Orthogonal and symplectic Yangians: linear and quadratic evaluations

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Outline

- $g\ell(n)$, $so(n)$ and $sp(2m)$ algebras
- RLL Yang-Baxter relation
- Truncated Yangians $\mathcal{Y}^{(p)}(G)$
- General solution
- Degenerate solutions, Fusion

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Nucl.Phys.B (2018)
and Theor.Math.Phys. (to be published)

Orthogonal Algebra

$g\ell(n)$ algebra:

$$[G^{a_1}{}_{b_1}, G^{a_2}{}_{b_2}] = \delta^{a_1}_{b_2} G^{a_2}{}_{b_1} - \delta^{a_2}_{b_1} G^{a_1}{}_{b_2}, \qquad [G_1, G_2] = [P_{12}, G_1] = -[P_{12}, G_2],$$

$so(n)$ algebra differs by the additional restriction $G_{ab} = -G_{ba}$:

$$[G^{a_1}{}_{b_1}, G^{a_2}{}_{b_2}] = \delta^{a_1}_{b_2} G^{a_2}{}_{b_1} - \delta^{a_2}_{b_1} G^{a_1}{}_{b_2} + \varepsilon^{a_1 a_2} G_{b_1 b_2} - \varepsilon_{b_1 b_2} G^{a_2 a_1},$$

here $\varepsilon_{ab} = \delta_{ab}$ ($\varepsilon^{ab} = \delta^{ab}$ is used for $so(n)$ and $\varepsilon_{ab} = (-1)^a \delta_{ab}$ for $so(m, n-m)$. In abstract notations this algebra relation looks like:

$$[G_1, G_2] = [P_{12} - K_{12}, G_1] = -[P_{12} - K_{12}, G_2],$$

where the invariant operator K_{12} has components:

$$K^{a_1 a_2}_{b_1 b_2} = \varepsilon^{a_1 a_2} \varepsilon_{b_1 b_2}.$$

preserves the bilinear form $[\mathrm{x}, \mathrm{y}] = \sum_{a=1}^{k} (\mathrm{x}_a \mathrm{y}_{-a} - \mathrm{x}_{-a} \mathrm{y}_a)$, x_a – coordinates and $p_a = x_{-a}$ – momenta of phase space. Generators

$$G_b^a = \mathrm{x}_a \partial_b - \varepsilon_a \varepsilon_b \mathrm{x}_{-b} \partial_{-a}, \qquad \varepsilon_a = sign(a), \qquad a, b = \pm 1, \pm 2, \ldots \pm m,$$

are skew-symmetric:

$$G_{ab} = -\varepsilon_a \varepsilon_b G_{-b,-a},$$

w.r.t. the metric

$$\varepsilon_{ab} = \varepsilon_a \delta_{a,-b}, \qquad \varepsilon^{bc} = -\varepsilon_b \delta^{b,-c}, \qquad \varepsilon_{ab} \varepsilon^{bc} = \delta_a^c$$

and form the algebra:

$$[G_{ab}, G_{cd}] = \delta_{bc} G_{ad} - \delta_{ad} G_{cb} + \varepsilon_c \varepsilon_b \delta_{b,-d} G_{a,-c} - \varepsilon_a \varepsilon_b \delta_{a,-c} G_{-b,d}.$$

Introducing a discrete parameter ϵ: ($\epsilon = +1$ in orthogonal and $\epsilon = -1$ in symplectic case):

$$\varepsilon_{ab} = \epsilon \varepsilon_{ba}, \qquad G_{ab} = -\epsilon G_{ba}, \quad \text{or} \quad K_{12}(G_1 + G_2) = (G_1 + G_2) K_{12},$$

one describes both algebras uniformly.

Yang-Baxter relation

The fundamental Yang-Baxter rquation:

$$R^{a_1 a_2}_{b_1 b_2}(u) R^{b_1 a_3}_{c_1 b_3}(u+v) R^{b_2 b_3}_{c_2 c_3}(v) = R^{a_2 a_3}_{b_2 b_3}(v) R^{a_1 b_3}_{b_1 c_3}(u+v) R^{b_1 b_2}_{c_1 c_2}(u).$$

In $s\ell(n)$-case the fundamental solution is given by

$$R^{a_1 a_2}_{b_1 b_2}(u) = u\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}, \qquad R(u) = uI + P.$$

In $so(n)$ the fundamental solution looks more complicated

$$R^{a_1 a_2}_{b_1 b_2}(u) = u(u-\alpha)\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + (u-\alpha)\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} - u\delta^{a_1 a_2}\delta_{b_1 b_2}, \qquad \alpha = 1 - n/2$$

In symplectic $sp(2m)$ case the fundamental solution looks quite similar:

$$R^{a_1 a_2}_{b_1 b_2}(u) = u(u+\beta)\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + (u+\beta)\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} - u\varepsilon_{a_2}\varepsilon_{b_2}\delta^{a_1 \bar{a}_2}\delta_{b_1 \bar{b}_2}, \qquad \beta = 1 + m$$

In the frame of Quantum inverse scattering method the new solution can be obtained by the fusion of fundamental ones:

$$T^{a_{01},\, a_1 \ldots a_n}_{a_{0,n+1},\, b_1 \ldots b_n}(u) = R^{a_{01} a_1}_{a_{02} b_1}(u) R^{a_{02} a_2}_{a_{03} b_2}(u) \ldots R^{a_{0n} a_n}_{a_{0,n+1} b_n}(u).$$

The irreducible parts of the Monodromy matrix $T(u)$ obtained by (anti)symmetrization of indices $a_1 \ldots a_n$ correspond to higher spin solutions of YBE.

The RLL-equation as a defining relation

RLL Yang-Baxter equation:

$$R^{a_1 a_2}_{b_1 b_2}(u) L^{b_1}_{c_1}(u+v) L^{b_2}_{c_2}(v) = L^{a_2}_{b_2}(v) L^{a_1}_{b_1}(u+v) R^{b_1 b_2}_{c_1 c_2}(u),$$

used to define the Yangian algebra.
Being given by the L-operator, acting in fundamental and an arbitrary representation in $g\ell(n)$-case one can rise the problem to determine $\mathfrak{R}$-operator, acting in two arbitrary representation spaces (indexes run (infinite-dimensional) range corresponding to an arbitrary representation).

In more complicated so(n) and sp(n) cases the more modest (the inverse) problem stands: to determine the most general L-operator, acting in fundamental and an arbitrary representation, if the fundamental R-matrix is given. Here indexes run the finite range, corresponding to fundamental representation.

