On supersymmetric Calogero-Moser type systems

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Overview

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   - $\vee$-systems
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Motivation and history

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Ansatz for $\mathcal{N}=4$ supercharges:

$$Q^a = p_l \psi^{al} + W_l \psi^{al} + F_{lmn} \Psi^{lmn}, \quad a = 1, 2,$$

where Ψ^{lmn} is cubic in fermionic variables $\psi^{bk}, \bar{\psi}_b^k$ $(b = 1, 2)$, and $W = W(x), F = F(x)$ are some functions.

Some previous works:

- Wyllard'00, $su(1,1|2)$ symmetry. Supercharges depend on two potentials W, F. Calogero–Moser system at a particular coupling parameter when $W = 0$
- Bellucci, Galajinsky, Latini'05 F satisfies WDVV; study of 2-3 particles systems
- Galajinsky, Lechtenfeld, Polovnikov'07,'09 Extension of ansatz for other root systems; study of 3-4 particle systems, $W = 0$ solutions
- Fedoruk, Ivanov, Lechtenfeld'10 $D(2,1;\alpha)$-symmetry, one particle spin system
- Krivonos, Lechtenfeld'11 N particles; extra bosonic variables
- F, Silantyev'12 A class of algebraic solutions with $W \neq 0$
- Krivonos, Lechtenfeld, Sutulin'18 Supercharges for any $\mathcal{N}$-Calogero-Moser system with many fermionic variables

Ansatz

Consider N quantum particles on a line with coordinates and momenta $\{x_j, p_j | j = 1, \dots, N\}$. To each particle we associate four fermionic variables

$$\{\psi^{aj}, \bar{\psi}^j_a | a = 1, 2, j = 1, \dots N\}.$$

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$$\{\psi^{aj}, \bar{\psi}^j_a | a = 1, 2, j = 1, \dots N\}.$$

We impose the following (anti)-commutator relations:

$$[x_j, p_k] = i\delta_{jk}, \quad \{\psi^{aj}, \bar{\psi}^k_b\} = -\frac{1}{2}\delta^{jk}\delta^a_b, \quad \{\psi^{aj}, \psi^{bk}\} = \{\bar{\psi}^j_a, \bar{\psi}^k_b\} = 0.$$

One can think of p_k as $p_k = -i\frac{\partial}{\partial x_k}$.

Let $\mathcal{N}=4$ supercharges be

$$Q^a = p_l\psi^{al} + iW_l\psi^{al} + iF_{lmn}\langle\psi^{bl}\psi_b^m\bar{\psi}^{an}\rangle,$$
$$\bar{Q}_a = p_l\bar{\psi}_a^l - iW_l\bar{\psi}_a^l + iF_{lmn}\langle\bar{\psi}_d^l\bar{\psi}^{dm}\psi_a^n\rangle,$$

where $W=W(x)$, $F=F(x)$, $W_l=\frac{\partial W}{\partial x_l}$, $F_{lmn}=\frac{\partial^3 F}{\partial x_l\partial_m\partial_n}$; $a=1,2$, $\langle,\rangle$ is the Weyl anti-symmetrisation, $\psi_b^k=\epsilon_{ba}\psi^{ak}$, $\bar{\psi}^{bk}=\epsilon^{ba}\bar{\psi}_a^k$ with $\epsilon_{12}=-\epsilon_{21}=\epsilon^{21}=-\epsilon^{12}=1, \epsilon_{11}=\epsilon_{22}=\epsilon^{11}=\epsilon^{22}=0.$

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Imposing $\mathcal{N}=4$ supersymmetry one obtains:

- (generalised) Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations for F,

$$F_{rjk}F_{kmn}=F_{rmk}F_{kjn}, \quad (r,j,k,m,n=1,\ldots,N).$$

- twisted period equations for W,

$$\partial_{kl}W+F_{klj}\partial_j W=0, \quad (k,l,j=1,\ldots,N).$$

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- twisted period equations for W,

$$\partial_{kl}W + F_{klj}\partial_j W = 0, \quad (k,l,j=1,\ldots,N).$$

We set $W=0$.

