Symmetires of generalized Calogero systems

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The talk is based on the results obtained in collaboration with

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Our goal and previous studies

The main goal of this talk is the the symmetry algebra of the nonlocal rational Calogero-type Hamiltonians using the Dunkl operator approach.

Our publications on the subject:

1. T. H., O. Lechtenfeld, A. Nersessian, *Superintegrability of generalized Calogero models with oscillator or Coulomb potential*, Phys. Rev. D **90**, 101701(R) (2014)
2. M. Feigin, T. H., *On the algebra of Dunkl angular momentum operators*; JHEP **11** 107 (2015)
3. T. H., A. Nersessian, *Runge-Lenz vector in Calogero-Coulomb problem*, Phys. Rev. A **92**, 022111 (2015)
4. F. Correa, T. H., O. Lechtenfeld, A. Nersessian, *Spherical Calogero model with oscillator/Coulomb potential: quantum case*; Phys. Rev. D **93**, 125009 (2016)
5. T. H., A. Nersessian, *Integrability and separation of variables in Calogero-Coulomb-Stark and two-center Calogero-Coulomb systems*, Phys.Rev. D **93** 045025 (2016)

Calogero-Moser model

The Calogero-Moser model (”free” Calogero model) describes $1d$ particles interacting with $1/r^2$ potential [Calogero (1969,1971), Moser (1975)],

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j} \frac{g(g \mp \hbar)}{(x_i - x_j)^2}.$$

Properties:

- Is integrable by the Lax and matrix model methods with N Liouville integrals.
- Is maximally superintegrable both in classical [Wojciechowsk (1983)] and quantum [Kuznetsov (1995)] cases with $N-1$ additional constants of motion.
- The quantum model can be solved using the exchange (Dunkl) operator formalism [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)].
- Superintegrability is preserved in the presence of the oscillator and Coulomb potentials.

Coxeter group extension

Calogero-Moser model associated with finite reflection (Coxeter) group:

$$H = \frac{p^2}{2} + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha(g_\alpha \mp \hbar)}{(\alpha, x)^2}$$

Properties:

- It describes as a single N-dimensional particle

$$x = (x_1, \dots x_N), \qquad p = (p_1, \dots, p_N), \qquad r = \sqrt{x^2}.$$

- $\mathcal{R}_+$ is a system of positive roots of the finite reflection group W.
- Coupling constants g_α form a W-invariant discrete function.

Finite reflection groups

A finite reflection group (Coxeter group) W is generated by reflections s_α across the selected hyperplanes $(x,\alpha)=0$ in $\mathbb{R}^N$:

$$w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_l}, \qquad w \in W$$
$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha,\alpha)}\alpha, \qquad \alpha \in \mathcal{R}$$

- The set of vectors α form the root system $\mathcal{R}$. Two vectors $\pm\alpha$ describe the same reflection.
- W-invariance of the root system: if $\alpha \in \mathcal{R}$ then $w(\alpha) \in \mathcal{R}$ since

$$w s_\alpha w^{-1} = s_{w(\alpha)}, \qquad w \in W,$$

- The reflection-invariance of the coupling constant,

$$g_{w(\alpha)} = g_\alpha .$$

Calogero-Coulomb model

The Calogero-Coulomb model [Khare (1996), Khare & Ghosh (1999)],

$$H_\gamma = \frac{p^2}{2} + \sum_{i<j} \frac{g(g \mp \hbar)}{(x_i - x_j)^2} - \frac{\gamma}{r}, \qquad r = \sqrt{x^2}$$

Properties:

- It is integrable [Calogero (1973)], [Khare (1996), Khare & Ghosh (1999)]
- The eigenfunctions have been calculated explicitly [Khare (1996)]
- The system is superintegrable with an analog of Runge-Lenz vector [T.H., Lechtenfeld & Nersessian (2014); T.H. & Nersessian (2015)]
- The superintegrability is preserved for the systems defined on hypersphere [Correa, T.H., Lechtenfeld & Nersessian (2016)]
- Admits integrable extensions for Stark potential and two-center Coulomb system [T.H. & Nersessian (2016)]