Yangian Y(a) is an infinite-dimensional Hopf algebra, associated with given algebra a in following sense: let R(u) is fundamental R-matrix related to algebra a, then ternary RTT-relation R(u-v)T(u)T(v)=T(v)T(u)R(u-v) generates Yangian defining relations for:

$$T_{ij}(u) = \sum_{k=0}^{\infty} t_{ij}^{(k)} u^{-k}, \qquad t_{ij}^{(0)} = \delta_{ij}.$$

The simplest example is $g\ell(n)$-algebra,

$$[t_{ij}^{(s+1)}, t_{k\ell}^{(p)}] - [t_{ij}^{(s)}, t_{k\ell}^{(p+1)}] = -(t_{kj}^{(s)} t_{i\ell}^{(p)} - t_{kj}^{(p)} t_{i\ell}^{(s)}).$$

This case admits the linear resolution: the series for T can be truncated at linear term:

$$T_{ij}(u) = u\delta_{ij} + t_{ij}^{(1)},$$

where $t_{ij}^{(1)}$ are generators of $g\ell(n)$-algebra.

The fundamental R-matrix can be written in the universal form:

$$\frac{1}{u^2v^2}R(u-v) = (\frac{1}{v}-\frac{1}{u})(\frac{1}{v}-\frac{1}{u}+\frac{\beta}{uv}) - (\frac{1}{uv^2}-\frac{1}{u^2v}+\frac{\beta}{u^2v^2})P - \epsilon(\frac{1}{uv^2}-\frac{1}{u^2v})K.$$

here $\beta = (\frac{n}{2}-\epsilon)$, $\epsilon = +1$ for SO and $\epsilon = -1$ for SP. Then the defining relations for the generators $(L^{(k)})^a_b$ of the Yangians $Y(\mathcal{G})$:

$$[L_1^{(k)},\ L_2^{(j-2)}] - 2[L_1^{(k-1)},\ L_2^{(j-1)}] + [L_1^{(k-2)},\ L_2^{(j)}] +$$
$$+\beta([L_1^{(k-1)},\ L_2^{(j-2)}] - [L_1^{(k-2)},\ L_2^{(j-1)}]) +$$
$$+ P\Big(L_1^{(k-1)}\ L_2^{(j-2)} - L_1^{(k-2)}\ L_2^{(j-1)} + \beta L_1^{(k-2)}\ L_2^{(j-2)}\Big) -$$
$$- \Big(L_2^{(j-2)}L_1^{(k-1)} - L_2^{(j-1)}L_1^{(k-2)} + \beta L_2^{(j-2)}\ L_1^{(k-2)}\Big)P +$$
$$+\epsilon\Big(K\,(L_1^{(k-2)}\ L_2^{(j-1)} - L_1^{(k-1)}\ L_2^{(j-2)}) - (L_2^{(j-1)}L_1^{(k-2)} - L_2^{(j-2)}L_1^{(k-1)})\,K\Big) = 0\,,$$

Inner automorphysms of Yangian algebra

The RLL-relation has translational symmetry: $u \to u + a$ as well as is invariant upon rescaling

$$L(u) \to f(u)L(u),$$

where $f(u) = 1 + b_1/u + b_2/u^2 + \ldots$ is scalar function. Consider

$$L(u) \to \frac{(u-a)^k}{u^k} L(u),$$

at $k = 1$ one has

$$L^{(1)} \to L^{(1)} - aI_n, \qquad L^{(2)} \to L^{(2)} - aL^{(1)}, \qquad L^{(3)} \to L^{(3)} - aL^{(2)}, \ldots$$

Taking $a = \frac{1}{n}\mathrm{Tr}L^{(1)}$ one can make $L^{(1)}$ traceless.

Orthogonal and Symplectic Yangians also can be truncated at some finite order k:

$$L(u) = I + u^{-1}L^{(1)} + \ldots + u^{-k}L^{(k)},$$

here $L^{(0)} = I$ unity operator.
We consider the linear:

$$L(u) = uI + G, \qquad G^{(1)} = G,$$

the quadratic:

$$L(u) = u^2 I + uG + H, \qquad G^{(1)} = G, \quad G^{(2)} = H,$$

and the cubic evaluation:

$$L(u) = u^3 I + u^2 G + uH + J, \qquad G^{(1)} = G, \quad C^{(2)} = H, \quad G^{(3)} = J.$$

It is convenient to assign the scale dimension to u: $[u] = 1$, then $[G^{(k)}] = k$.
In contrast with the $g\ell(n)$ case, generators G, H, J are not arbitrary, they are subjected to the symmetry constraints and to the additional restrictions following from the RLL-relation.

Linear resolution

We start with the linear ansatz:

$$L^a{}_b(u) = u\delta^a_b + G^a{}_b.$$

Then defining RLL-relation takes the form:

$$(u(u+\beta)I_{12} + (u+\beta)P_{12} - u\epsilon K_{12})(u+v+G_1)(v+G_2) =$$

$$= (v+G_2)(u+v+G_1)(u(u+\beta)I_{12} + (u+\beta)P_{12} - u\epsilon K_{12}),$$

here $I_{12} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}$, $P_{12} = \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}$, $K_{12} = \varepsilon^{a_1a_2}\varepsilon_{b_1b_2}$. One can be rewritten it:

$$(u+\beta)\Big([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]\Big) - \epsilon v[K_{12}, G_1 + G_2] -$$

$$-\epsilon K_{12}(G_1 - \beta)G_2 + \epsilon G_2(G_1 - \beta)K_{12} = 0.$$

It has to take place identically by powers of u and v, which implies three restrictions on generators G:

$$-v\mathfrak{C}^{(1,1)} = -\epsilon v[K_{12}, G_1 + G_2] = 0, \tag{1}$$

$$(u+\beta)\mathfrak{C}^{(1,2)} = (u+\beta)\Big([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]\Big) = 0, \tag{2}$$

$$-\mathfrak{C}^{(1,3)} = -\epsilon\Big(K_{12}(G_1 - \beta)G_2 - G_2(G_1 - \beta)K_{12}\Big) = 0. \tag{3}$$

The first constraint just tells that the Yangian generators must be ε-antisymmetric (up to the unity matrix)

$$G = g + \bar{G}, \qquad \bar{G}^{ba} = -\epsilon \bar{G}^{ab},$$

like the generators of the Lie algebra $so(n)$ or $sp(2m)$. *The scalar parameter g*

singled out above is just the trace of generator G and can be treated as a center of the algebra. It can be excluded by imposing the additional (unitarity) condition.

The second constraint just states that the first Yangian generator $G^{(1)} = G$ satisfies to $so(n)$ or $sp(2m)$ algebra relations. The finite-dimensional linearly truncated Yangian differs from the corresponding $so(n)$ or $sp(2m)$ Lie algebra by the additional third constraint, which specified unique (resolution) representation.
In linear case this is the usual spinor representation in orthogonal case and its (infinite-dimensional) analogue in the symplectic case.

