Time-dependent non-Hermitian systems a status update

Andreas Fring

Supersymmetry in Integrable Systems - SIS'18
Bogoliubov Laboratory of Theoretical Physics of the JINR
Dubna, Russia, 13 - 16 August, 2018

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A. Fring, T. Frith, Physical Review A 95, 010102(R) (2017)

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A. Fring, T. Frith, arXiv:1808.03547

Outline

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- Theoretical framework (key equations)

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- The role of $H(t)$
  - governs unitary time-evolution
  - not observable and not the energy operator

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- Special features and concrete models

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- Theoretical framework (key equations)
- The role of $H(t)$
  - governs unitary time-evolution
  - not observable and not the energy operator
- Solutions procedures for Dyson operator
- Special features and concrete models
- Conclusions

Theoretical framework (key equations):

Time-dependent Schrödinger eqn for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \qquad \text{and} \qquad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

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Time-dependent Dyson operator

$$\phi(t) = \eta(t)\Psi(t)$$

$\Rightarrow$ Time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

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$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t)$$

$\Rightarrow$ Time-dependent quasi-Hermiticity relation

$$H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$$

[from conjugating Dyson relation and $\rho(t) := \eta^\dagger(t)\eta(t)$)]

$H(t)$ governs unitary time-evolution:

Hermitian:

$$\phi(t) = u(t,t')\phi(t'), \qquad u(t,t') = T\exp\left[-i\int_{t'}^{t} ds h(s)\right]$$

with

$$h(t)u(t,t') = i\hbar\partial_t u(t,t'), \quad u(t,t')u(t',t'') = u(t,t''), \quad u(t,t) = \mathbb{I}$$

$$\left\langle u(t,t')\phi(t') \middle| u(t,t')\tilde{\phi}(t') \right\rangle = \left\langle \phi(t) \middle| \tilde{\phi}(t) \right\rangle$$

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Non-Hermitian:

$$\Psi(t) = U(t,t')\Psi(t'), \qquad U(t,t') = T\exp\left[-i\int_{t'}^{t} ds H(s)\right]$$

$$H(t)U(t,t') = i\hbar\partial_t U(t,t'),\ U(t,t')U(t',t'') = U(t,t''),\ U(t,t) = \mathbb{I}$$

$$\left\langle U(t,t')\Psi(t') \middle| U(t,t')\tilde{\Psi}(t') \right\rangle_\rho = \left\langle \Psi(t) \middle| \tilde{\Psi}(t) \right\rangle_\rho$$

Relation between $u(t,t')$ and $U(t,t')$:

$$U(t,t') = \eta^{-1}(t)u(t,t')\eta(t')$$

or the generalized Duhamel's formula

$$\begin{aligned} U(t,t') &= u(t,t') - \int_{t'}^{t} \frac{d}{ds}\left[U(t,s)u(s,t')\right] ds \\ &= u(t,t') - i\hbar \int_{t'}^{t} U(t,s)\left[H(s) - h(s)\right] u(s,t') ds \end{aligned}$$

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Relation between Green's functions:

$$G_h(t,t') := -iu(t,t')\theta(t-t') \quad G_H(t,t') := -iU(t,t')\theta(t-t')$$

$$G_U(t,t') = G_u(t,t') + i \int_{-\infty}^{\infty} G_U(t,s)\left[H(s) - h(s)\right] G_u(s,t') ds$$

$H(t)$ is nonobservable and not the energy operator

Observables $o(t)$ in the Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian $\mathcal{O}(t)$ are quasi Hermitian

$$o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t) \,.$$

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Then we have

$$\langle \phi(t) \, | o(t)\phi(t) \rangle = \langle \Psi(t) \, | \rho(t)\mathcal{O}(t)\Psi(t) \rangle \,.$$

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Then we have

$$\left\langle \phi(t) \left| o(t)\phi(t) \right.\right\rangle = \left\langle \Psi(t) \left| \rho(t)\mathcal{O}(t)\Psi(t) \right.\right\rangle \,.$$

Since $H(t)$ is not quasi/pseudo Hermitian it is not an observable.
The observable energy operator is

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t).$$

Three scenarios:

1. $\partial_t \eta = 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

   Technically reduces to time-independent case.