$D(2,1;\alpha)$ superalgebra

[Frappat, Sorba, Sciarrino “Dictionary on Lie superalgebras”, 1996]

$D(2,1;\alpha)$ has 8 odd generators Q^{abc}, and 9 even ones J^{ab}, I^{ab}, T^{ab} $(a,b,c=1,2)$. The latter form mutually commuting $sl(2)$ subalgebras; $[T^{ab},I^{cd}]=[I^{cd},J^{ef}]=[T^{ab},J^{ef}]=0$.
The (anti)-commutation relations of $D(2,1;\alpha)$ read

$$\{Q^{ace},Q^{bdf}\}=-2\big(\epsilon^{ef}\epsilon^{cd}T^{ab}+\alpha\epsilon^{ab}\epsilon^{cd}J^{ef}-(1+\alpha)\epsilon^{ab}\epsilon^{ef}I^{cd}\big),$$

$$[T^{ab},T^{cd}]=-i\big(\epsilon^{ac}T^{bd}+\epsilon^{bd}T^{ac}\big),$$

$$[J^{ab},J^{cd}]=-i\big(\epsilon^{ac}J^{bd}+\epsilon^{bd}J^{ac}\big),\quad [I^{ab},I^{cd}]=-i\big(\epsilon^{ac}I^{bd}+\epsilon^{bd}I^{ac}\big),$$

$$[T^{ab},Q^{cdf}]=i\epsilon^{c(a}Q^{b)df},\quad [J^{ab},Q^{cdf}]=i\epsilon^{f(a}Q^{|cd|b)},$$

$$[I^{ab},Q^{cdf}]=i\epsilon^{d(a}Q^{|c|b)f},$$

where we symmetrise over two indices inside $(\dots)$ with indices inside $|\dots|$ being unchanged.

Root systems

Let $V = \mathbb{R}^N$, $u, \gamma \in V$ and $(\,,\,)$ the standard bilinear form in V.

Definition

Let R be a set of non-zero vectors in V s.t

1. $R \cap \mathbb{R}\gamma = \{-\gamma, \gamma\}$,
2. $s_\gamma R = R$,

$\forall \gamma \in R$. The set R is called a (Coxeter) root system with associated finite Coxeter group $W = \langle s_\gamma | \gamma \in R \rangle$.

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Let W be irreducible: A_N, B_N, D_N, $E_{6,7,8}$, F_4, $H_{3,4}$, $I_2(m)$.

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We fix $(\gamma, \gamma) = 2$ for all $\gamma \in R$.
For example,
$B_N = \{\pm(e_i \pm e_j), (1 \leq i < j \leq N), \pm\sqrt{2}e_i, (1 \leq i \leq N)\}$.

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For example,
$B_N=\{\pm(e_i\pm e_j),(1\leq i<j\leq N),\pm\sqrt{2}e_i,(1\leq i\leq N)\}$.

Let also $R=R_+\cup(-R_+)$.

1^{st} representation of $D(2,1;\alpha)$

Let

$$F = \frac{\lambda}{2} \sum_{\gamma \in R_+} (\gamma, x)^2 \log(\gamma, x), \quad \lambda \in \mathbb{C}.$$

Let $\{e_k\}_{k=1}^N$ be the standard basis of V with the corresponding coordinates $\{x_k\}_{k=1}^N$.

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Lemma

For any irreducible Coxeter root system R in a Euclidean space V and for any $u,v\in V$

$$\sum_{\gamma\in R_+}(\gamma,u)(\gamma,v)=h(u,v),$$

where h is the Coxeter number of W.

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Corollary

$$x_k F_{klm} = \lambda h \, \delta_{lm}.$$

Re-denote generators:

$$Q^a=-Q^{21a},\quad \bar{Q}^a=-Q^{22a},\quad S^a=Q^{11a},\quad \bar{S}^a=Q^{12a},$$

$$K=T^{11},\quad H=T^{22},\quad D=-T^{12}=T^{21}.$$

1st ansatz

Consider the following anzatz for the supercharges:

$$Q^a = p_r\psi^{ar} + iF_{rjk}\langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle,$$

$$\bar{Q}_c = p_l\bar{\psi}^l_c + iF_{lmn}\langle\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c\rangle,$$

where Weyl anti-symmetrization can be simplified to

$$F_{rjk}\langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle = F_{rjk}(\psi^{br}\psi^j_b\bar{\psi}^{ak} - \frac{1}{2}\psi^{ar}\delta^{jk}),$$

and

$$F_{lmn}\langle\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c\rangle = F_{lmn}(\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c - \frac{1}{2}\bar{\psi}^l_c\delta^{nm}).$$

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Note that under Hermitian conjugation † defined by

$$\psi^j_a{}^\dagger = \bar{\psi}^{aj}, \psi^{aj\dagger} = \bar{\psi}^j_a, p_j{}^\dagger = -p_j, x_j{}^\dagger = x_j, i^\dagger = -i, (AB)^\dagger = B^\dagger A^\dagger$$

we have $Q^{a\dagger} = \bar{Q}_a$.