Quadratic ansatz

In general case the number of constraints is $(p+1)^2 - 1 = p(p+2)$, here p is the number of Yangian $Y^{(p)}(\mathcal{G})$ generators. $L(u) = u^2 I + uG + H$:

$$\mathfrak{C}^{(2,1)} = [K_{12}, G_1 + G_2] = 0,$$

$$\mathfrak{C}^{(2,2)} = ([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]) = 0,$$

$$\mathfrak{C}^{(2,3)} = K_{12}(H_1 + H_2 + (G_1 - \beta)G_2) - (G_2(G_1 - \beta) + H_1 + H_2)K_{12} = 0,$$

$$\mathfrak{C}^{(2,4)} = ([G_1, H_2] + (H_1 - H_2)P_{12} - \epsilon[K_{12}, H_2]) = 0,$$

$$\mathfrak{C}^{(2,5)} = ([H_1, G_2] + (H_1 - H_2)P_{12} - \epsilon[K_{12}, H_1]) = 0,$$

$$\mathfrak{C}^{(2,6)} = K_{12}(H_1(G_2 + \beta) + (G_1 - \beta)H_2) - (H_2(G_1 - \beta) + (G_2 + \beta)H_1)K_{12} = 0.$$

$$\mathfrak{C}^{(2,7)} = \Big([H_1, H_2] + (G_2 H_1 - H_2 G_1)P_{12} - \epsilon K_{12}(G_1 - \beta)H_2 + H_2(G_1 - \beta)\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(2,8)} = \epsilon K_{12}(H_1 - \beta G_1 + \beta^2)H_2 - H_2(H_1 - \beta G_1 + \beta^2)\epsilon K_{12} = 0.$$

Symmetric and antisymmetric constraints

The set of defining equations above is equivalent to the following set of equations with definite symmetry with respect to $1 \leftrightarrow 2$:

$$\mathfrak{A}\mathfrak{C}^{(2,1)} = 0, \qquad \mathfrak{S}\mathfrak{C}^{(2,1)} = \mathfrak{C}^{(2,1)} = [P_{12} - \epsilon K_{12}, G_1 + G_2] = 0,$$

$$\mathfrak{S}\mathfrak{C}^{(2,2)} = \mathfrak{S}\mathfrak{C}^{(2,1)}, \qquad \mathfrak{A}\mathfrak{C}^{(2,2)} = [G_1, G_2] - \frac{1}{2}[P_{12} - \epsilon K_{12}, G_1 - G_2] = 0,$$

$$\mathfrak{A}\mathfrak{C}^{(2,3)}_L = (1 - \epsilon P_{12})\mathfrak{C}^{(2,3)} = \Big([G_1, G_2] - \beta(G_1 - G_2)\Big) K_{12} = \mathfrak{A}\mathfrak{C}^{(2,2)} K_{12},$$

$$\mathfrak{A}\mathfrak{C}^{(2,3)}_R = \mathfrak{C}^{(2,3)}(1 - \epsilon P_{12}) = K_{12}\Big([G_1, G_2] + \beta(G_1 - G_2)\Big) = K_{12}\mathfrak{A}\mathfrak{C}^{(2,2)},$$

$$\mathfrak{S}\mathfrak{C}^{(2,3)} = [P_{12} - \epsilon K_{12}, H_1 + H_2 - \frac{1}{2}(G_1^2 + G_2^2)] + \{\mathfrak{C}^{(2,1)}, G_1 + G_2\} = 0,$$

$$\mathfrak{S}\mathfrak{C}^{(2,4)} = [G_1, H_2 - \frac{1}{2}G_2^2] + [G_2, H_1 - \frac{1}{2}G_1^2] - \mathfrak{S}\mathfrak{C}^{(2,3)} - \frac{1}{2}\{G_1 - G_2, \mathfrak{S}\mathfrak{C}^{(2,2)}\},$$

$$\mathfrak{A}\mathfrak{C}^{(2,4)} = [G_1, H_2] - [G_2, H_1] - [P_{12} - \epsilon K_{12}, H_1 - H_2] = 0,$$

$$\mathfrak{AC}_L^{(2,6)} = \mathfrak{C}^{(2,6)}(1-\epsilon P_{12}) = K_{12}\mathfrak{AC}^{(2,4)}, \qquad \mathfrak{AC}_R^{(2,6)} = (1-\epsilon P_{12})\mathfrak{C}^{(2,6)} = \mathfrak{AC}^{(2,4)}K_{12},$$

$$\mathfrak{SC}^{(2,6)} = \frac{1}{2}(1+\epsilon P_{12})\mathfrak{C}^{(2,6)}(1+\epsilon P_{12}) = [P_{12} - \epsilon K_{12}, \{H_1, G_2\} + \{G_1, H_2\}],$$

$$\mathfrak{SC}^{(2,7)} = \mathfrak{SC}^{(2,4)}P_{12} - \frac{\epsilon}{2}\{K_{12}, \mathfrak{SC}^{(2,4)}\} - \frac{\epsilon}{2}\mathfrak{SC}^{(2,6)},$$

$$\mathfrak{AC}^{(2,7)} = [H_1, H_2] + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, H_2\} - \{G_2, H_1\}] - \frac{\epsilon}{4}\{K_{12}, \mathfrak{AC}^{(2,4)}\},$$

$$\mathfrak{AC}_L^{(2,8)} = \mathfrak{C}^{(2,8)}(1-\epsilon P_{12}) = \epsilon K_{12}\Big(\mathfrak{AC}^{(2,7)} + \frac{\epsilon - n}{4}\mathfrak{AC}^{(2,4)}\Big),$$

$$\mathfrak{SC}^{(2,8)} = [K_{12}, \{H_1, H_2\} - \beta\epsilon(H_1 + H_2)] - \frac{\beta}{2}\{K_{12}, \mathfrak{SC}^{(2,4)}\} + \frac{\beta\epsilon}{2}\mathfrak{SC}^{(2,6)},$$

So one deduces that the independent constraints are: $\mathfrak{SC}^{(2,1)}$, $\mathfrak{SC}^{(2,3)}$, $\mathfrak{SC}^{(2,6)}$ and $\mathfrak{SC}^{(2,8)}$ ("symmetric" constraints) and $\mathfrak{AC}^{(2,2)}$, $\mathfrak{C}^{(2,4)} = P_{12}\mathfrak{C}^{(2,5)}P_{12}$ and $\mathfrak{AC}^{(2,7)}$ ("algebra" constraints).