   [C. Figueira de Morisson Faria, A. Fring; J. of Phys. A 39 (2006) 9269]

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   - Heisenberg picture: time-dependent observables
   - Schrödinger picture: time-dependent states
   - Metric picture: time-dependent metric operators

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3. $\partial_t \eta \neq 0$, $\partial_t H \neq 0$, $\partial_t h \neq 0$

   - Solve full quasi-Hermiticity relation for $\rho(t)$
     $\Rightarrow \eta(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$
   - Solve full time-dependent Dyson equation $\eta(t)$
     $\Rightarrow \rho(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$

Making sense of the broken $\mathcal{PT}$-regime:

Two-level system

$$H = -\frac{1}{2}\left[\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right]$$

with eigensystem

$$E_{\pm} = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \qquad \varphi_{\pm} = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

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with $\mathcal{PT}$-symmetry $\mathcal{PT} = \tau\sigma_z$; $\tau : i \to -i$

$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_\pm = e^{i\phi}\varphi_\pm \qquad \text{for} \quad |\lambda| > |\kappa|$$

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Claim: This system has real energies for $|\lambda(t)| < |\kappa(t)|$!

Two-dimensional system with infinite dimensional Hilbert space

$$H_K = aK_1 + bK_2 + i\lambda K_3, \qquad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$\begin{aligned} K_1 &= \frac{1}{2}\left(p_x^2 + x^2\right), \quad K_2 = \frac{1}{2}\left(p_y^2 + y^2\right), \quad K_3 = \frac{1}{2}\left(xy + p_x p_y\right) \\ K_4 &= \frac{1}{2}\left(x p_y - y p_x\right) \end{aligned}$$

$$\begin{array}{lll} [K_1, K_2] = 0, & [K_1, K_3] = iK_4, & [K_1, K_4] = -iK_3, \\ [K_2, K_3] = -iK_4, & [K_2, K_4] = iK_3, & [K_3, K_4] = i(K_1 - K_2)/2 \end{array}$$

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- H_K is $\mathcal{PT}$-symmetric: $[\mathcal{PT}_\pm, H_K] = 0$

  $\mathcal{PT}_\pm : x \to \pm x,\ y \to \mp y,\ p_x \to \mp p_x,\ p_y \to \pm p_y,\ i \to -i$

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- H_K is $\mathcal{PT}$-symmetric: $[\mathcal{PT}_\pm, H_K] = 0$

  $\mathcal{PT}_\pm : x \to \pm x,\ y \to \mp y,\ p_x \to \mp p_x,\ p_y \to \pm p_y,\ i \to -i$
- H_K is quasi-Hermitian: $h_K = \eta H_K \eta^{-1}$

$$h_K = \frac{1}{2}(a+b)\left(K_1 + K_2\right) + \frac{1}{2}\sqrt{(a-b)^2 - \lambda^2}\left(K_1 - K_2\right)$$

with $\eta = e^{2\theta K_4}$, $\theta = \operatorname{arctanh}[\lambda/(b-a)]$

Spontaneously broken $\mathcal{PT}$-symmetry for $a = b$:

Eigenenergies:

$$E_{n,m} = E^*_{m,n} = a(1 + n + m) + i\frac{\lambda}{2}(n - m)$$

Eigenfunctions:

$$\varphi_{n,m}(x,y) = \frac{e^{-\frac{x^2}{2}-\frac{y^2}{2}}}{2^{n+m}\sqrt{n!m!\pi}} \left[\sum_{k=0}^{n} \binom{n}{k} H_k(x) H_{n-k}(y)\right] \times \left[\sum_{l=0}^{m} (-1)^l \binom{m}{l} H_l(y) H_{m-l}(x)\right]$$

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Claim: This system has real energies for $a(t), \lambda(t)$!

Time-dependent system:

$$H(t) = \frac{a(t)}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + i\frac{\lambda(t)}{2}\left(xy + p_x p_y\right), \; a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod_{i=1}^{4} e^{\gamma_i(t) K_i}, \qquad \gamma_i \in \mathbb{R}$$

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Time-dependent Dyson equations is satisfied when

Constraint:

$$\gamma_1 = \gamma_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh\gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh\gamma_3 \sinh\gamma_4,$$

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$$h(t) = a(t)\left(K_1 + K_2\right) + \frac{\lambda(t)}{2} \frac{\sinh \gamma_4}{\cosh \gamma_3}\left(K_1 - K_2\right)$$

Solution: $\gamma_4 = \mathrm{arcsinh}\left(\kappa \,\mathrm{sech}\, \gamma_3\right), \; \chi(t) := \cosh \gamma_3, \; \kappa = const$
with dissipative Ermakov-Pinney equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2 \chi = \frac{\kappa^2 \lambda^2}{\chi^3}$$

Lewis Riesenfeld invariants:

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar \left[I_{\mathcal{H}}(t), \mathcal{H}(t) \right] = 0, \quad \text{for} \;\; \mathcal{H} = h = h^{\dagger}, H \neq H^{\dagger}$$

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The invariants I_H is quasi-Hermitian:

$$I_h(t) = \eta(t) I_H(t) \eta^{-1}(t)$$

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Solution to time-dependent Schrödinger equation:

$$\begin{aligned} I_{\mathcal{H}}(t) \left|\phi_{\mathcal{H}}(t)\right\rangle &= \Lambda \left|\phi_{\mathcal{H}}(t)\right\rangle, & \left|\Psi_{\mathcal{H}}(t)\right\rangle = e^{i\hbar\alpha(t)} \left|\phi_{\mathcal{H}}(t)\right\rangle \\ \dot{\alpha} &= \left\langle \phi_{\mathcal{H}}(t)\right| i\hbar\partial_t - \mathcal{H}(t) \left|\phi_{\mathcal{H}}(t)\right\rangle, & \dot{\Lambda} = 0 \end{aligned}$$

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Procedure:

1. Construct $I_h(t)$
2. Construct $I_H(t)$
3. Find $\eta(t)$ from similarity transformation

With Ansätze:

$$I_H(t) = \sum_{i=1}^{4} \alpha_i(t) K_i, \qquad I_h(t) = \sum_{i=1}^{4} \beta_i(t) K_i, \qquad h(t) = \sum_{i=1}^{4} b_i(t) K_i,$$

where $\alpha_i = \alpha_i^r + i\alpha_i^i \in \mathbb{C}$, $b_i, \beta_i, \alpha_i^r, \alpha_i^i \in \mathbb{R}$.

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We find

$$\gamma_3 = \arctan\left[\frac{\tanh\left[q_2 - \int_0^t \lambda(s)ds\right]}{\sqrt{1 - q_3^2 \operatorname{sech}\left[q_2 - \int_0^t \lambda(s)ds\right]^2}}\right]$$

$$\gamma_4 = -\operatorname{arccot}\left[\frac{1}{q_3}\cosh\left[q_2 - \int_0^t \lambda(s)ds\right]\right]$$

$q_2, q_3 = const$

With γ_3 we obtain a solution to the Ermakov-Pinney equation

$$\chi(t) = \cosh\gamma_3 = \sqrt{\frac{\cosh^2\left[q_2 - \int_0^t \lambda(s)ds\right] - q_3^2}{1 - q_3^2}}$$

where $\kappa = q_3/\sqrt{1-q_3^2}$, $|q_3| < 1$.

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Explicit form of the Hermitian Hamiltonian:

$$h(t) = f_+(t)K_1 + f_-(t)K_2$$

with

$$f_\pm(t) = a(t) \pm \frac{q_3\sqrt{1-q_3^2}\lambda(t)}{1+\cosh\left[2q_2 - 2\int_0^t \lambda(s)ds\right] - 2q_3^2}$$

Solution for time-dependent Schrödinger equation

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with phase

$$\alpha_n(t) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{a(s)}{\varkappa^2(s)} ds$$

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We compute

$$\langle \tilde{\varphi}_n(x,t)| K_1 |\tilde{\varphi}_m(x,t)\rangle = 2^{n-2} n!(2n+1)\sqrt{\pi}\frac{a^2(1+\varkappa^4) + \varkappa^2\dot{\varkappa}^2}{a^2\varkappa^2}\delta_{n,m}$$

$$\langle \tilde{\varphi}_n(x,t)|\tilde{\varphi}_n(x,t)\rangle = 2^n n!\sqrt{\pi} := N$$

right hand side does not depend on t:

$$\frac{d}{dt}\left[\frac{a^2(1+\varkappa^4)+\varkappa^2\dot{\varkappa}^2}{a^2\varkappa^2}\right]=\frac{2\dot{\varkappa}}{a^2}\left(\ddot{\varkappa}-\frac{\dot{a}}{a}\dot{\varkappa}+a^2\varkappa-\frac{a^2}{\varkappa^3}\right)=0$$

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take previous solution

$$\varkappa(t)=\sqrt{\tilde{\kappa}\cos\left[2\int_0^t a(s)ds\right]+\sqrt{1+\tilde{\kappa}^2}}$$

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For $\hat{\varphi}_n(x,t)=\tilde{\varphi}_m(x,t)/\sqrt{N}$ we compute

$$\langle\hat{\varphi}_n(x,t)|\,K_1\,|\hat{\varphi}_m(x,t)\rangle=\left(n+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}^2}\delta_{n,m}$$

