# Multi-Hamiltonian formulation for extended Chern-Simons

**Dmitry S. Kaparulin**

Tomsk State University

based on: arXiv: 1711.07897, 1806.00936

Supersymmetry in integrable systems – SIS'18,
Dubna, August 14, 2018

# Motivation

Higher derivative theories is a class of models whose Lagrangian depends on second and higher time derivatives of the generalized coordinates,

$$S[q^i(t)] = \int L(q^i, \dot{q}^i, \ddot{q}^i, \ldots)dt, \qquad \overset{(k)}{q}{}^i \equiv \frac{d^k q^i}{dt^k}\,. \tag{1}$$

Even though the canonical energy is unbounded, higher derivative theories can have bounded conserved quantities that prevent collapse of classical solutions.

No collapse of classical solutions does not automatically mean that the spectrum of energies has the vacuum state with minimal energy.

$\Rightarrow$ To have well-defined vacuum state with minimal energy, higher derivative model should admit *alternative* Hamiltonian formulation with bounded Hamiltonian.

in mechanics — Bolonek and Kosinski, 2005

in field theory — DK, Lyakhovich, and Sharapov, 2014

# Warm-up example: Pais-Uhlenbeck oscillator

Lagrangian

$$L = \frac{1}{2(\omega_1^2 - \omega_2^2)}(\ddot{q}^2 - (\omega_1^2 + \omega_2^2)\dot{q}^2 + \omega_1^2\omega_2^2 q^2)\,. \tag{2}$$

First-order equations

$$\dot{q} = q^1\,, \qquad \dot{q}^1 = q^2\,, \qquad \dot{q}^2 = q^3\,, \qquad \dot{q}^3 = -(\omega_1^2 + \omega_2^2)q^2 - \omega_1^2\omega_2^2 q\,. \tag{3}$$

Two-parameter series of Hamiltonians

$$H(\beta) = \frac{1}{2(\omega_1^2 - \omega_2^2)}\sum_{a=1}^{2}\beta_a((q^3 + \omega_a^2 q^1)^2 + (\omega_1^2 + \omega_2^2 - \omega_a^2)(q^2 + \omega_a^2 q)^2)\,. \tag{4}$$

Poisson brackets

$$\begin{gathered}\{q^1, q\} = 1/\beta_1 + 1/\beta_2\,, \qquad \{q^2, q^1\} = -\{q^3, q\} = \omega_2^2/\beta_1 + \omega_1^2/\beta_2\,, \\ \{q^2, q^3\} = \omega_2^4/\beta_1 + \omega_1^4/\beta_2\,, \qquad \{q^2, q\} = \{q^3, q^1\} = 0\,.\end{gathered} \tag{5}$$

Bolonek and Kosinski, 2005

Generalizations Damaskinsky and Sokolov, 2006; Masterov, 2015 and 2016

# Outline

- Extended Chern-Simons model & conserved quantities
- Constrained Hamiltonian formulation
- Alternative Poisson brackets
- Alternative Hamiltonian action
- Stable interactions
- Summary

# Extended Chern-Simons model

Extended Chern-Simons a class of theories of vector field $A = A_\mu dx^\mu$ on $3d$ Minkowski space with the action functional

$$S[A] = \frac{1}{2}\int *A \wedge \left(\alpha_1 m * dA + \alpha_2 * d * dA + m^{-1} * d * d * dA\right), \qquad (6)$$

$d$ is the de-Rham differential, $*$ is the Hodge star operator, $m$ is a dimensional constant, and $\alpha_1, \alpha_2$ are dimensionless constant real parameters.

Deser and Jackiw, 1999

The theory admits a two-parameter series of conserved quantities

$$E(\alpha,\beta) = \frac{1}{2}\int d^2x\left[\beta_2 m^{-2} G_\mu G_\mu + 2m^{-1}\beta_1 G_\mu F_\mu + (\beta_1\alpha_2 - \beta_2\alpha_1)F_\mu F_\mu\right], \quad (7)$$

where $\beta_1, \beta_2$ are parameters, $F = *dA$, and $G = *d * dA$; $\beta_1 = 1, \beta_2 = 0$ corresponds to the canonical energy.

