Quaternion-Kähler deformations of $\mathcal{N} = 4$ supersymmetric mechanics

Evgeny Ivanov (BLTP JINR)

SIS '18 ,   August 13 - 16, 2018,   Dubna

Based on joint works with Luca Mezincescu (Miami)

Outline

Motivations

Deformations of $\mathcal{N} = 4$ SQM: first type

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Generalizations

Summary and Outlook

Outline

Motivations

Deformations of $\mathcal{N} = 4$ SQM: first type

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Generalizations

Summary and Outlook

Outline

Outline

Motivations

Deformations of $\mathcal{N}=4$ SQM: first type

QK $\mathcal{N}=4$ SQM as a deformation of HK SQM models

Generalizations

Summary and Outlook

Outline

Motivations

Deformations of $\mathcal{N} = 4$ SQM: first type

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Generalizations

Summary and Outlook

Motivations

Supersymmetric Quantum Mechanics (SQM) (Witten, 1981) is the simplest ($d = 1$) supersymmetric theory:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended $\mathcal{N} > 2, d = 1$ SUSY is specific: dualities between various supermultiplets, nonlinear "cousins" of off-shell linear multiplets, etc.
- $\mathcal{N} = 4$ SQM: $\{Q_\alpha, \bar{Q}^\beta\} = 2\delta_\alpha^\beta H\,, \alpha = 1, 2,$ is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.
- Being motivated by an interest in theories with the "curved" rigid supersymmetries (see, e.g., Festuccia, Seiberg, 2011), it is important to study various non-trivial deformations of SQM models.

Deformations of $\mathcal{N} = 4$ SQM: first type

The first type of deformed SQM amounts to choosing some semi-simple supergroups instead of higher-rank $d = 1$ super-Poincare:

A. Standard extension:

$$(\mathcal{N} = 2\,,\ d = 1) \quad \Rightarrow \quad (\mathcal{N} > 2\,,\ d = 1 \text{ Poincaré})\,,$$

B. Non-standard extension:

$$(\mathcal{N} = 2\,,\ d = 1) \equiv u(1|1) \quad \Rightarrow \quad su(2|1) \ \subset \ su(2|2) \ \subset \ \ldots .$$

In the case B, the closure of supercharges contains, besides H, also internal symmetry generators.

- The deformed $\mathcal{N} = 4$ SQM is associated with $su(2|1)$ (Bellucci & Nersessian, 2003, 2004; Smilga, 2004; Römelsberger, 2006, 2007; I. & Sidorov, 2014, 2016; I., Sidorov & Toppan, 2015).
- Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized (Fedoruk & I., 2017; Fedoruk, I., Lechtenfeld & Sidorov, 2017).
- The analogous deformations of $\mathcal{N} = 8$ SQM are associated with $su(2|2)$ and $su(4|1)$ (I., Lechtenfeld & Sidorov, 2016, 2018).

This type of deformations, with focus on the $su(4|1)$ case, will be discussed in the talk by Stepan Sidorov, so I will concentrate on another type.

QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $\mathcal{N} = 4$ SQM models proceeds from the general Hyper-Kähler (HK) subclass of the latter. The deforrmed models are $\mathcal{N} = 4$ supersymetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models (I. & Mezincescu, 2017).

Both HK and QK $\mathcal{N} = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach (I. & Lechtenfeld, 2003).

HK manifolds are bosonic targets of sigma models with **rigid** $\mathcal{N} = 2, d = 4$ SUSY (Alvarez-Gaume, Freedman, 1980, 1981). After coupling these models to **local** $\mathcal{N} = 2, d = 4$ SUSY in the supergravity framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds (Bagger, Witten, 1983).

Both types of the manifolds are $4n$ dimensional, but their holonomy groups are in $Sp(n)$ and $Sp(1) \times Sp(n)$, respectively.

