_{i}\,\partial_A\psi^{\beta i}\left(\gamma_B\right)_{\alpha\beta}\mathcal{E}_C{}^K\mathcal{E}_D{}^L F_{KL}\,\mathcal{E}_M{}^P\mathcal{E}_N{}^Q F_{PQ} + \partial_A\left(U^A\mathcal{L}_{WZ}\right) - \\
& -\epsilon^{ABCDMN}\psi^{\alpha}_{i}\,\partial_A\psi^{\beta i}\varepsilon^{\mu}_{j}\partial_B\psi^{\nu j}\epsilon_{\alpha\beta\mu\nu}\mathcal{E}_C{}^K\mathcal{E}_D{}^L F_{KL}\,\mathcal{E}_M{}^P\mathcal{E}_N{}^Q F_{PQ} = \\
\sim\; & \epsilon^{ABCDMN}\varepsilon^{\alpha}_{i}\,\partial_A\psi^{\beta i}\left(\gamma_B\right)_{\alpha\beta}\mathcal{F}_{CD}\,\mathcal{F}_{MN} + \partial_A\left(U^A\mathcal{L}_{WZ}\right) + \\
& +\epsilon^{ABCDMN}\epsilon_{\alpha\beta\mu\nu}\partial_B\left(\varepsilon^{\mu}_{j}\psi^{\nu j}\,\psi^{\alpha}_{i}\partial_A\psi^{\beta i}\right)\mathcal{F}_{CD}\,\mathcal{F}_{MN},
\end{aligned}$$

which is full divergence due to previously established Bianchi identities.

The unbroken supersymmetry of the action

Using the usual formula for the transformation of the superfield components,

$$\delta^\star_Q f = -\epsilon^\alpha_i D^i_\alpha \mathbf{f}| \equiv -\epsilon^\alpha_i \nabla^i_\alpha \mathbf{f}| + H^{\mu\nu}\partial_{\mu\nu} f,\ H^{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon^\lambda_i V_\lambda{}^{[\mu}\psi^{\nu]i}.$$

With equation of motion for the auxiliary field taken into account, $B^{ij} = 0$, the transformation laws for the basic components are

$$\begin{aligned}\delta^\star_Q \psi^\alpha_i &= -\frac{1}{2}\epsilon^{\beta i} V_\beta{}^\alpha + H^{\rho\sigma}\partial_{\rho\sigma}\psi^\alpha_i,\\ \delta^\star_Q V_\alpha{}^\beta &= -4\mathrm{i}\epsilon^\gamma_i \triangle_{\gamma\alpha}\psi^{i\beta} - 4\mathrm{i}\epsilon^\gamma_i Z_\gamma{}^\beta \triangle_{\alpha\mu}\psi^{\nu i}\left(Z^{-1}\right)_\nu{}^\mu + \\ &\quad +2\mathrm{i} Z_\alpha{}^\beta \epsilon^\gamma_i \triangle_{\gamma\mu}\psi^{\nu i}\left(Z^{-1}\right)_\nu{}^\mu + H^{\rho\sigma}\partial_{\rho\sigma} V_\alpha{}^\beta.\end{aligned}$$

Here $\triangle_{\alpha\beta} = \partial_{\alpha\beta} + \frac{1}{4} V_\alpha{}^\mu V_\beta{}^\nu \partial_{\mu\nu}$. If one wants to check the invariance of the action up to first order in the fermions, the $H^{\mu\nu}$ terms are not relevant. Then one may check, that variation of the Lagrangian, which in spinor notations reads

$$\begin{aligned}\mathcal{L} &= -(1 - \mathrm{i}\partial_{\alpha\beta}\psi^\alpha_i\psi^{\beta i})\Big(1 + \sqrt{1 - \frac{\mathrm{tr}(F^2)}{4} - \frac{(\mathrm{tr}(F^2))^2}{64} + \frac{\mathrm{tr}(F^4)}{16} - \frac{(\mathrm{tr}(F^3))^2}{576}}\Big) + \\ &\quad +\frac{1}{8}\big(-4\psi^{[\alpha}_i\partial_{\mu\nu}\psi^{\nu]i}(F^2)_\alpha{}^\mu + \mathrm{tr}(F^2)\psi^\alpha_i\partial_{\alpha\beta}\psi^{\beta i}\big),\end{aligned}$$

vanishes due to the Bianchi identities.

Conclusion

In vector notations, the full action of the $N = 2$, $d = 6$ Born-Infeld theory, with all fermions taken into account, reads

$$S = -\int d^6x \det\mathcal{E}\left(1 + \sqrt{-\det\left(\eta_{AB} + F_{AB}\right)}\right) + \\ +\frac{1}{16}\int d^6x \det\mathcal{E}\,\epsilon^{ABCDMN}\psi_i^{\alpha}\,\mathcal{D}_A\psi^{\beta i}\left(\mathcal{E}^{-1}\right)_B{}^K\left(\gamma_K\right)_{\alpha\beta}F_{CD}F_{MN}.$$

It was derived within the nonlinear realizations formalism, which allowed us to find all necessary ingredients and, in particular, the transformation laws, the derivatives, covariant with respect to both supersymmetries, and the irreducibility conditions of the multiplet from the assumption that they should be covariant with respect broken supersymmetry and and the $SO(4)$ automorphism group of the $N = (2, 0)$, $d = 6$ superalgebra. As a consequence of these constraints, the Bianchi identities and relation of the physical field strength to the multiplet components were found. Then the knowledge of the broken supersymmetry allowed to write the covariant ansatz for the action and prove its invariance with respect to the unbroken supersymmetry, in the lowest order in fermions. One of important issues for further study is the consistency of the superfield constraints in all orders in the auxiliary field and the fermions. It is also desirable to explicitly check that the fermionic forms in the Bianchi identities are strictly those determined by the broken supersymmetry.