DK, Karataeva and Lyakhovich, 1999

The general representative in the series (7) is bounded if

$$\beta_2 > 0, \quad -\beta_1^2 + \alpha_2\beta_1\beta_2 - \alpha_1\beta_2^2 > 0. \qquad (8)$$

# First-order equations

Lagrange equations for the extended Chern-Simons ,

$$\frac{\delta S}{\delta A} \equiv (\alpha_1 m * d + \alpha_2 * d * d + m^{-1} * d * d * d)A = 0\,, \qquad (9)$$

involve the third-time derivatives of $A$.

Introduce new variables $F_i, G_i$ that absorb the time derivatives of $A$

$$F_i = \varepsilon_{ij}(\dot{A}_j - \partial_j A_0)\,, \quad G_i = -\ddot{A}_i + \partial_i \dot{A}_0 + \partial_j(\partial_j A_i - \partial_i A_j)\,, \quad i,j = 1,2\,, \qquad (10)$$

with $\varepsilon_{ij}$ being the $2d$ Levi-Civita symbol. In so doing, $F_i = (*dA)_i, G_i = (*dF)_i$.

The first-order equations of motion in terms of the fields $A_\mu, F_i, G_i$ are

$$\begin{gathered} \dot{A}_i = \partial_i A_0 - \varepsilon_{ij} F_j\,, \qquad \dot{F}_i = \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big]\,, \\ \dot{G}_i = \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k) + m(\alpha_2 G_j + \alpha_1 m F_j)\big]\,, \end{gathered} \qquad (11)$$

and we have one constraint

$$\Theta \equiv \varepsilon_{ij}\partial_i\big(m^{-1}G_j + \alpha_2 F_j + \alpha_1 m A_j\big) = 0\,. \qquad (12)$$

# Constrained Hamiltonian formulation

Constrained Hamiltonian formulation for extended Chern-Simons reads

$$\dot{Z}_i = \{Z_i, H(\alpha,\beta)\}_{\alpha,\beta}\,, \qquad \Theta = 0\,. \tag{13}$$

where $Z_i = \{A_i, F_i, G_i\}$.

We seek the Hamiltonian in the form

$$H(\alpha,\beta,k) = E(\alpha,\beta) + \int d^2x (k_0 A_0 + k_1 F_0 + k_2 G_0)\Theta\,, \tag{14}$$

where $k_0, k_1, k_2$ are some constants; $k_0 \neq 0$.

Substituting (14) into (13), we get defining equations for the Poisson bracket

$$\begin{aligned} \{A_i, H(\alpha,\beta,k)\}_{\alpha,\beta,k} &= \partial_i A_0 - \varepsilon_{ij} F_j\,, \\ \{F_i, H(\alpha,\beta,k)\}_{\alpha,\beta,k} &= \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big]\,, \\ \{G_i, H(\alpha,\beta,k)\}_{\alpha,\beta,k} &= \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k) + m(\alpha_2 G_j + \alpha_1 m F_j)\big]\,, \end{aligned} \tag{15}$$

# Solution for Poisson brackets and $k_0, k_1, k_2$

$$
\begin{aligned}
\{G_i(\vec{x}), G_j(\vec{y})\}_{\alpha,\beta,\gamma} &= \frac{(\alpha_1-\alpha_2^2)\beta_1+\alpha_1\alpha_2\beta_2}{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2} m^3\varepsilon_{ij}\delta(\vec{x}-\vec{y})\,,\\
\{F_i(\vec{x}), G_j(\vec{y})\}_{\alpha,\beta,\gamma} &= \frac{\alpha_2\beta_1-\alpha_1\beta_2}{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2} m^2\varepsilon_{ij}\delta(\vec{x}-\vec{y})\,,\\
\{A_i(\vec{x}), G_j(\vec{y})\}_{\alpha,\beta,\gamma} &= \{F_i(\vec{x}), F_j(\vec{y})\}_{\beta,\gamma} = -\frac{\beta_1}{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2} m\varepsilon_{ij}\delta(\vec{x}-\vec{y})\,,\\
\{A_i(\vec{x}), F_j(\vec{y})\}_{\alpha,\beta,\gamma} &= \frac{\beta_2}{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2} \varepsilon_{ij}\delta(\vec{x}-\vec{y})\,,\\
\{A_i(\vec{x}), A_j(\vec{y})\}_{\alpha,\beta,\gamma} &= \frac{\gamma}{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2} m^{-1}\varepsilon_{ij}\delta(\vec{x}-\vec{y})\,;
\end{aligned}
\tag{16}
$$