Harmonic $\mathcal{N}=4, d=1$ superspace

- Ordinary $\mathcal{N}=4, d=1$ superspace:

$$(t, \theta^i, \bar\theta_k), \quad i, k, = 1, 2;$$

- Harmonic extension:

$$(t, \theta^i, \bar\theta_k) \quad \Rightarrow \quad (t, \theta^i, \bar\theta_k, u_j^\pm),\ u^{+i} u_i^- = 1,\ u_i^\pm \in SU(2)_{Aut}.$$

- Analytic basis:

$$(t_A, \theta^+, \bar\theta^+, u_k^\pm, \theta^-, \bar\theta^-) \equiv (\zeta, u^\pm, \theta^-, \bar\theta^-)$$

$$\theta^\pm = \theta^i u_i^\pm,\ \bar\theta^\pm = \bar\theta^k u_k^\pm,\ \ t_A = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)$$

- Analytic superspace and superfields:

$$D^+ = \frac{\partial}{\partial\theta^-},\ \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}, \quad D^+\Phi = \bar D^+\Phi = 0 \ \Rightarrow\ \Phi = \Phi(\zeta, u^\pm)$$

- Harmonic derivatives:

$$D^{\pm\pm} = u_\alpha^\pm \frac{\partial}{\partial u_\alpha^\mp} + \theta^\pm \frac{\partial}{\partial\theta^\mp} + \bar\theta^\pm \frac{\partial}{\partial\bar\theta^\mp} + 2i\theta^\pm\bar\theta^\pm \frac{\partial}{\partial t_{\mathrm{A}}},$$

$$[D^+, D^{++}] = [\bar D^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++}\Phi(\zeta, u^\pm) \text{ is analytic}$$

Basic $\mathcal{N} = 4, d = 1$ multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

- Described off-shell by an analytic superfield $q^{+a}(\zeta, u)$:

$$(4,4,0) \quad \Longleftrightarrow \quad q^{+a}(\zeta, u) \propto (f^{ia}, \chi^a, \bar{\chi}^a), \; a = 1, 2,$$

$$\begin{aligned} &(a) \;\; D^+ q^{+a} = \bar{D}^+ q^{+a} = 0 && (Grassmann \;\; analyticity), \\ &(b) \;\; D^{++} q^{+a} = 0 && (Harmonic \;\; analyticity), \end{aligned}$$

$$(a) + (b) \quad \Longrightarrow \quad q^{+a} = f^{ka} u^+_k + \theta^+ \chi^a - \bar{\theta}^+ \bar{\chi}^a - 2i\theta^+ \bar{\theta}^+ \dot{f}^{ka} u^-_k \,.$$

- Free off-shell action:

$$S_{free} \sim \int dt d^4\theta du \, q^{+a} q^-_a \sim \int dt \left(\dot{f}^{ia} \dot{f}_{ia} - \frac{i}{2} \bar{\chi}^a \dot{\chi}_a \right), \quad q^{-a} := D^{--} q^{+a}$$

- Nonlinear $d = 1$ sigma model action:

$$S_{free} \sim \int dt d^4\theta du \, \mathcal{L}(q^{+a}, q^{-b}, u^{\pm}).$$

- In bosonic sector: HKT (“Hyper-Kähler with torsion”) sigma model. In components, the torsion appears in a term quartic in fermions.

How to construct general HK $\mathcal{N}=4, d=1$ sigma models? No torsion in this case, the geometry involves only Riemann curvature tensor. The answer was given in Delduc, I., 2010.

- The basic superfields are real analytic, $q^{+A}(\zeta, u) = f^{iA}u_i^+ + ... , i = 1,2, \; A = 1, \ldots 2n$, it encompasses just $4n$ fields $f^{iA}(t)$ parametrizing the target bosonic manifold, $\widetilde{(q_A^+)} = \Omega^{AB}q_B^+$, with $\Omega^{AB} = -\Omega^{BA}$ a constant symplectic metric.

- The linear constraint $D^{++}q^{+A} = 0$ is promoted to a nonlinear one
$$D^{++}q^{+A} = \Omega^{AB}\,\frac{\partial L^{+4}(q^{+C}, u^{\pm})}{\partial q^{+B}}.$$

- The superfield action is bilinear as in the free case,
$$S_{HK} \sim \int dtd^4\theta du\,\Omega^{AB}q_B^+q_A^- = \int dt\left[g_{iA\,kB}(f)\dot{f}^{iA}\dot{f}^{kB} + \ldots\right],$$
the whole interaction appears only on account of nonlinear deformation of the q^{+A}-constraint.

- L^{+4} is an analytic hyper-Kähler potential (Galperin, I., Ogivetsky, Sokatchev, 1986): every L^{+4} produces the component HK metric $g_{iA\,kB}(f)$ and, *vice versa*, each HK metric originates from some HK potential L^{+4}.

From $\mathcal{N} = 4$ HK SQM to its QK deformation

The harmonic superspace approach supplies the most natural arena for defining $\mathcal{N} = 4$ QK SQM. Basic new features of these models as compared to their HK prototypes are as follows.