In quadratic case the Yangian algebra form two generators: $G^{(1)} = G$ and $G^{(2)} = H$. The appearance of the new generator H lifts the third constraint on G^2 specifying the resolution representation in linear case and expresses the symmetry restriction on new generator H instead. So, like in the first constraint $\mathfrak{C}^{(2,1)}$ specifies the ε-symmetric part of G to be proportional to the unity operator, $\mathfrak{C}^{(2,3)}$ fixes the ε-symmetric part of H:

$$H = h + \frac{1}{2}\bar{G}^2 + \bar{H}, \quad K_{12}(\bar{H}_1 + \bar{H}_2) = 0 = (\bar{H}_1 + \bar{H}_2)K_{12} \quad \Leftrightarrow \quad \bar{H}_{ab} = -\epsilon \bar{H}_{ba}.$$

One can rewrite the remaining set of $p(p+1)$ equations in terms of parameters g, h and "independent" (ε-antisymmetric) generators $\bar{G}$ and $\bar{H}$:

$$\bar{\mathfrak{C}}^{(2,2)} = [\bar{G}_1 + P_{12} - \epsilon K_{12}, \bar{G}_2] = 0,$$

$$\bar{\mathfrak{C}}^{(2,4)} = [\bar{G}_1 + P_{12} - \epsilon K_{12}, \bar{H}_2] = 0,$$

$$\bar{\mathfrak{C}}^{(2,5)} = -P_{12}\bar{\mathfrak{C}}^{(2,4)}P_{12} = [\bar{H}_1, \bar{G}_2 + P_{12} - \epsilon K_{12}] = 0,$$

$$\bar{\mathfrak{C}}^{(2,7)} = [\bar{H}_1, \bar{H}_2] - \frac{1}{4}[P_{12} - \epsilon K_{12}, \bar{G}_2^3 + 2\{\bar{H}_2, \bar{G}_2\} - 4g\bar{H}_2 - 2g\bar{G}_2^2 + 4h\bar{G}_2] +$$

$$+\frac{1}{4}(\bar{G}_2\bar{G}_1^2 - \bar{G}_2^2\bar{G}_1)P_{12} + \frac{\epsilon}{4}(\bar{G}_2^2 K_{12}\bar{G}_2 - \bar{G}_2\bar{K}_{12}G_2^2) = 0,$$

define the Yangian algebra between generators $\bar{G}$ and $\bar{H}$ and the two higher "symmetric" constraints:

$$2\bar{\mathfrak{C}}^{(2,6)} = -[K_{12}, \{\bar{H}_1, \bar{G}_1\} + \{\bar{H}_2, \bar{G}_2\} - (\beta + g)(\bar{G}_1^2 + \bar{G}_2^2)] = 0,$$

and

$$2\bar{\mathfrak{C}}^{(2,8)} = [K_{12}, \Big(-\bar{H}_1^2 - \bar{H}_2^2 + \frac{1}{4}(\bar{G}_1^4 + \bar{G}_2^4) + (h + (g + \epsilon)\frac{\beta}{2})(\bar{G}_1^2 + \bar{G}_2^2)\Big)] = 0,$$

define restrictions:

$$\{\bar{H}, \bar{G}\} + 2\beta\bar{H} = (\beta + g)(\bar{G}^2 + \beta\bar{G}) - c^{(2,6)},$$

and

$$\bar{H}^2 = -\frac{1}{4}\bar{G}^4 - \frac{n}{4}\bar{G}^3 - \beta(\beta + g)\bar{H} - (h + \frac{1}{2} + \frac{\beta}{2}(n + g + 2\epsilon)\bar{G}^2 +$$

$$+(\frac{\beta}{8}(n - \epsilon)(n - 4\epsilon) + \frac{\beta^2}{2}(n + 2g - \frac{3\epsilon}{2}) + \frac{m_2}{2})\bar{G} + \frac{1}{2}c^{(2,8)} - \frac{n + \epsilon}{4}m_2,$$

specifying the particular Yangian resolution representation in quadratic case.

Let us summarize: the quadratic ($p = 2$) resolution of the Yangian $Y(\mathcal{G})$ is determined by eight constraint. The four (p^2) constraints $\mathfrak{C}^{(2,2)}$, $\mathfrak{C}^{(2,4)}$, $\mathfrak{C}^{(2,5)}$ and $\mathfrak{C}^{(2,7)}$ contain commutator between generators and form the Yangian algebra (like in $g\ell(n)$ case). All these constraints are antisymmetric with respect to auxiliary space index $1 \leftrightarrow 2$ (their symmetric parts reduce to lower-dimensional constraints). Remaining four ($2p$) constraints are symmetric: two of them with the lower dimension $\mathfrak{C}^{(2,1)}$ and $\mathfrak{C}^{(2,3)}$ impose the restrictions on ε-symmetric parts of generators G and H (relate them to the numerical parameters and to the lower generator(s)). These restrictions correspond to (1) of the linear case, which just declares the difference between *so*, *sp* and $g\ell$. So Yangians $Y_O^{(p)}$ and $Y_S^{(p)}$ like $Y(g\ell(n))$ consist of p^2 commutator algebra relations, but obey also p symmetry constraint, which fix the ε-symmetric part of each generator and p additional algebraic relations, which fix the anticommutators of the highest generator with the remaining ones.

Composite solutions (with the lower number of generators)

Along with the most general solution obtained above there exist also particular ones, corresponding to the case when one or more highest dimensional Yangian generators ($\bar{H}$ in the case under consideration) are absent. There are two possibilities: the first one is trivial, when the quadratic ansatz is in fact linear ($L(u) = (u + a)(u + b + G)$. The second degeneracy corresponds to the case of the single generator $\bar{G}$ ($\bar{H} = 0$ or $\bar{H} = a\bar{G}$, where a is (a dimensionful) numerical parameter). So let us set:

$$\bar{H} = a\bar{G},$$

then one has using $\mathfrak{C}^{(2,3)}$:

$$H = h + \frac{1}{2}\bar{G}^2 + a\bar{G}.$$

$\mathfrak{C}^{(2,4)}$ and $\mathfrak{C}^{(2,5)}$ then are reduced to $\mathfrak{C}^{(2,2)}$. The next constraint $\mathfrak{C}^{(2,6)}$ then tells:

$$2a = g + \beta,$$

and the further restrictions come from $\mathfrak{C}^{(2,7)}$ and $\mathfrak{C}^{(2,8)}$:

The first leads to:

$$W = -\bar{G}_2(P_{12} - \epsilon K_{12})\bar{G}_2 - \epsilon(P_{12} - \epsilon K_{12})\bar{G}_2 + (\epsilon \bar{G}_2 + 1)\bar{G}_1 = 0,$$

or in components:

$$W^{a_2}{}_{dc_1c_2} = (\bar{G}^{a_2}{}_d + \epsilon\delta^{a_2}_d)\bar{G}_{c_1c_2} + (\bar{G}^{a_2}{}_{c_2} + \epsilon\delta^{a_2}_{c_2})\bar{G}_{dc_1} + (\bar{G}^{a_2}{}_{c_1} + \epsilon\delta^{a_2}_{c_1})\bar{G}_{c_2d} =$$

$$= \frac{1}{2}\Big(\{\bar{G}^{a_2}{}_d, \bar{G}_{c_1c_2}\} + \{\bar{G}^{a_2}{}_{c_2}, \bar{G}_{dc_1}\} + \{\bar{G}^{a_2}{}_{c_1}, \bar{G}_{c_2d}\}\Big),$$

so called cyclic constraint, while the last relation takes the form:

$$[K_{12}, (\bar{G}_1^2 + \bar{G}_2^2)^2 + b(\bar{G}_1^2 + \bar{G}_2^2)] = 0,$$

which is equivalent to (3).
Compare now at $g\ell(n)$ case, the quadratic resolution of $Y(g\ell(n))$ is specified by two unconstrained generators $G^{(1)}$, $G^{(2)}$ which obey some algebra. The quadratic solution $L_1(u) = u^2 + uG_1^{(1)} + G_1^{(2)}$ corresponds to the fusion of two linear ones $L_{13}(u) = u + a + G_{13}^{(1)}$ and $L_{14}(u) = u + b + G_{14}^{(1)}$ at $G_1^{(2)} = (a + G_{13}^{(1)})(b + G_{14}^{(1)})$.