Dunkl operators

Define the Dunkl operators as a deformations of derivative [Dunkl (1988)]:

$$\nabla_i = \partial_i - \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha \alpha_i}{x_\alpha} s_\alpha, \qquad \text{where} \qquad \alpha_i = (\alpha, e_i), \qquad x_\alpha = (x, \alpha)$$

- They provide the deformed momentum operator (Dunkl momentum)

$$\pi_i = -\imath \nabla_i$$

- With coordinates they define Cherednik algebra with commutation rules:

$$[\pi_i, \pi_j] = 0, \qquad [\pi_i, x_j] = -\imath S_{ij},$$

$$S_{ij} = \delta_{ij} + \sum_{\alpha\in\mathcal{R}_+} \frac{2g_\alpha \alpha_i \alpha_j}{\alpha^2} s_\alpha.$$

- The reflection-invariant element:

$$S = -\sum_{\alpha\in\mathcal{R}_+} g_\alpha s_\alpha : \qquad [S, s_\alpha] = 0.$$

Dunkl operators for A_{N-1} root system

In case of the simplest A_{N-1} Coxeter root system:

- There are $\frac{N(N-1)}{2}$ positive roots

$$\mathcal{R}_+ = \{e_i - e_j | i > j\},$$

- The Weyl group coincides with the symmetric group of permutations: $W = \mathcal{S}_{N-1}$,
- The Dunkl operator

$$\nabla_i = \partial_i - \sum_{j \neq i} \frac{g}{x_i - x_j} s_{ij},$$

- s_{ij} are pairwise permutations $x_i \leftrightarrow x_j$, $\qquad S_{ij} = \begin{cases} -g s_{ij}, & \text{for } i \neq j, \\ 1 - \sum_{k \neq i} S_{ik}, & \text{for } i = j. \end{cases}$
- The invariant element is $S = \sum_{i<j} S_{ij}$

Nonlocal Calogero Hamiltonians

The nonlocal Calogero model is [T.H, Lechtenfeld, Nersessian (2014)]:

$$\mathcal{H}_0 = \frac{\pi^2}{2} - \frac{\gamma}{r} = \frac{p^2}{2} + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha(g_\alpha - s_\alpha)}{2x_\alpha^2}$$

$$= \frac{p^2}{2} + \sum_{i<j} \frac{g(g-s_{ij})}{(x_i - x_j)^2} \qquad \text{in } A_{N-1} \text{ case}$$

- On (anti)symmetric wavefunctions it reduces to the Calogero model,

$$\psi(s_\alpha x) = \pm\psi(x),$$

  with $+(-)$ sign for bosons (fermions).
- The Dunkl momenta are (commuting) integrals [Polychronakos (1992)],

$$\mathcal{H}_0 = \frac{\pi^2}{2}, \qquad [\mathcal{H}_0, \pi_i] = 0.$$

The nonlocal Calogero-oscillator and Calogero-Coulomb [T.H, Lechtenfeld, Nersessian (2014)] models:

$$\mathcal{H}_\omega = \mathcal{H}_0 + \frac{w^2 r^2}{2}, \qquad \mathcal{H}_\gamma = \mathcal{H}_0 - \frac{\gamma}{r}$$

Dunkl angular momentum tensor

Define the Dunkl angular momentum operator [Feigin (2003), Feigin & T.H. (2015)],

$$L_{ij} = x_i \pi_j - x_j \pi_i.$$

- The nonlocal Calogero Hamiltonian (with/without central potential $V(r)$) preserves it:

$$[\mathcal{H}_0, L_{ij}] = [\mathcal{H}_\omega, L_{ij}] = [\mathcal{H}_\gamma, L_{ij}] = 0$$