1. QK SQM model corresponding to $4n$ dimensional QK manifold requires $n + 1$ multiplets $(4, 4, 0)$ described by analytic superfields $q^{+a}(\zeta, w^{\pm}), (a = 1, 2), Q^{+r}(\zeta, w^{\pm}), (r = 1, \ldots 2n)$. An extra superfield $q^{+a}(\zeta, w^{\pm})$ is $d = 1$ analog of $\mathcal{N} = 2, d = 4$ “conformal compensator”.

2. QK SQM actions are invariant under local $\mathcal{N} = 4, d = 1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w_i^{\pm}$.

3. For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar{\theta}^-, w^{\pm})$ which is a general $\mathcal{N} = 4, d = 1$ superfield.

4. Besides the (q^+, Q^+) superfield part, the correct action should contain a “comological term” involving the vielbein superfield only.

Minimal local $\mathcal{N} = 4, d = 1$ SUSY

By analogy with the $\mathcal{N} = 2, d = 4$ case we postulate that local $\mathcal{N} = 4, d = 1$ SUSY preserves Grassmann analyticity,

$$\delta t = \Lambda(\zeta, w)\,,\ \delta\theta^+ = \Lambda^+(\zeta, w)\,,\ \delta\bar\theta^+ = \bar\Lambda^+(\zeta, w)\,,$$
$$\delta w_i^+ = \Lambda^{++}(\zeta, w) w_i^-\,,\ \delta w_i^- = 0\,,$$
$$\delta\theta^- = \Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\,,\quad \delta\bar\theta^- = \bar\Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\,,$$

The explicit structure of the minimal set of analytic parameters is as follows

$$\begin{aligned}
\Lambda &= 2b + \ldots \\
\Lambda^+ &= \lambda^i w_i^+ + \ldots \\
\Lambda^{++} &= \tau^{(ik)} w_i^+ w_k^+ + \ldots \\
\Lambda^- &= \lambda^i w_i^- + \ldots
\end{aligned}$$

Here, $b(t)$, $\tau^{(ik)}(t)$ and $\lambda^i(t)$, $\bar\lambda^i(t)$ are arbitrary local parameters, bosonic and fermionic, respectively. The local $\mathcal{N} = 4, d = 1$ supergroup obtained is isomorphic to the classical (having no central charges) “small” $\mathcal{N} = 4$ superconformal symmetry.

How to generalize $(4,4,0)$ superfields $q^{+A}(\zeta, w)$ to local SUSY?

- The simplest possibility is to keep the linear constraint

$$D^{++} q^{+a} = 0\,, \quad D^{++} Q^{+r} = 0\,.$$

- It is covariant under the transformations

$$\delta D^{++} = -\Lambda^{++} D^0\,,\ \delta q^{+a} = \Lambda_0\, q^{+a}\,,\ \delta Q^{+r} = \Lambda_0\, Q^{+r}\,,$$
$$\Lambda^{++} = D^{++} \Lambda_0\,.$$

- To construct invariant actions, one needs the transformations of the integration measures $\mu_H := dtdwd^2\theta^+ d^2\theta^-$, $\mu^{(-2)} := dtdwd^2\theta^+$,

$$\delta \mu^{(-2)} = 0\,, \quad \delta \mu_H = \mu_H\, 2\Lambda_0\,,$$

and that of harmonic derivative D^{--},

$$\delta D^{--} = -(D^{--}\Lambda^{++}) D^{--}\,.$$

Simplest invariant action

The basic part of the total invariant action of the analytic superfields $q^{+a}(\zeta, w)$, $a = 1, 2$, and $Q^{+r}(\zeta, w)$, $r = 1, 2, \ldots 2n$, can be written as

$$S_{(2)} = \int \mu_H \, E \, \mathcal{L}_{(2)}(q, Q) \,, \quad \mathcal{L}_{(2)}(q, Q) = \gamma q^{+a} q^-_a - Q^{+r} Q^-_r \,,$$
$$q^-_a := D^{--} q^+_a \,, \quad Q^-_r := D^{--} Q^+_r \,,$$

and $\gamma = \pm 1$. The new object is vielbein E which is harmonic-independent, $D^{++} E = D^{--} E = 0$, and transforms as

$$\delta E = (-4\Lambda_0 + 2D^{--}\Lambda^{++}) E \,, \qquad D^{++}(-4\Lambda_0 + 2D^{--}\Lambda^{++}) = 0 \,.$$