$$
k_0 = \frac{\beta_2^2\alpha_1-\beta_2\beta_1\alpha_2+\beta_1^2}{\beta_1-\beta_2\alpha_2-\gamma\alpha_1}\,,\qquad k_1 = \frac{\beta_2\beta_1+\gamma(\beta_2\alpha_1-\beta_1\alpha_2)}{\beta_1-\beta_2\alpha_2-\gamma\alpha_1}\,,
$$
$$
k_2 = \frac{\beta_2^2+\gamma\beta_1}{\beta_1-\beta_2\alpha_2-\gamma\alpha_1}\,. \tag{17}
$$

Here, $\gamma$ is accessory parameter.

# Critical points for Poisson bracket

Solution for Poisson bracket is not defined if parameters $\alpha, \beta, \gamma$ meet the relations:

$$\beta_2^2\alpha_1 - \beta_2\beta_1\alpha_2 + \beta_1^2 = 0\,, \qquad \beta_1 - \beta_2\alpha_2 - \gamma\alpha_1 = 0\,. \tag{18}$$

The first relation implies that the energy of the system,

$$E(\alpha,\beta) = \frac{1}{2}\int d^2x\big[\beta_2 m^{-2}G_\mu G_\mu + 2m^{-1}\beta_1 G_\mu F_\mu + (\beta_1\alpha_2 - \beta_2\alpha_1)F_\mu F_\mu\big]\,, \tag{19}$$

is a degenerate quadratic form in the variables $F, G$.

$\Rightarrow$ degenerate conserved quantity can't serve as Hamiltonian.

The second relation implies that the constraint $\Theta$ is unable to generate gauge symmetry of the vector field $A$,

$$\{A_i(\vec{x}), \Theta(\vec{y})\}_{\alpha,\beta,\gamma} = \frac{\beta_1 - \beta_2\alpha_2 - \gamma\alpha_1}{\beta_2^2\alpha_1 - \beta_2\beta_1\alpha_2 + \beta_1^2}\,\partial_i\delta(\vec{x}-\vec{y}) = 0\,. \tag{20}$$

$\Rightarrow$ there is a critical value of $\gamma$ such that constraint $\Theta$ can't serve as gauge generator.

# Alternative Hamiltonian (final form)

The final form for total Hamiltonian reads

$$H(\alpha,\beta,\gamma) = E(\alpha,\beta) + \int d^2x\,\Theta\Big[\frac{\beta_1^2-\alpha_2\beta_1\beta_2+\alpha_1\beta_2^2}{\beta_1-\alpha_2\beta_2-\alpha_1\gamma}A_0 + \frac{\beta_1\beta_2+\alpha_1\beta_2\gamma-\alpha_2\beta_1\gamma}{\beta_1-\alpha_2\beta_2-\alpha_1\gamma}m^{-1}\varepsilon_{ij}\partial_iA_j + \frac{\beta_2^2+\beta_1\gamma}{\beta_1-\alpha_2\beta_2-\alpha_1\gamma}m^{-2}\varepsilon_{ij}\partial_iF_j\Big]\,, \qquad (21)$$

where

$$\beta_2^2\alpha_1-\beta_2\beta_1\alpha_2+\beta_1^2\neq 0\,,\qquad \beta_1-\beta_2\alpha_2-\gamma\alpha_1\neq 0\,, \qquad (22)$$

and Poisson bracket is defined by equations (16).