Cubic ansatz

$$\mathfrak{C}^{(3,1)} = \epsilon[K_{12}, G_1 + G_2] = 0,$$

$$\mathfrak{C}^{(3,2)} = ([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]) = 0,$$

$$\mathfrak{C}^{(3,3)} = \epsilon K_{12}(H_1 + H_2 + (G_1 - \beta)G_2) - (G_2(G_1 - \beta) + H_1 + H_2)\epsilon K_{12} = 0,$$

$$\mathfrak{C}^{(3,4)} = ([G_1, H_2] + (H_1 - H_2)P_{12} - \epsilon[K_{12}, H_2]) = 0,$$

$$\mathfrak{C}^{(3,5)} = ([H_1, G_2] + (H_1 - H_2)P_{12} - \epsilon[K_{12}, H_1]) = 0,$$

$$\mathfrak{C}^{(3,6)} = \epsilon K_{12}(J_1 + J_2 + H_1(G_2 + \beta) + (G_1 - \beta)H_2) -$$
$$(H_2(G_1 - \beta) + (G_2 + \beta)H_1 + J_1 + J_2)\epsilon K_{12} = 0.$$

$$\mathfrak{C}^{(3,7)} = [J_1, G_2] + (J_1 - J_2)P_{12} - \epsilon K_{12}(J_2 + H_1(G_2 + \beta) + (G_1 - \beta)H_2) -$$
$$-(H_2(G_1 - \beta) + (G_2 + \beta)H_1 + J_2)\epsilon K_{12} = 0,$$

$$\mathfrak{C}^{(3,8)} = \Big([H_1, H_2] + (J_1 - J_2 + G_2 H_1 - H_2 G_1)P_{12} -$$
$$-\epsilon K_{12}(J_2 + (G_1 - \beta)H_2) - (H_2(G_1 - \beta) + J_2)\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,9)} = [G_1, J_2] + (J_1 - J_2)P_{12} - \epsilon[K_{12}, J_2],$$

$$\mathfrak{C}^{(3,10)} = \Big([H_1, J_2] + (G_2 J_1 - J_2 G_1)P_{12} - \epsilon K_{12}(G_1 - \beta)J_2) - J_2(G_1 - \beta)\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,11)} = \Big([J_1, H_2] + (G_2 J_1 - J_2 G_1)P_{12} - \epsilon K_{12}((G_1 - 2\beta)J_2 + (H_1 - \beta G_1 + \beta^2)H_2) -$$

$$-(H_2(H_1 - \beta G_1 + \beta^2) + J_2(G_1 - 2\beta))\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,12)} = \Big(-\epsilon K_{12}((G_1 - 3\beta)J_2 + H_1 H_2 - 2\beta(G_1 - \beta)H_2 - \beta H_1(G_2 + \beta) + J_1 G_2) -$$

$$-(J_2(G_1 - 3\beta) + H_2 H_1 - 2\beta(G_1 - \beta)H_2 - \beta H_1(G_2 + \beta) + G_2 J_1)\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,13)} = \Big([J_1, J_2] + (H_2 J_1 - J_2 H_1)P_{12} - \epsilon K_{12}(H_1 - \beta G_1 + \beta^2)J_2 +$$

$$+J_2(H_1 - \beta G_1 + \beta^2)\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,14)} = \Big(-\epsilon K_{12}((J_1 - \beta H_1 + \beta^2 G_1 - \beta^3)H_2 + (H_1 - 2\beta G_1 + 3\beta^2)J_2) +$$

$$+(H_2(J_1 - \beta H_1 + \beta^2 G_1 - \beta^3) + J_2(H_1 - \beta G_1 + \beta^2))\epsilon K_{12}\Big) = 0,$$

$$\mathfrak{C}^{(3,15)} = -\epsilon K_{12}(J_1 - \beta H_1 + \beta^2 G_1 - \beta^3)J_2 + J_2(J_1 - \beta H_1 + \beta^2 G_1 - \beta^3)\epsilon K_{12} = 0.$$

The above set of equations is equivalent to:

$$\mathfrak{SC}^{(3,1)} = \mathfrak{C}^{(3,1)} = [P_{12} - \epsilon K_{12}, G_1 + G_2] = 0,$$

$$\mathfrak{AC}^{(3,2)} = [G_1, G_2] - \frac{1}{2}[P_{12} - \epsilon K_{12}, G_1 - G_2] = 0,$$

$$\mathfrak{SC}^{(3,3)} = [P_{12} - \epsilon K_{12}, H_1 + H_2 - \frac{1}{2}(G_1^2 + G_2^2)] = 0,$$

$$\mathfrak{SC}^{(3,4)} = [G_1, H_2 - \frac{1}{2}G_2^2] + [G_2, H_1 - \frac{1}{2}G_1^2] = 0,$$

$$\mathfrak{AC}^{(3,4)} = [G_1, H_2] - [G_2, H_1] - [P_{12} - \epsilon K_{12}, H_1 - H_2] = 0,$$

$$\mathfrak{SC}^{(3,6)} = [P_{12} - \epsilon K_{12}, J_1 + J_2 + \frac{1}{2}(\{H_1, G_2\} + \{G_1, H_2\})] = 0,$$

$$\mathfrak{AC}^{(3,7)} = [G_1, J_2] - [G_2, J_1] - [P_{12} - \epsilon K_{12}, J_1 - J_2] = 0,$$

$$\mathfrak{SC}^{(3,7)} = [G_1, J_2] + [G_2, J_1] + [P_{12} - \epsilon K_{12}, J_1 + J_2] = 0,$$

$$\mathfrak{AC}^{(3,8)} = [H_1, H_2] + [P_{12} - \epsilon K_{12}, \frac{1}{2}(J_2 - J_1) + \frac{1}{4}(\{G_1, H_2\} - \{G_2, H_1\})\}] = 0,$$