One more important term in the action is the “cosmological term”:

$$S_\beta = \beta \int \mu_H \sqrt{E} \,, \quad \delta S_\beta = \beta \int \mu_H \, D^{--}\Lambda^{++} \sqrt{E} = 0 \,.$$

The simplest locally $\mathcal{N}=4$ supersymmetric action so reads

$$S_{HP} \sim S_{(2)} + S_\beta = \int \mu_H \big[E\,\mathcal{L}_{(2)} + \beta\sqrt{E}\big].$$

Why should the “cosmological” term S_β be added?

$$E = E_{bos} + E_{ferm}\,,$$

$$\begin{aligned}
E_{bos} &= e + \theta^+\theta^- M - \bar\theta^+\bar\theta^- \bar M + \theta^+\bar\theta^-(\mu - i\dot e) + \bar\theta^+\theta^-(\mu + i\dot e)\\
&+ 4i(\theta^+\bar\theta^+ w_i^- w_k^- - \theta^+\bar\theta^- w_i^- w_k^+ - \theta^-\bar\theta^+ w_i^- w_k^+ + \theta^-\bar\theta^- w_i^+ w_k^+)L^{(ik)}\\
&+ 4\theta^+\bar\theta^+\theta^-\bar\theta^-[D + 2\dot L^{(ik)} w_i^+ w_k^-],
\end{aligned}$$

$$\begin{aligned}
E_{ferm} &= (\theta^- w_i^+ - \theta^+ w_i^-)\phi^i - (\bar\theta^- w_i^+ - \bar\theta^+ w_i^-)\bar\phi^i + 4i\theta^-\bar\theta^-(\theta^+ w_i^+\sigma^i - \bar\theta^+ w_i^+\bar\sigma^i)\\
&+ 2i\theta^+\bar\theta^+[\theta^- w_i^-(2\sigma^i - \dot\phi^i) - \bar\theta^- w_i^-(2\bar\sigma^i - \dot{\bar\phi}^i)]\,.
\end{aligned}$$

The fields $(e, \phi^i, \bar\phi^i, L^{ik})$ are gauge fields of $\mathcal{N}=4, d=1$ supergravity, $(D, \sigma^i, \bar\sigma^i)$ are additional fields, Lagrange multipliers. So we deal with a kind of **non-minimal** $\mathcal{N}=4, d=1$ supergravity in the present case.

- Pass to the bosonic limit:

$$q^{+a} \Rightarrow f^{ia} w_i^+ - 2i\theta^+\bar{\theta}^+ \dot{f}^{ia} w_i^- \,,\; Q^{+r} \Rightarrow F^{ir} w_i^+ - 2i\theta^+\bar{\theta}^+ \dot{F}^{ir} w_i^- \,,\; E \Rightarrow E_{bos}$$

- In this limit,

$$\begin{aligned} L_{HP} \;\Rightarrow\; & \frac{1}{2} e \left(\dot{F}^{ir} \dot{F}_{ir} - \gamma\, \dot{f}^{ia} \dot{f}_{ia} \right) + L_{ik} \left[F^{(ir} \dot{F}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right] \\ & + \frac{1}{4} D \Big(\gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta \frac{1}{\sqrt{e}} \Big) \\ & + \frac{\beta}{4} \, \frac{1}{e^{3/2}} \left[L^{ik} L_{ik} - \frac{1}{8} \left(M\bar{M} + \mu^2 + \dot{e}^2 \right) \right]. \end{aligned}$$

- The auxiliary fields $M, \bar{M}$ and μ fully decouple and can be put equal to zero by their equations of motion. Also, $e(t)$ is an analog of $d = 1$ vierbein, so it is natural to choose the gauge

$$e = 1\,.$$

- Then the bosonic Lagrangian becomes

$$\begin{aligned} L_{HP} \;\Rightarrow\; & \frac{1}{2} \left(\dot{F}^{ir} \dot{F}_{ir} - \gamma\, \dot{f}^{ia} \dot{f}_{ia} \right) + L_{ik} \left[F^{(ir} \dot{F}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right] \\ & + \frac{1}{4} D \Big(\gamma f^{ia} f_{ia} - F^{ir} F_{ir} + \beta \Big) + \frac{\beta}{4} L^{ik} L_{ik}. \end{aligned}$$