- L_{ij} satisfy $so(N)$ relations deformed with $\delta_{ij} \to S_{ij}$:

$$[L_{ij}, L_{kl}] = \imath L_{ik} S_{lj} + \imath L_{jl} S_{ki} - \imath L_{il} S_{kj} - \imath L_{jk} S_{li}.$$

- Their Casimir element is an analog of angular momentum square:

$$\mathcal{I} = L^2 + S(S - N + 2), \qquad L^2 = \sum_{i<j} L_{ij}^2$$

$$[L_{ij}, \mathcal{I}] = 0.$$

Structure of deformed $so(N)$ and $iso(N)$ algebras

- The essential different between deformed $so(N)$ is not a Lie algebra: the relation

$$[L_{ij}, L_{kl}] = \imath L_{ik}S_{lj} + \imath L_{jl}S_{ki} - \imath L_{il}S_{kj} - \imath L_{jk}S_{li} \quad \text{(a)}$$

  does not imply the Jacobi identity.

- There is a crossing relations among L_{ij}:

$$L_{ij}(L_{kl} + \imath S_{kl}) + L_{jk}(L_{il} + \imath S_{il}) + L_{ki}(L_{jl} + \imath S_{jl}) = 0 \quad \text{(b)}$$

- Appart from (a) and (b), there is no other relation between the deformed $so(N)$ generators [Feigin (2003), Feigin & T.H. (2015)]

- The symmetry of $\mathcal{H}_0$ is generated by π_i, L_{ij} forming deformation of Euclidean $ISO(N)$ generators:

$$[L_{ij}, \pi_k] = \imath\pi_i S_{kj} - \imath\pi_j S_{ki}$$

$$L_{ij}\pi_k + L_{jk}\pi_i + L_{ki}\pi_j = 0$$

Nonlocal Calogero-oscillator system I

The nonlocal Calogero-oscillator model is [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)]:

$$\mathcal{H}_\omega = \frac{p^2}{2} + \sum_{i<j} \frac{g(g - s_{ij})}{(x_i - x_j)^2} + \frac{\omega^2 r^2}{2} \qquad \text{in } A_{N-1} \text{ case}$$

The symmetries are generated by the Dunkl-operator deformation of the $SU(N)$ generators [Feigin & T.H. (2015)]; extension to sphere: [Correa, T. H., Lechtenfeld & Nersessian, (2016)]:

$$[\mathcal{H}_\omega, L_{ij}] = 0, \qquad [\mathcal{H}_\omega, I_{ij}] = 0.$$

- The additional integrals are provided by the Dunkl-deformed Fradkin tensor:

$$I_{ij} = x_i x_j + \pi_i \pi_j \qquad (\omega = 1)$$

- The commutation relations with the Dunkl angular momenta generators,

$$[L_{ij}, I_{kl}] = -\imath I_{ik} S_{jl} - \imath S_{jk} I_{il} + \imath I_{jk} S_{il} + \imath S_{ik} I_{jl}$$

Nonlocal Calogero-oscillator system II

- The commutations between deformed Fradkin tensor components,

$$[I_{ij}, I_{kl}] = \imath\Big(S_{jl}L_{ik} + L_{il}S_{jk} + L_{jk}S_{il} + S_{ik}L_{jl}\Big) + [S_{ij}, S_{kl}]\,.$$

- This set contains "Liouville" (commuting) integrals, including the Hamiltonian [Mathieu & Xudous (2001)], for $1/\sin^2$ [Bernard, Gaudin, Haldane & Pasquier (1993)]

$$D_i = I_{ii} + \sum_{j\neq i} \operatorname{sgn}(i-j)\, s_{ij}$$

$$[D_i, D_j] = 0$$

- The integrals of local Calogero-Coulomb model are symmetric polynomials on L_{ij}, I_{ij}, s_{ij}, for example:

$$\mathcal{D}_k = \sum_i (D_i)^k, \qquad \mathcal{I}_k = \sum_{i<j} (I_{ij})^k$$

- In particular,

$$\mathcal{D}_1 = -2\mathcal{H}_\omega$$

Generalized Polychronakos-Frahm chain

- Freeze particle positions of $\mathcal{H}_\omega$ at potential minimum $x = x^0$: $\dfrac{\partial V}{\partial x_i} = 0$ [Polychronakos (1993)]
- Consider $\hbar$-decomposition $(g = w = 1)$ [Frahm (1993); Mathieu & Xudous (2001)]

$$\mathcal{H}_\omega = V + \hbar H_1 - \frac{\hbar^2 \partial^2}{2}, \qquad V = \sum_i \frac{x_i^2}{2} + \sum_{i<j} \frac{1}{(x_i - x_j)^2},$$

- The first-order term describes full exchange Calogero-chain model

$$H_1 = \sum_{i<j} \frac{1}{(x_i^0 - x_j^0)^2} s_{ij}$$

- Upon adding internal $SU(n)$ spin degrees of freedom and replacing coordinate exchanges to spin exchange ($k = 1, \ldots, n$)

$$s_{ij} \to P_{ij}, \qquad P_{ij} | \ldots k_i \ldots k_j \ldots \rangle = | \ldots k_j \ldots k_i \ldots \rangle,$$

we get Polychronakos-Frahm chain with onsite $SU(n)$ spins $\vec{J}$:

$$\mathcal{H}_{\rm PF} = \sum_{i<j} \frac{P_{ij}}{(x_i^0 - x_j^0)^2} \sim \sum_{i<j} \frac{\vec{J}_i \cdot \vec{J}_j}{(x_i^0 - x_j^0)^2}$$

Symmetries of generalized Calogero chain

- Symmetries of the generalized Calogero chain H_1 correspond to $\hbar = 0$ limit of generalized Calogero-oscillator system $\mathcal{H}_\omega$

$$L^0_{ij} = x^0_i \pi^0_j - x^0_j \pi^0_i \,, \qquad I^0_{ij} = x^0_i x^0_j + \pi^0_i \pi^0_j$$

$$\pi^0_i = \sum_{j \neq i} \frac{\imath}{x^0_i - x^0_j} s_{ij}$$

$$[H_1, L^0_{ij}] = [H_1, I^0_{ij}] = [H_1, s_{ij}] = 0$$

- Their commutations are inherited from the commutations of the deformed $su(N)$ formed by L_{ij} and I_{ij}. The result in a degenerate deformed $su(N)$ algebra.

- Symmetrization simplifies the expression, like

$$\mathcal{I} = \sum_{i<j} I^0_{ij} s_{ij} = 3H_1 + 2S + \sum_{i<j} x^0_i x^0_j s_{ij} + \sum_{i,j,k,l}{}' \frac{s_{ijlk}}{(x^0_i - x^0_k)(x^0_j - x^0_l)}$$

- Problem: This set of integrals does not contain the chain Hamiltonian H_1.

Liouville integrals

- Take $\hbar$ decomposition of "diagonal" integral

$$D_i = D_i^0 + \hbar D_i^1 - \hbar^2 \partial_i^2$$

$$D_i^0 = I_{ii}^0 + \sum_{j\neq i} \mathrm{sgn}(i-j) s_{ij}, \qquad D_i^1 = I_{ii}^1 = -\imath\{\partial_i, \pi_i^0\}$$

and its symmetric polynomial

$$\mathcal{D}_k = \sum_i (D_i)^k = \mathcal{D}_k^0 + \hbar \mathcal{D}_k^1 + \hbar^2 \mathcal{D}_k^2 + \dots$$

- Their first-order terms $\mathcal{D}_k^1$ define nontrivial commuting integrals

$$\mathcal{D}_k^1 = \sum_i \sum_{l=0}^{k-1} (D_i^0)^l D_i^1 (D_i^0)^{k-l-1}, \qquad [\mathcal{D}_k^1, \mathcal{D}_l^1] = 0$$

- The first term of this set is chain Hamiltonian: $\mathcal{D}_1^1 = -2H^1$

- Problems: Is the set of integrals L_{ij}^0, I_{ij}^0, $\mathcal{D}_k^1$ complete? What are the commutations between them?