- Constrained Hamiltonian formulation exists for almost all $\alpha,\beta,\gamma$. The parameters are two constants $\beta$, while $\gamma$ is accessory quantity;
- Canonical Ostrogradski's Hamiltonian is included in the series for

$$\beta_1=0\,,\qquad \beta_2=\gamma=0\,; \qquad (23)$$

- Alternative Hamiltonian is bounded if

$$\beta_2>0\,,\qquad \beta_2^2\alpha_1-\beta_2\beta_1\alpha_2+\beta_1^2>0\,. \qquad (24)$$

# Alternative first-order action

For all the admissible values of the parameters $\alpha, \beta, \gamma$, the Poisson bracket is a non-degenerate tensor

$$\det(\{Z_i, Z_j\}) = (\beta_1 - \beta_2\alpha_2 - \gamma\alpha_1)^2(\beta_2^2\alpha_1 - \beta_2\beta_1\alpha_2 + \beta_1^2)^2 \neq 0\,. \tag{25}$$

So, the Poisson bracket define the series of Hamiltonian action functionals

$$\begin{aligned} S(\alpha,\beta,\gamma) = \int \Big\{ & \frac{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_2\beta_2^2}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}(\alpha_1 m A_i + 2\alpha_2 F_i + 2m^{-1}G_i)\varepsilon_{ij}\dot{A}_j + \\ & + \frac{\beta_1^2 + ((\alpha_2^2 - \alpha_1)\beta_1 - \alpha_1\alpha_2\beta_2)\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma} m^{-1}\varepsilon_{ij}F_i\dot{F}_j + \\ & + 2\frac{\beta_1\beta_2 + (\alpha_2\beta_1 - \alpha_1\beta_2)\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma} m^{-2}\varepsilon_{ij}G_i\dot{F}_j + \\ & + \frac{\beta_2^2 + \beta_1\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma} m^{-3}\varepsilon_{ij}G_i\dot{G}_j - H(\alpha,\beta,\gamma)\Big\} d^3x\,. \end{aligned}$$

The series includes the canonical Ostrogradski action ($\beta_1 = 1, \beta_2 = \gamma = 0$),

$$S_{\text{Ost.}} = \int \Big\{(\alpha_1 m A_i + 2\alpha_2 F_i + 2m^{-1}G_i)\varepsilon_{ij}\dot{A}_j - m^{-1}\varepsilon_{ij}F_i\dot{F}_j - A_0\Theta - H_{\text{Ost.}}\Big\} d^3x\,.$$

# Stable interactions

a) with spinor field

$$
\begin{aligned}
&(\alpha_1 m * d + \alpha_2 * d * d + m^{-1} * d * d * d)A = J_\psi\,, \qquad J_\psi = e\overline{\psi}\gamma_\mu\psi\,, \\
&(i\gamma^\mu D_\mu - m)\psi = 0\,, \qquad D_\mu = \partial_\mu - ie(\beta_1 A + \beta_2 * dA)_\mu\,, \\
&E(e;\alpha,\beta) = E(\alpha,\beta) + \int d^2x\overline{\psi}(i\gamma_i D_i - m)\psi\,.
\end{aligned}
\tag{26}
$$

Abakumova, DK, Lyakhovich, 2018

b) with Einstein's gravity

$$
\begin{aligned}
&(\alpha_1 m * d + \alpha_2 * d * d + m^{-1} * d * d * d)A = 0\,, \\
&R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}(A, g; \alpha, \beta)\,, \\
&E(A, g; \alpha, \beta) = R^{00} - \frac{1}{2}g^{00}R - T^{00}(A, g; \alpha, \beta)\,.
\end{aligned}
$$

DK, Karataeva, Lyakhovich, 2018

# Summary

- Extended Chern-Simons of third order admits a two-parameter series of conserved quantities. This series can include bounded representatives.
- Two-parameter series of Hamiltonian formulations exists such that any bounded quantity can serve as Hamiltonian.
- Canonical Ostrogradski's formulation is included in the series. The canonical Hamiltonian is unbounded.
- There are interactions such that stability of free theory is preserved. The interaction vertices are non-Lagrangian but Hamiltonian.

Thank you for your attention