$$\mathfrak{AC}^{(3,10)} = [H_1, J_2] - [H_2, J_1] + \frac{1}{2}[P_{12} - \epsilon K_{12}, \{G_1, J_2\} - \{G_2, J_1\}] = 0,$$

$$\mathfrak{SC}^{(3,10)} = [H_1, J_2] + [H_2, J_1] + \frac{\epsilon}{2}[K_{12}, \{H_1, H_2\} - \beta\epsilon(H_1 + H_2)] = 0,$$

$$\mathfrak{SC}'^{(3,12)} = [K_{12}, \{H_1, H_2\} - \epsilon\beta(H_1 + H_2) + \{G_1, J_2\} + \{J_1, G_2\}] = 0,$$

$$\mathfrak{AC}^{(3,13)} = [J_1, J_2] + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{H_1, J_2\} - \{H_2, J_1\}] = 0,$$

$$\mathfrak{SC}^{(3,14)} = \epsilon[K_{12}, \{J_1, H_2\} + \{J_2, H_1\} - 2\beta\epsilon(J_1 + J_2) + 2\beta^2\epsilon(H_1 + H_2)] = 0,$$

$$\mathfrak{SC}^{(3,15)} = \epsilon[K_{12}, \{J_1, J_2\} + \frac{\beta\epsilon}{2}\{H_1, H_2\} + \frac{\beta^2\epsilon}{4}(3\beta - 2\epsilon)(H_1 + H_2)].$$

Again, constraints $\mathfrak{AC}^{(3,2)}$, $\mathfrak{SC}^{(3,4)}$, $\mathfrak{AC}^{(3,4)}$, $\mathfrak{AC}^{(3,7)}$, $\mathfrak{SC}^{(3,7)}$, $\mathfrak{AC}^{(3,8)}$, $\mathfrak{AC}^{(3,10)}$, $\mathfrak{SC}^{(3,10)}$ and $\mathfrak{AC}^{(3,13)}$ express the Yangian algebra $Y^{(3)}(\mathcal{G})$. The lower-dimensional symmetric constraints: $\mathfrak{SC}^{(3,1)}$, $\mathfrak{SC}^{(3,3)}$ and $\mathfrak{SC}^{(3,6)}$ restrict the ε-symmetric parts of the generators:

$$G = g + \bar{G}, \qquad H = h + \frac{1}{2}\bar{G}^2 + \bar{H}, \qquad J = j + \frac{1}{2}\{\bar{H}, \bar{G}\} - \frac{\beta + g}{2}\bar{G}^2 + \bar{J}.$$

Substituting this solution to the remaining independent $p(p+1) = 12$ constraints:

$$\bar{\mathfrak{AC}}^{(3,2)} = [\bar{G}_1, \bar{G}_2] - \frac{1}{2}[P_{12} - \epsilon K_{12}, \bar{G}_1 - \bar{G}_2] = 0,$$

$$\bar{\mathfrak{SC}}^{(3,4)} = [\bar{G}_1, \bar{H}_2] + [\bar{G}_2, \bar{H}_1] = 0,$$

$$\bar{\mathfrak{AC}}^{(3,4)} = [\bar{G}_1, \bar{H}_2] - [\bar{G}_2, \bar{H}_1] - [P_{12} - \epsilon K_{12}, \bar{H}_1 - \bar{H}_2] = 0,$$

$$\bar{\mathfrak{S}\mathfrak{C}}^{(3,7)} = [\bar{G}_1, \bar{J}_2] + [\bar{G}_2, \bar{J}_1] = 0,$$

$$\bar{\mathfrak{A}\mathfrak{C}}^{(3,7)} = [\bar{G}_1, \bar{J}_2] - [\bar{G}_2, \bar{J}_1] - [P_{12} - \epsilon K_{12}, \bar{J}_1 - \bar{J}_2] = 0,$$

$$\bar{\mathfrak{A}\mathfrak{C}}^{(3,8)} = [\bar{H}_1, \bar{H}_2] + [P_{12} - \epsilon K_{12}, \frac{1}{8}(\bar{G}_1^3 - \bar{G}_2^3) - \frac{1}{2}(\bar{J}_1 - \bar{J}_2) - \frac{g}{2}(\bar{H}_1 - \bar{H}_2) + \frac{h}{2}(\bar{G}_1 - \bar{G}_2)] +$$

$$+\frac{1}{8}\Big(\bar{G}_2[P_{12} - \epsilon K_{12}, \bar{G}_2]\bar{G}_2 - \bar{G}_1[P_{12} - \epsilon K_{12}, \bar{G}_1]\bar{G}_1\Big) = 0,$$

$$\bar{\mathfrak{S}\mathfrak{C}}^{(3,10)} = [\bar{H}_1, \bar{J}_2] + [\bar{H}_2, \bar{J}_1] + \frac{1}{2}([\bar{G}_1^2, \bar{J}_2] + [\bar{G}_2^2, \bar{J}_1]) - \frac{1}{2}\{(\bar{G}_1 - \bar{G}_2), [\bar{H}_1, \bar{H}_2]\} +$$

$$+\frac{1}{4}\{(\bar{G}_1^2 - \bar{G}_2^2), [\bar{G}_1, \bar{H}_2]\} - \frac{1}{4}[[\bar{G}_1, \bar{G}_2], [\bar{G}_1, \bar{H}_2]] +$$

$$+\frac{\beta + g}{2}([\bar{G}_1^2, \bar{H}_2] + [\bar{G}_2^2, \bar{H}_1]) + \frac{\epsilon}{8}[K_{12}, \bar{G}_1^4 + \bar{G}_2^4 - 4\beta(\beta + \epsilon)(\bar{G}_1^2 + \bar{G}_2^2)] = 0,$$

$$\bar{\mathfrak{A}\mathfrak{C}}^{(3,10)} = [\bar{H}_1, \bar{J}_2] - [\bar{H}_2, \bar{J}_1] + \{(\bar{G}_1 + \bar{G}_2), \frac{1}{2}[\bar{H}_1, \bar{H}_2]\} + \frac{g(g+\beta)}{2}[\bar{G}_1, \bar{G}_2] +$$

$$+\{[\bar{G}_1, \bar{G}_2], j + \frac{1}{2}(\bar{J}_2 + \bar{J}_2) + \frac{1}{4}(\{\bar{H}_1, \bar{G}_1\} + \{\bar{H}_2, \bar{G}_2\}) - \frac{\beta + g}{4}(\bar{G}_1^2 + \bar{G}_2^2)\} +$$

$$+[P_{12} - \epsilon K_{12}, \frac{1}{2}(\bar{H}_1^2 - \bar{H}_2^2) - g(\bar{J}_1 - \bar{J}_2) - \frac{g}{2}(\{\bar{H}_1, \bar{G}_1\} - \{\bar{H}_2, \bar{G}_2\})],$$