Dunkl Runge-Lenz vector

Define the Dunkl-operator deformation of Runge-Lenz vector [Feigin & T.H. (in preparation)]

$$A_i = \frac{1}{2}\sum_{j=1}^{N}\{L_{ij},\pi_j\} + \frac{\imath}{2}[\pi_i, S] - \frac{\gamma x_i}{r}$$

- It extends pervious construction for the root system $\mathcal{R} = \mathcal{A}_{N-1}$ [T.H. & Nersessian (2015)]. Possesses extensions to const. curv. spaces [Correa, T.H., Lechtenfeld & Nersessian (2016)].
- It is an integral of motion of nonlocal Calogero-Coulomb Hamiltonian:

$$[\mathcal{H}_\gamma, A_i] = 0$$

- Explicit expressions via coordinate and Dunkl momentum:

$$A_i = x_i\left(\pi^2 - \frac{\gamma}{r}\right) - \pi_i\left(rp_r - \imath\tfrac{N-3}{2}\right)$$
$$A_i = \left(\pi^2 - \frac{\gamma}{r}\right)x_i - \left(rp_r - \imath\tfrac{N+3}{2}\right)\pi_i$$

Properties of Runge-Lenz vector

It behaves as a vector under the Dunkl rotations

$$[A_i, L_{kl}] = \imath A_k S_{li} - \imath A_l S_{ki}$$

Useful relation:

$$A_i A_j = \left(\pi^2 - \frac{\gamma}{r}\right) x_i x_j \left(\pi^2 - \frac{\gamma}{r}\right) + \left(r p_r - \imath \tfrac{N+3}{2}\right) \pi_i \pi_j \left(r p_r - \imath \tfrac{N-3}{2}\right) - \left(\pi^2 - \frac{\gamma}{r}\right) x_i \pi_j \left(r p_r - \imath \tfrac{N-3}{2}\right) - \left(r p_r - \imath \tfrac{N+3}{2}\right) \pi_i x_j \left(\pi^2 - \frac{\gamma}{r}\right)$$

Its consequences:

- The commutation rule between the components:

$$[A_i, A_j] = -2\imath \mathcal{H}_\gamma L_{ij}$$

- Expression for deformed Runge-Lenz vector square:

$$A^2 = \gamma^2 + 2\mathcal{H}_\gamma \left(\mathcal{I} - S + \tfrac{(N-1)^2}{4}\right)$$

Deformed $so(N+1)$ symmetry

1. Extend L_{ij} by encapsulating A_i into extra $(N+1)$th dimension:

$$\tilde{L}_{ij} = L_{ij} \qquad \text{for} \quad i, j \le N$$
$$\tilde{L}_{i\,N+1} = -\tilde{L}_{N+1\,i} = \frac{A_i}{\sqrt{-2\mathcal{H}_\gamma}}, \qquad \tilde{L}_{N+1\,N+1} = 0$$

2. Do not change the root system $\mathcal{R}$ and reflection group W so that:

$$\tilde{S}_{ij} = S_{ij} \qquad \text{for} \quad i, j \le N$$
$$\tilde{S}_{i\,N+1} = \tilde{S}_{N+1\,i} = 0 \qquad \tilde{S}_{N+1\,N+1} = 1$$
$$\tilde{S} = S$$

Then all commutators between L_{ij} and A_i are unified into deformed $so(N+1)$ algebra:

$$[\tilde{L}_{ij}, \tilde{L}_{kl}] = \imath\tilde{L}_{ik}\tilde{S}_{lj} + \imath\tilde{L}_{jl}\tilde{S}_{ki} - \imath\tilde{L}_{il}\tilde{S}_{kj} - \imath\tilde{L}_{jk}\tilde{S}_{li}$$