Solution for

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$$\Psi_h^{n,m}(x,y,t) = \hat{\varphi}_n^+(x,t)\hat{\varphi}_m^-(y,t)$$

with $a \to f^\pm$, $\varkappa \to \varkappa_\pm$, $\tilde{\kappa} \to \tilde{\kappa}_\pm$, $\alpha_n \to \alpha_n^\pm$

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with $a \to f^\pm$, $\varkappa \to \varkappa_\pm$, $\tilde{\kappa} \to \tilde{\kappa}_\pm$, $\alpha_n \to \alpha_n^\pm$
gives real instantaneous energy expectation values

$$\begin{aligned} E^{n,m}(t) &= \langle \Psi_h^{n,m}(t)|\, h(t)\, |\Psi_h^{n,m}(t)\rangle = \langle \Psi_H^{n,m}(t)|\, \rho(t)\tilde{H}(t)\, |\Psi_H^{n,m}(t)\rangle \\ &= f_+(t)\left(n+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}_+^2} + f_-(t)\left(m+\frac{1}{2}\right)\sqrt{1+\tilde{\kappa}_-^2} \end{aligned}$$

for any given fields $a(t)$, $\lambda(t) \in \mathbb{R}$, constants $\tilde{\kappa}_\pm \in \mathbb{R}$, $|q_3| < 1$

Symmetry ensuring reality of $E(t)$

Back to time-dependent two-level system

$$H(t) = -\frac{1}{2}\left[\omega\mathbb{I} + \alpha\kappa(t)\sigma_z + i\kappa(t)\sigma_x\right] \quad h(t) = -\frac{1}{2}\left[\omega\mathbb{I} + \chi(t)\sigma_z\right]$$

either known $\kappa(t)$ unknown $\chi(t)$ or unknown $\kappa(t)$ known $\chi(t)$.

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Energy operator

$$\tilde{H}(t) = -\frac{1}{2}\left\{\omega\mathbb{I} + \frac{\chi}{\delta}\left[i\left(\alpha\xi - 1\right)\sigma_x + i\left(\hat{\xi}\sqrt{1-\alpha^2}\right)\sigma_y + (\xi - \delta)\sigma_z\right]\right\}$$

with instantaneous expectation values

$$\tilde{E}_\pm(t) = \left\langle \psi_\pm(t) \middle| \tilde{H}(t)\eta^2\psi_\pm(t) \right\rangle = -\frac{1}{2}\left[\omega \pm \chi(t)\right]$$

Time-dependent $\widetilde{\mathcal{PT}}$-symmetry

Solve

$$\left[\widetilde{\mathcal{PT}}, \tilde{H}\right] = 0, \qquad \widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}, \qquad \widetilde{\mathcal{PT}}^{2} = \mathbb{I}.$$

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Solve

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We find

$$\widetilde{\mathcal{PT}} = \frac{1}{\sqrt{(\xi-\delta)^2 + (\alpha^2-1)\hat{\xi}^2}}\left[i\left(\sqrt{1-\alpha^2}\hat{\xi}\right)\sigma_y + (\xi-\delta)\sigma_z\right]\tau$$

$$\tilde{\varphi}_\pm \sim \begin{pmatrix} (1\mp 1)\delta - \xi \\ \sqrt{1-\alpha^2}\hat{\xi} + i(1-\alpha\xi) \end{pmatrix}$$

$$\begin{aligned} \tilde{\omega}_+ &= \arctan\left[\frac{2\sqrt{1-\alpha^2}(1-\alpha\xi)\hat{\xi}}{1+\xi(\xi-2\alpha+\xi\alpha^2)+(\alpha^2-1)\hat{\xi}^2}\right], \\ \tilde{\omega}_- &= \arctan\left[\frac{\sqrt{1-\alpha^2}(1-\alpha\xi)\hat{\xi}}{2\delta^2-3\delta\xi+\xi^2+(\alpha^2-1)\hat{\xi}^2}\right] + \pi \end{aligned}$$

Hamiltonians of Euclidean-Lie algebraic type

$$E_2: \quad [u,J]=iv, \quad [v,J]=-iu, \quad [u,v]=0$$

$\mathcal{PT}$-symmetries:

$$\begin{array}{lllll}
\mathcal{PT}_1: & J\to -J, & u\to -u, & v\to -v, & i\to -i \\
\mathcal{PT}_2: & J\to -J, & u\to u, & v\to v, & i\to -i \\
\mathcal{PT}_3: & J\to J, & u\to v, & v\to u, & i\to -i \\
\mathcal{PT}_4: & J\to J, & u\to -u, & v\to v, & i\to -i \\
\mathcal{PT}_5: & J\to J, & u\to u, & v\to -v, & i\to -i
\end{array}$$