Let also

$$K = x^2, \quad D = -\frac{1}{4}\{x_j, p_j\} = -\frac{1}{2}x_j p_j + \frac{iN}{2},$$

$$J^{ab} = J^{ba} = i\psi^{aj}\bar{\psi}^{bj} + i\psi^{bj}\bar{\psi}^{aj},$$

$$I^{11} = -i\psi^j_a\psi^{aj}, \quad I^{22} = i\bar{\psi}^{aj}\bar{\psi}^j_a, \quad I^{12} = -\frac{i}{2}[\psi^j_a, \bar{\psi}^{aj}],$$

$$S^a = -2x_j\psi^{aj}, \quad \bar{S}_a = -2x_j\bar{\psi}^j_a.$$

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Remark

If $\alpha = -1$ then this ansatz for $su(1,1|2)$ subalgebra generated by Q, S, J, K, D is a particular case of ansatz from [Galajinsky, Lechtenfeld, Polovnikov'09].

Hamiltonian

Theorem

For any $a, b \in \{1,2\}$ we have $\{Q^a, \bar{Q}_b\} = -2H\delta^a_b$, where the Hamiltonian H is given by

$$H = \frac{p^2}{4} - \frac{\partial_r F_{jlk}}{2}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk} + \frac{1}{4}\delta^{rj}\delta^{lk}) + \frac{1}{16}F_{rjk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{rk}.$$

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Theorem

Let

$$\lambda = -\frac{2\alpha+1}{h},$$

where h is the Coxeter number of the root system R. Then $D(2,1;\alpha)$ relations are satisfied.

Proposition

The Hamiltonian H has the following form:

$$4H = -\Delta + \sum_{\gamma \in R_+} \frac{\lambda(\lambda+1)(\gamma,\gamma)}{(\gamma,x)^2} + \Psi,$$

where

$$\Psi = 2\lambda \sum_{\gamma \in R_+} \frac{\gamma_r \gamma_j \gamma_k \gamma_l}{(\gamma,x)^2} \psi^{br} \psi_b^j \bar{\psi}_d^l \bar{\psi}^{dk} - 4\lambda \sum_{\gamma \in R_+} \frac{\gamma_r \gamma_j}{(\gamma,x)^2} \psi_b^r \bar{\psi}^{bj}$$

with $\gamma_k = (\gamma, e_k)$.

The bosonic part is Olshanetsky-Perelomov generalised Calogero-Moser Hamiltonian associated with R. E.g. for $R = B_N$ we have $-\Delta + \sum_{i<j} \frac{2\lambda(\lambda+1)}{(x_i \pm x_j)^2} + \sum_i \frac{\lambda(\lambda+1)}{x_i^2}$

2nd ansatz

Now let the supercharges be of the form (no antisymmetrisation in the third order terms)

$$Q^a = p_r \psi^{ar} + iF_{rjk}\psi^{br}\psi^j_b\bar{\psi}^{ak},$$

$$\bar{Q}_c = p_l \bar{\psi}^l_c + iF_{lmn}\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c.$$

Let also

$$K = x^2, \quad D = -\frac{1}{2}x_j p_j + \frac{i}{2}(\alpha+1)N,$$

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Note: This ansatz leads to the terms of the form $\frac{1}{x}p$.

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where $\partial_\gamma = (\gamma, \partial_x)$ *and*

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E.g. for $R = B_N$ the bosonic part is

$$-\Delta + \sum_{i<j} \frac{2\lambda}{x_i \pm x_j}(\partial_i \pm \partial_j) + \sum_i \frac{2\lambda}{x_i}\partial_i$$

The fermionic term Ψ is the same as in the 1^{st} representation.