Deformed $so(N+1)$ Casimir

- The Casimir element of Dunkl–deformed $so(N+1)$ algebra is given by the standard formula:

$$\tilde{\mathcal{I}} = \tilde{L}^2 + S(S-N+1) = -\frac{\gamma^2}{2\mathcal{H}_\gamma} - \frac{(N-1)^2}{4}$$

- As a result, the nonlocal Calogero-Coulomb Hamiltinain is expressed via it,

$$\mathcal{H}_\gamma = -\frac{2\gamma^{-2}}{\tilde{\mathcal{I}} + \frac{(N-1)^2}{4}}$$

- In the absence of Calogero interaction, it reduces to the well-known relations between the Coulomb Hamiltonuin and the Casimir element of its $so(N+1)$ symmetry generators.

Crossing relations

We have:

1. The following relation for three vectors $u = x, \pi, A$:

$$L_{ij}u_k + L_{jk}u_i + L_{ki}u_j = 0,$$

which reduces at $g_\alpha = 0$ and $N = 3$ to

$$\vec{L} \cdot \vec{x} = \vec{L} \cdot \vec{p} = \vec{L} \cdot \vec{A}$$

2. The crossing relation for the deformed $so(N)$ algebra,

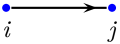
$$L_{ij}(L_{kl} + \imath S_{kl}) + L_{jk}(L_{il} + \imath S_{il}) + L_{ki}(L_{jl} + \imath S_{jl}) = 0$$

As a result, we get the crossing relation for the $so(N+1)$ case:

$$\tilde{L}_{ij}(\tilde{L}_{kl} + \imath \tilde{S}_{kl}) + \tilde{L}_{jk}(\tilde{L}_{il} + \imath \tilde{S}_{il}) + \tilde{L}_{ki}(\tilde{L}_{jl} + \imath \tilde{S}_{jl}) = 0$$

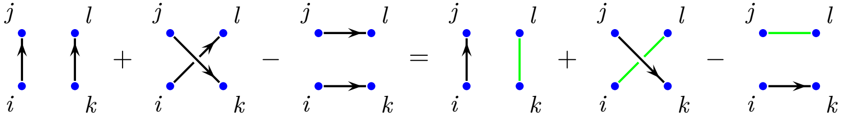
Graphical description of crossing relation

Graphical representation:

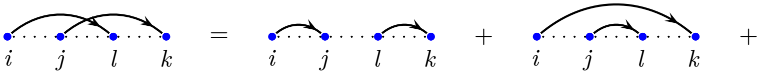
$\tilde{L}_{ij} =$ 

$\tilde{S}_{ij} =$ 

Each term is a products of two operators, with index l is on the right. The change of the operator order affects the right-hand side only:



The change of the operator order does not affect here:



$+$ [first-order terms in $\tilde{L}_{ij}$]

PBW basis in deformed $so(N+1)$ algebra

- Applying successively the commutation and crossing relations among $\tilde{L}_{ij}$ and $\tilde{S}_{ij}$ one can prove that the monomials

$$\tilde{L}^{n_1}_{i_1 j_1} \ldots \tilde{L}^{n_k}_{i_k j_k} w, \qquad w = s_\alpha s_\beta \ldots s_\gamma \in W$$

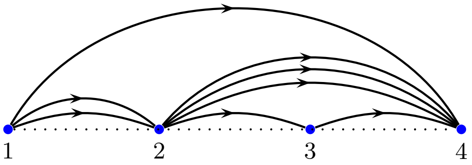
$$i_s < i_{s'} < j_s \quad \Rightarrow \quad j_{s'} \leq j_s \qquad (*)$$

  are (linearly) independent and form the Poincaré–Birkhoff–Witt (PBW) basis of deformed $so(N+1)$ algebra.

- The condition (*) means that the monomial diagram does not contain intersecting bonds (i_s, j_s).

- An example of nonintersecting monomial:

$$\tilde{L}^2_{12} \tilde{L}_{14} \tilde{L}_{23} \tilde{L}^3_{24} \tilde{L}_{34} \ =$$



Thank you!