Invariant Hamiltonians:

$$\begin{aligned}
H_{\mathcal{PT}_i}(t) &= \mu_{JJ}(t)J^2+\mu_J(t)J+\mu_u(t)u+\mu_v(t)v+\mu_{uJ}(t)uJ \\
&\quad +\mu_{vJ}(t)vJ+\mu_{uu}(t)u^2+\mu_{vv}(t)v^2+\mu_{uv}(t)uv
\end{aligned}$$

$$\begin{array}{lll}
\mathcal{PT}_1: & (\mu_J,\mu_u,\mu_v)\in i\mathbb{R}, & (\mu_{JJ},\mu_{uJ},\mu_{vJ},\mu_{uu},\mu_{vv},\mu_{uv})\in\mathbb{R}, \\
\mathcal{PT}_2: & (\mu_J,\mu_{uJ},\mu_{vJ})\in i\mathbb{R}, & (\mu_u,\mu_v,\mu_{JJ},\mu_{uu},\mu_{vv},\mu_{uv})\in\mathbb{R}, \\
\mathcal{PT}_3: & (\mu_{JJ},\mu_J,\mu_{uv})\in\mathbb{R}, & \mu_u=\mu_v^*,\mu_{uJ}=\mu_{vJ}^*,\mu_{uu}=\mu_{vv}^* \\
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\mathcal{PT}_5: & (\mu_v,\mu_{vJ},\mu_{uv})\in i\mathbb{R}, & (\mu_J,\mu_u,\mu_{JJ},\mu_{uJ},\mu_{uu},\mu_{vv})\in\mathbb{R}
\end{array}$$

Solution for time-dependent Dyson relation:
Ansatz:

$$\eta(t) = e^{\tau(t)v} e^{\lambda(t)J} e^{\rho(t)u}$$

$$\begin{aligned} h_{\mathcal{PT}_1} = {} & J^2 \mu_{JJ} + \frac{[\mu_{vJ} \tanh\lambda - \mu_J \mu_{vJ}] \sinh\lambda}{2\mu_{JJ}} u - \frac{\mu_J \mu_{uJ} \tanh\lambda \operatorname{sech}\lambda}{2\mu_{JJ}} v \\ & + \left(\mu_{uu} - \frac{\mu_{uJ}^2 \tanh^2\lambda}{4\mu_{JJ}} \right) u^2 + \left(\mu_{uu} + \frac{\cosh^2(\lambda)\mu_{vJ}^2 - \mu_{uJ}^2}{4\mu_{JJ}} \right) v^2 \\ & + \mu_{uv} uv + \frac{\mu_{uJ}}{2} \operatorname{sech}\lambda \{u, J\} + \frac{\mu_{vJ}}{2} \cosh\lambda \{v, J\} \end{aligned}$$

with 7 constraining relations

$$\mu_v = \frac{\mu_J \mu_{vJ} - \dot{\mu}_{vJ} \tanh\lambda}{2\mu_{JJ}} - \frac{\mu_{uJ}}{2} \quad \tau = \frac{\mu_{vJ} \sinh\lambda}{2\mu_{JJ}} \quad \rho = \frac{\mu_{uJ} \tanh\lambda}{2\mu_{JJ}}$$

$$\mu_u = \frac{\mu_J \mu_{uJ} - \dot{\mu}_{uJ} \tanh\lambda}{2\mu_{JJ}} + \frac{\mu_{vJ}}{2} \quad \mu_{vv} = \mu_{uu} + \frac{\mu_{vJ}^2 - \mu_{uJ}^2}{4\mu_{JJ}} \quad \mu_{uv} = \frac{\mu_{uJ} \mu_{vJ}}{2\mu_{JJ}}$$

$$\lambda = -\int^t \mu_J(s) ds$$

Systems solved so far:

- non-Hermitian Swanson model
- one-site lattice Yang-Lee model
- non-Hermitian spin $1/2$, 1 and $3/2$ models
- two dimensional system with infinite Hilbert space
- general E_2-Lie algebraic Hamiltonians
- quasi-exactly solvable models of E_2-Lie algebraic type

Conclusions

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Thank you for your attention

Спасибо за внимание