- At $\beta \neq 0$ L^{ik} can be eliminated by its algebraic equation of motion, while D serves as the Lagrange multiplier giving rise to the constraint relating f^{ia} and F^{ir}:

$$L^{ik} = -2\frac{1}{\beta}\left[F^{(ir}\dot{F}_r^{k)} - \gamma f^{(ia}\dot{f}_a^{k)}\right], \quad \gamma f^{ia}f_{ia} - F^{ir}F_{ir} + \beta = 0\,.$$

- Assuming that f^{ia} starts with a constant (compensator!), one uses local $SU(2)$ freedom, $\delta f^{ia} = \tau^i{}_l f^{la}$, to gauge away the triplet from f^{ia},

$$f^{(ia)} = 0 \;\rightarrow\; f^i_a = \sqrt{2}\,\delta^i_a\,\omega\,.$$

- Then the constraint can be solved as

$$(a)\,\gamma = 1 \;\Rightarrow\; \beta < 0\,, \quad \omega = \frac{|\beta|^{1/2}}{2}\sqrt{1 + \frac{1}{|\beta|}F^2}\,,$$

$$(b)\,\gamma = -1 \;\Rightarrow\; \beta > 0\,, \quad \omega = \frac{\beta^{1/2}}{2}\sqrt{1 - \frac{1}{\beta}F^2}\,.$$

- The final form of the bosonic action for $\gamma = 1$ is

$$L_{HP} = \frac{1}{2}\Big[(\dot{F}\dot{F}) + \frac{2}{|\beta|}\,(F_{r(i}\dot{F}_{j)}{}^{r})(F_{s}^{(i}\,\dot{F}^{\,sj)}) - \frac{1}{|\beta|}\;\frac{1}{1+\frac{1}{|\beta|}\,F^2}(F\dot{F})(F\dot{F})\Big].$$

The option $\gamma = -1$ is recovered by the replacement $|\beta| \rightarrow -|\beta|$.

- These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally “flat” $4n$ dimensional QK manifolds, respectively:

$$\widetilde{\mathbb{HP}}^{\mathrm{n}} = \frac{Sp(1,n)}{Sp(1)\times Sp(n)}\,, \quad \mathbb{HP}^{\mathrm{n}} = \frac{Sp(1+n)}{Sp(1)\times Sp(n)}\,.$$

- Thus $\mathcal{N} = 4$ mechanics constructed is just superextensions of these QK $d = 1$ sigma models.

Fermionic sector

- What about fermionic fields? One can choose the gauge in which the $d = 1$ "gravitino" field equals zero. In this gauge, the fermionic fields from q^{+a} are expressed in terms of those from Q^{+r} by the constraint which is a fermionic counterpart of the bosonic constraint:

$$\chi^a = -\frac{\gamma}{\sqrt{2}\omega}\delta^a_i F^{ir}\chi_r\,, \quad \bar{\chi}^a = -\frac{\gamma}{\sqrt{2}\omega}\delta^a_i F^{ir}\bar{\chi}_r\,.$$

- Then the fermionic part of the total Lagrangian is given by

$$\begin{aligned} L^f_{HP} \quad &= \quad -\frac{i}{4}G^{[sr]}\left(\dot{\chi}_s\bar{\chi}_r - \chi_s\dot{\bar{\chi}}_r\right) + \frac{i\gamma}{4\omega^2}F^{i(s}\dot{F}^{r)}_i\chi_s\bar{\chi}_r \\ &\quad - \frac{1}{2\beta}\left(G^{[sr]}G^{[fg]} + G^{[fs]}G^{[gr]}\right)\chi_s\chi_r\bar{\chi}_f\bar{\chi}_g\,, \end{aligned}$$

where

$$G^{[sr]} := \Omega^{[sr]} + \frac{\gamma}{2\omega^2}F^{i[s}F^{r]}_i\,.$$

The second term involves the $Sp(n)$ part of spin connection, while the last term can be expressed through the curvature of this connection.