Gauge relation

Denote by H_1 the Hamiltonian from the 1^{st} representation and denote by H_2 the one from the 2^{nd} representation (multiplied by 4):

$$H_1 = -\Delta + \sum_{\gamma\in R_+} \frac{\lambda(\lambda+1)(\gamma,\gamma)}{(\gamma,x)^2} + \Psi,$$

$$H_2 = -\Delta + \sum_{\gamma\in R_+} \frac{2\lambda}{(\gamma,x)}\partial_\gamma + \Psi.$$

Proposition

Let δ be the function $\delta = \prod_{\beta\in R_+}(\beta,x)^\lambda$. Then H_1 and H_2 are related by the gauge transformation

$$\delta^{-1}\circ H_2\circ\delta = H_1.$$

$\vee$-systems

Let $V \cong \mathbb{C}^N$ and $\mathcal{A} \subset V$ be a finite set of non-collinear covectors. Define a bilinear form $G_\mathcal{A}$ on V by

$$G_\mathcal{A}(u,v) = \sum_{\gamma\in\mathcal{A}} \gamma(u)\gamma(v), \quad u,v \in V,$$

and assume that $G_\mathcal{A}$ is non-degenerate. Then $V \cong V^*$ and $\gamma \in V^*$ corresponds to $\gamma^\vee \in V$ s.t $G_\mathcal{A}(\gamma^\vee, u) = \gamma(u)$ for any $u \in V$.

Definition (Veselov '99)

$\mathcal{A}$ is a $\vee$-system if for any $\gamma \in \mathcal{A}$ and $\pi \subset V^*$, $\dim \pi = 2$

$$\sum_{\beta\in\mathcal{A}\cap\pi} \beta(\gamma^\vee)\beta = \mu\gamma,$$

for $\mu = \mu(\gamma,\pi) \in \mathbb{C}$.

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Theorem (Veselov'99; FV'07)

F satisfies WDVV equations if and only if $\mathcal{A}$ is a $\vee$-system.

Supersymmetric extension of $\vee$-systems

Let us apply a linear transformation to $\mathcal{A}$ so that the bilinear form becomes the standard one in V:

$$G_{\mathcal{A}}(u,v)=(u,v), \quad u,v\in V.$$

We identify V and V^* so that $\gamma(u)=(\gamma^\vee,u)=(\gamma,u)$ for any $\gamma\in\mathcal{A}, u\in V$.

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Consider the 1st ansatz: $Q^a = p_r \psi^{ar} + iF_{rjk} \langle \psi^{br} \psi_b^j \bar{\psi}^{ak} \rangle$, $\bar{Q}_c = p_l \bar{\psi}_c^l + iF_{lmn} \langle \bar{\psi}_d^l \bar{\psi}^{dm} \psi_c^n \rangle$.

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Theorem

We have $\{Q^a, \bar Q_b\} = -\frac{1}{2}H_1\delta^a_b$, *where the Hamiltonian* H_1 *is*

$$H_1 = -\Delta + \frac{\lambda}{2}\sum_{\gamma\in\mathcal{A}}\frac{(\gamma,\gamma)^2}{(\gamma,x)^2} + \frac{\lambda^2}{4}\sum_{\gamma,\beta\in\mathcal{A}}\frac{(\gamma,\gamma)(\beta,\beta)(\gamma,\beta)}{(\gamma,x)(\beta,x)} + \Psi,$$

where $\Psi = \sum_{\gamma\in\mathcal{A}}\frac{2\lambda\gamma_r\gamma_j\gamma_l\gamma_k}{(\gamma,x)^2}\psi^{br}\psi^j_b\bar\psi^l_d\bar\psi^{dk} - \sum_{\gamma\in\mathcal{A}}\frac{2\lambda\gamma_r\gamma_j(\gamma,\gamma)}{(\gamma,x)^2}\psi^r_b\bar\psi^{bj}$.
All the relations of $D(2,1;\alpha)$ *are satisfied if* $\lambda = -(2\alpha+1)$.

Consider the 2nd ansatz:

$$Q^a = p_r \psi^{ar} + iF_{rjk}\psi^{br}\psi^j_b\bar{\psi}^{ak},$$

$$\bar{Q}_c = p_l \bar{\psi}^l_c + iF_{lmn}\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c.$$

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Theorem

We have $\{Q^a, \bar{Q}_b\} = -\frac{1}{2}H_2\delta^a_b$, *where the Hamiltonian* H_2 *is*

$$H_2 = -\Delta + \lambda \sum_{\gamma\in\mathcal{A}} \frac{(\gamma,\gamma)}{(\gamma,x)}\partial_\gamma + \Psi.$$

All the relations of $D(2,1;\alpha)$ *are satisfied if* $\lambda = -(2\alpha+1)$.

Proposition

We have gauge relation

$$\delta^{-1} \circ H_2 \circ \delta = H_1,$$

where $\delta = \prod_{\beta \in \mathcal{A}} (\beta, x)^{\frac{\lambda}{2}(\beta,\beta)}$.

Thank you for your attention!