$$\mathfrak{A}\bar{\mathfrak{C}}^{(3,13)} = [\bar{J}_1, \bar{J}_2] + \frac{1}{2}(\{\bar{G}_1, [\bar{H}_1, J_2]\} - \{\bar{G}_2, [\bar{H}_2, J_1]\}) + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{\bar{H}_1, \bar{J}_1\} - \{\bar{H}_2, \bar{J}_2\}] +$$

$$+ \frac{1}{8}[P_{12} - \epsilon K_{12}, \{\bar{G}_1^2, \bar{J}_2\} - \{\bar{G}_2^2, \bar{J}_1\}] + \frac{\beta + g}{2}([\bar{G}_1^2, \bar{J}_2] - [\bar{G}_2^2, \bar{J}_1]) - \frac{h}{2}[P_{12} - \epsilon K_{12}, \bar{J}_1 - \bar{J}_2] +$$

$$+ \frac{1}{4}[\{\bar{G}_1, \bar{H}_1\} - \{\bar{G}_2, \bar{H}_2\}] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \{\bar{H}_1, \{\bar{H}_2, G_2\}\} - \{\bar{H}_2, \{\bar{H}_1, G_1\}\}] +$$

$$+ \frac{1}{16}[P_{12} - \epsilon K_{12}, \{\bar{G}_1^2, \{\bar{H}_2, G_2\}\} - \{\bar{G}_2^2, \{\bar{H}_1, G_1\}\}] + \frac{h}{4}[P_{12} - \epsilon K_{12}, \{\bar{H}_2, G_2\} - \{\bar{H}_1, G_1\}] -$$

$$- \frac{\beta + g}{4}([\bar{G}_1^2, \{\bar{H}_2, G_2\}] - [\bar{G}_2^2, \{\bar{H}_1, G_1\}]) + \frac{\beta + g}{8}[P_{12} - \epsilon K_{12}, \{\bar{H}_2, \bar{G}_1^2\} - \{\bar{H}_1, \bar{G}_2^2\})] +$$

$$+ [P_{12} - \epsilon K_{12}, \frac{(\beta + g)h + j}{4}(\bar{G}_1^2 - \bar{G}_2^2) + \frac{j}{4}(\bar{H}_1 - \bar{H}_2)] + \frac{(\beta + g)^2}{4}[\bar{G}_1^2, \bar{G}_2^2],$$

The remaining three constraints: $\mathfrak{S}\mathfrak{C}^{(3,12)}$, $\mathfrak{S}\mathfrak{C}^{(3,14)}$ and $\mathfrak{S}\mathfrak{C}^{(3,15)}$ impose the algebraic restrictions on $\bar{J}^2$, $\{\bar{J}, \bar{H}\}$ and $\{\bar{J}, \bar{G}\}$, which specify the particular resolution representation.

Second order evaluation

$$L(u) = u^2 + uG + H,$$

$$G = g + \bar{G}, \qquad H = h + \frac{1}{2}(\bar{G}^2 + \beta\bar{G}) + \bar{H}.$$

$\bar{G}$ obeys the Lie algebra relation, $\bar{H}$ transforms as the adjoint representation.

$$\{\bar{G}, \bar{H}\} + 2\beta\bar{H} - g(\bar{G}^2 + \beta\bar{G}) = c^{(2.6)},$$

$$[\bar{H}_1, \bar{H}_2] + \frac{1}{8}[W_{12}, \bar{G}_1 - \bar{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \chi_1 - \chi_2 - 4g(\bar{H}_1 - \bar{H}_2)\alpha(\bar{G}_1 - \bar{G}_2)] = 0$$

$$\alpha = 4h + \beta^2 + 1 - 2\epsilon\beta + m_2\epsilon/2,$$

$$\bar{H}^2 = c^{(2.8)} + \frac{1}{4}\bar{G}^4 - g\beta\bar{H} + \beta G^3 + (\frac{5}{4}\beta^2 + h)\bar{G}^2 + (\frac{\beta^3}{2} + 2h\beta)\bar{G}.$$

Center of the Yangian $\mathcal{Y}^{(2)}$

Center is generated by

$$C(u) = L^t(u-\beta)L(u) =$$

$$= (u^2 + ug + h)(h + (u-\beta)^2 + (u-\beta)g) + (\beta - u)c^{(2.6)} - c^{(2.8)}.$$

The elements g, h, $c^{(2.6)}$ and $c^{(2.8)}$ are central.

Lie algebra resolution

Consider first the trivial case:

$$\rho(\bar{H}) = a\bar{G}, \qquad \rho(G) = G = g + \bar{G},$$

and

$$L(u) = u^2 + u(g + \bar{G}) + h + \frac{1}{2}(\bar{G}^2 + \beta\bar{G}),$$

The sufficient condition is

$$W_{a_1 b_1 a_2 b_2} = \bar{G}_{[a_1 b_1} \bar{G}_{a_2 b_2)},$$

The all central elements then are expressed in terms of $m_2 = \frac{1}{n} tr(\bar{G}^2)$:

$$g^2 = -\beta^2 - \frac{m_2}{8}, \qquad 4h = 2\beta^2 - 1 + 2\beta\epsilon - \frac{m_2}{2}.$$

The condition $W_{12} = 0$ implies that the graded-antisymmetric part of $\bar{G}^3$ is proportional to $\bar{G}$:

$$\chi = \bar{G}^3 + (2\beta + \epsilon)\bar{G}^2 + \frac{\epsilon}{2}(4\beta - m_2)\bar{G} - \frac{m_2}{2} = 0.$$

are realized as follows:

$$c^a c^b + \epsilon c^b c^a = \varepsilon^{ba}, \qquad c_a = \varepsilon_{ab} c^b, \qquad \Rightarrow \qquad c_1 c^b + \epsilon c^b c_a = \delta_a^b,$$

$$c_a c_b + \epsilon c_b c_a = \varepsilon_{ab}, \qquad c_a c^a = \frac{n}{2} = \epsilon c^b c_b.$$

$$G_{ab} = \frac{\epsilon}{2}\varepsilon_{ab} - c_a c_b = -\epsilon G_{ba}$$

$$(G^2 + \beta G)_{ab} = \frac{\epsilon}{4}(n - \epsilon)\varepsilon_{ab} = \frac{\epsilon}{2}(\beta + \frac{\epsilon}{2})\varepsilon_{ab}.$$

The metric for $Sp(n)$ and $O(n)$ $(n = 2k)$ is convenient to choose as:

$$\varepsilon_{ab} = \varepsilon_a \delta_{a,-b}, \qquad a, b = -\frac{n}{2}, \ldots, -1, 1, \ldots, \frac{n}{2}.$$

$i, j, k = 1, 2, \ldots, \frac{n}{2}.$

of the Lie algebra representation $|0\rangle$ is realized as follows:

$$G_{-i,-j}|0\rangle = 0, \qquad G_{-i,j}|0\rangle = 0, \quad i<j, \qquad G_{-i,i}|0\rangle = h_i|0\rangle = 0.$$