Generalizations

- The basic step in generalizing to $\mathcal{N}=4$ mechanics with an arbitrary QK manifold is to pass to nonlinear harmonic constraints

$$D^{++}q^{+a} - \gamma\frac{1}{2}\frac{\partial}{\partial q^{+}_{a}}\Big[\hat{\kappa}^2(w^- \cdot q^+)^2\mathcal{L}^{+4}\Big] = 0\,,$$
$$D^{++}Q^{+r} + \frac{1}{2}\frac{\partial}{\partial Q^{+}_{r}}\Big[\hat{\kappa}^2(w^- \cdot q^+)^2\mathcal{L}^{+4}\Big] = 0\,,$$
$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4}\Big(\frac{Q^{+r}}{\hat{\kappa}(w^- q^+)}, \frac{q^{+a}}{(w^- q^+)}, w_i^-\Big), \quad \hat{\kappa} := \frac{\sqrt{2}}{|\beta|^{1/2}}.$$

- The invariant superfield action is the same as in the $\mathbb{HP}^n$ case

$$\mathbb{S}_{QK} \sim \Big[\tilde{S}_{(2)} + S_{\beta}\Big] = \int \mu_H\Big[E\,\tilde{\mathcal{L}}_{(2)} + \beta\sqrt{E}\Big],$$
$$\tilde{\mathcal{L}}_{(2)} = \gamma q^{+a}q^{-}_{a} - Q^{+r}Q^{-}_{r}\,, \quad q^{-}_{a} = D^{--}q^{+a}\,, \quad Q^{-}_{r} = D^{--}Q^{+r}\,.$$

- The bosonic action precisely coincides with $d=1$ reduction of the general QK sigma model action derived from $\mathcal{N}=2, d=4$ supergravity-matter action in I., Valent, 2000. This coincidence proves that we have constructed most general QK $\mathcal{N}=4$ mechanics.

- One more possibility is to consider the following generalization of the $\mathbb{HP}^n$ action

$$S^{loc}(q,Q) = \int \mu_H \sqrt{E}\mathcal{F}(X,Y,w^-),\; X := \sqrt{E}\,(q^{+a}q^-_a)\,,\; Y := \sqrt{E}\,(Q^{+r}Q^-_r),$$
$$D^{++}q^{+a} = D^{++}Q^{+r} = 0 \quad \Rightarrow \quad D^{\pm\pm}X = D^{\pm\pm}Y = 0\,.$$

- When $E = const$, it is reduced to the particular form of the HKT action $\int \mu_H \mathcal{F}(q^{+A}, q^{-B}, w^{\pm})$, while for $\mathcal{F}(X,Y,w^-) = \gamma\,X - Y + \beta$ just to $\mathbb{HP}^n$ action. So the target geometry associated with $S^{loc}(q,Q)$ is expected to be a kind of QKT, i.e. “Quaternion-Kähler with torsion”. To date, not too much known about such geometries...

Summary and Outlook

- $\mathcal{N}=4, d=1$ harmonic superspace methods allow one to construct a new class of deformed $\mathcal{N}=4$ supersymmetric mechanics models, those with $d=1$ Quaternion-Kähler sigma models as a bosonic core. The basic distinguishing feature of these models is *local* $\mathcal{N}=4, d=1$ supersymmetry.

- The superfield and component actions are now known for both general $\mathcal{N}=4$ QK mechanics, and the maximally "flat" $\mathbb{HP}^{\mathrm{n}}$ mechanics.

- A few generalizations of QK mechanics can be constructed, in particular "Quaternion-Kähler with torsion" (QKT) models.

- *Some further lines of study*:

(a) To construct the Hamiltonian formalism for the new class of $\mathcal{N}=4$ mechanical systems, to perform quantization, at least for the simplest case of $\mathbb{HP}^{\mathrm{n}}$ mechanics, to find the energy spectra.

Last news (I & Mezincescu, 2018, in preparation): Noether currents were calculated and shown to be vanishing on-shell (like in the case of spinning particles, see, e.g., Pashnev & Sorokin, 1991),

$$Q^i = \bar{Q}^i = H = J_{kl} = 0\,.$$

(b) To explicitly construct some other $\mathcal{N}=4$ QK SQM models, e.g. associated with symmetric QK manifolds ("Wolf spaces").

(c) To construct locally supersymmetric versions of other off-shell $\mathcal{N}=4, d=1$ multiplets (such as $(3,4,1)$, $(1,4,3)$, etc) and of the associated SQM systems (Landau-type, Calogero-Moser-type and others).

(d) Links between the two types of SQM deformations?

- E. Ivanov, L. Mezincescu, *Quaternion-Kähler* $\mathcal{N} = 4$ *supersymmetric mechanics*, JHEP **1712** (2017) 016.

- E. Ivanov, L. Mezincescu, *QK* $\mathcal{N} = 4$ *quantum mechanics: Hamiltonian formalism and supercharges* (in preparation).