The algebra implies:

$$[G_{ab}, G^m_{cd}] = -\varepsilon_{cb}G^m_{ad} + \varepsilon_{ad}G^m_{cb} + \varepsilon_{ac}G^m_{bd} - \varepsilon_{db}G^m_{ca},$$

so one deduces:

$$G^m_{-i,-j}|0\rangle = 0, \qquad G^m_{-i,j}|0\rangle = 0, \quad i<j,$$

$$G^m_{-i,i}|0\rangle = h^{(m)}_{+i}|0\rangle = 0, \qquad G^m_{i,-i}|0\rangle = h^{(m)}_{-i}|0\rangle = 0,$$

$$h^{(1)}_{+1} = h_i, \qquad h^{(1)}_{-1} = -\epsilon h_i,$$

$h^{(m)}_{\pm i}$ is calculated iteratively:

$$h^{(m+1)}_{+i} = (\epsilon h_i - 2\beta + i - \epsilon)h^{(m)}_{+i} + (\epsilon - 1)h^{(m)}_{-i} + \sum_{k<i}\epsilon h^{(m)}_{+k} + \sum_{k>i}(\epsilon h^{(m)}_{-k} + h^{(m)}_{+k}),$$

$$h^{(m+1)}_{-i} = -\epsilon h^{(m)}_{-i}h_i + (1-i)h^{(m)}_{-i} + \sum_{k<i}h^{(m)}_{-k},$$

So for any Lie algebra representation obeying:

$$G^2 + 2\beta G - m_2 = 0,$$

has weights subjected to $\frac{n}{2} - 1$ conditions

$$(h_i - h_{i-1})(\epsilon(h_i + h_{i-1} - \beta + i - 1) = 0.$$

The oscillator (spinor) representation admits two solutions:

$$h_i = -\frac{1}{2}, \quad i = 1, \ldots, \frac{n}{2}, \qquad c_{-i}|0\rangle = 0,$$

and

$$h_i = -\frac{1}{2}, \quad i = 1, \ldots, \frac{n}{2} - 1, \qquad h_{\frac{n}{2}} = +\frac{1}{2},$$

and the highest weight vector $|\tilde{0}\rangle$:

$$c_{-i}|\tilde{0}\rangle = 0, \quad i = 1, \ldots, \frac{n}{2} - 1, \qquad c_{\frac{n}{2}}|\tilde{0}\rangle = 0.$$

Any Lie algebra representation obeying

$$W_{ab,cd} = G_{ab}G_{cd} + G_{ac}G_{db} + G_{ad}G_{bc} + G_{cd}G_{ab} + G_{db}G_{ac} + G_{bc}G_{ad} = 0,$$

has weights

$$(h_1, \ldots, h_{\frac{n}{2}}) = (1, \ldots, h, 0 \ldots, 0).$$

is realized as follows:

$$x_a\partial_b - \epsilon\partial_b x_a = [x_a, \partial_b]_{-\epsilon} = \varepsilon_{ab}, \qquad [x_a, x_b]_{-\epsilon} = 0 = [\partial_a, \partial_b]_{-\epsilon}.$$

$\rho: \quad \mathcal{Y}^{(2)}(\mathcal{G}) \to \mathcal{H}$

$$\rho(G_{ab}) = \tilde{G}_{ab} = x_a\partial_b - \epsilon x_b\partial_a.$$

All conditions are fulfilled.

$$W_{a_1 b_1 a_2 b_2} = \bar{G}_{[a_1 b_1}\bar{G}_{a_2 b_2)},$$

The all central elements then are expressed in terms of $m_2 = \frac{1}{n}tr(\bar{G}^2)$:

$$g^2 = -\beta^2 - \frac{m_2}{8}, \qquad 4h = 2\beta^2 - 1 + 2\beta\epsilon - \frac{m_2}{2}.$$

The condition $W_{12} = 0$ implies that the graded-antisymmetric part of $\bar{G}^3$ is proportional to $\bar{G}$:

$$\chi = \bar{G}^3 + (2\beta + \epsilon)\bar{G}^2 + \frac{\epsilon}{2}(4\beta - m_2)\bar{G} - \frac{m_2}{2} = 0.$$

$$[\partial_a, x_b]_{-\epsilon} = \varepsilon_{ab}, \qquad G_{ba} = x_a\partial_b - \epsilon x_b\partial_a.$$

Highest weight vector $|0\rangle$

$$\psi(x) = (x_{-1})^{\lambda},$$

weights

$$h_1 = -\epsilon\lambda = h, \qquad h_i = 0, \qquad i = 2, \dots, \frac{n}{2}.$$

$$(h_1, \dots, n_{\frac{n}{2}} = (h, 0, \dots, 0).$$

In orthogonal case the canonical pairs are bosonic and λ is an arbitrary number. In the symplectic case the canonical pairs are fermionic and ψ is either constant or is proportional to the first power of x_{-1}, so $\lambda = 0$ or $\lambda = 1$.

The highest weight $|0\rangle$ of the Yangian algebra $\mathcal{Y}^{(2)}$

$$[\partial_a, x_b]_{-\epsilon} = \varepsilon_{ab}, \qquad G_{ba} = x_a\partial_b - \epsilon x_b\partial_a.$$

Highest weight vector $|0\rangle$

$$G_{-i,-j}|0\rangle = 0, \qquad G_{-i,j}|0\rangle = 0, \quad i<j, \qquad G_{-i,i}|0\rangle = h_i|0\rangle,$$

$$H_{-i,-j}|0\rangle = 0, \qquad H_{-i,j}|0\rangle = 0, \quad i<j, \qquad H_{-i,i}|0\rangle = \bar{b}h_i|0\rangle.$$

6-th constraint implies:

$$2\bar{h}_i[h_i + \epsilon(i-1-\beta)] - 2\epsilon\sum_{k=1}^{i-1}\bar{h}_k - g(h_{-i}^{(2)} - \epsilon\beta h_i) = c^{(2.6)},$$

while 8-th constraint gives:

$$c^{(2.8)} = -\frac{1}{4}h_{-i}^{(4)} - \beta h_{-i}^{(3)} - (\frac{5}{4}+h)h_{-i}^{(2)} + \epsilon\beta(\frac{\beta^2}{2}+2h)h_i -$$

$$-\sum_{k<i}\Big[\frac{1}{4}(h_{-k}^{(3)} - h_{-i}^{(3)} + \epsilon h_i h_{-k}^{(2)} - \epsilon h_k h_{-i}^{(2)} + \frac{1}{2}(-\epsilon(h_k - h_i)(2h+\frac{\beta}{3}) + \beta(h_{-k}^{(2)} - h_{-i}^{(2)})\Big].$$