Twistorial particle with continuous spin

Sergey Fedoruk

BLTP, JINR, Dubna, Russia

based on
J. Buchbinder, SF, A. Isaev, A. Rusnak
JHEP 1807 (2018) 031,
arXiv:1805.09706 [hep-th]

International Workshop
"Supersymmetry in Integrable Systems" (SIS'18)
13-16 August 2018, BLTP JINR, Dubna, Russia

In this talk there will be presented twistorial formulation of the $D = 4$ continuous spin particle.

The unitary irreducible representations of the Poincaré group are defined by values of the Casimir operators $P^m P_m$ and $W^m W_m$, $W_m = \frac{1}{2}\,\varepsilon_{mnkl} P^n M^{kl}$. In case of continuous spin representations these operators take the following value
[E. Wigner, 1947; V. Bargmann, E. Wigner, 1948]

$$P^m P_m = 0\,, \qquad W^m W_m = -\mu^2\,,$$

where the constant $\mu \in \mathbb{R}$, $\mu \neq 0$ is a dimensionful parameter.

In contrast to other physical states having a positive or zero mass,
continuous spin representation includes an infinite number of massless spin states.
Confusion in the name of this representation:
helicity in continuous spin representations takes standard discrete values $0, 1/2, 1, \ldots, \infty$.

This property of the continuous spin particles is very attractive at the present time by reason of the intensive development of higher-spin theory [M. Vasiliev, 1989, and others].
Lately a lot of research has been carried out on the continuous spin particles
(see, for example, [X. Bekaert, N. Boulanger, J. Mourad, 2006; P. Schuster, N. Toro, 2013;
V. Rivelles, 2015; R. Metsaev, 2017; X. Bekaert, E. Skvortsov, 2017;
M. Khabarov, Yu. Zinoviev, 2018; K. Alkalaev, M. Grigoriev, 2018]).

Space-time formulation

Up to now, all the considered formulations of the continuous spin particles have been the space-time ones. The fields of the continuous spin in this formulation are described by the function $\Phi(x, y)$ defined on the space which is parametrized by commuting four-vector x^m (the position coordinates on Minkowski space) and additional commuting vector variables y^m. The conditions describing the irreducible representation with continuous spin have the form [E. Wigner, 1947; V. Bargmann, E. Wigner, 1948]

$$\frac{\partial}{\partial x^m}\frac{\partial}{\partial x_m}\Phi = 0\,, \qquad \frac{\partial}{\partial x^m}\frac{\partial}{\partial y_m}\Phi = 0\,, \qquad \frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m}\Phi = \mu^2\,\Phi\,, \qquad -i\,y^m\,\frac{\partial}{\partial x^m}\,\Phi = \Phi\,.$$

The Wigner-Bargmann space-time field formulation of the continuous spin massless particle is reproduced by means of the one-dimensional dynamical model with the following Lagrangian in the first-order formalism

$$L_{sp.-time} = p_m\dot{x}^m + w_m\dot{y}^m + ep_mp^m + e_1p_mw^m + e_2\left(w_mw^m + \mu^2\right) + e_3\left(p_my^m - 1\right).$$

Here the quantities $p_m(\tau)$ and $w_m(\tau)$ are the momenta for $x^m(\tau)$ and $y^m(\tau)$;
their nonvanishing Poisson brackets are $\{x^m, p_n\} = \delta^m_n$, $\{y^m, w_n\} = \delta^m_n$.

The variables $e(\tau)$, $e_1(\tau)$, $e_2(\tau)$, $e_3(\tau)$ in the Lagrangian are the Lagrange multipliers for the constraints

$$T := p_mp^m \;\approx\; 0\,, \quad T_1 := p_mw^m \;\approx\; 0\,, \quad T_2 := w_mw^m + \mu^2 \;\approx\; 0\,, \quad T_3 := p_my^m - 1 \;\approx\; 0\,.$$

All these constraints are first class and
the Wigner-Bargmann equations arise as result of quantization of the particle model.

Twistorial formulation of continuous spin particle

We turn to constructing the twistorial formulation of the space-time system. To obtain this, we will follow the standard prescriptions of the twistor approach [R. Penrose, 1967].

The first step in this twistor program is to resolve the constraint $p_m p^m \approx 0$.
Introducing commuting Weyl spinor π_α, $\bar\pi_{\dot\alpha} = (\pi_\alpha)^*$, we represent light-like vector $p_m = p_{\alpha\dot\alpha}$ ($p_{\alpha\dot\beta} = \frac{1}{\sqrt{2}} p_m(\sigma^m)_{\alpha\dot\beta}$) by the Cartan-Penrose relation

$$p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}\,.$$

As a result of $\pi^\alpha\pi_\alpha \equiv 0$, the constraint $p_m p^m \approx 0$ is satisfied automatically. In the twistor formulation, the spinor π_α determines half of the components of the Penrose twistor.

The second step for obtaining the twistor formulation consists in a spinor representation of the four-vector components $w_m = w_{\alpha\dot\alpha}$ which should resolve the constraints $p_m w^m \approx 0$, $w_m w^m \approx -\mu^2$. For this, it is necessary to introduce in addition one more commuting Weyl spinor ρ_α, $\bar\rho_{\dot\alpha} = (\rho_\alpha)^*$. Then we represent $w_{\alpha\dot\alpha}$ in the form

$$w_{\alpha\dot\alpha} = \pi_\alpha \bar\rho_{\dot\alpha} + \rho_\alpha \bar\pi_{\dot\alpha}\,.$$

This expression solves the constraint $p_m w^m \approx 0$.

The constraint $w_m w^m \approx -\mu^2$ leads to the necessity of imposing in twistor space the following constraint:

$$\mathcal{M} \ := \ \pi^\alpha \rho_\alpha \, \bar\rho_{\dot\alpha} \bar\pi^{\dot\alpha} - M^2 \ \approx \ 0\,,$$

where the real constant M is defined by

$$M^2 = \mu^2/2\,.$$

As the next step we introduce canonically conjugated momenta for π_α, $\bar\pi_{\dot\alpha}$ and ρ_α, $\bar\rho_{\dot\alpha}$. To do this, we define the momenta ω^α, $\bar\omega^{\dot\alpha}$ and η^α, $\bar\eta^{\dot\alpha}$ by the following incidence relations:

$$\omega^\alpha = \bar\pi_{\dot\alpha} x^{\dot\alpha\alpha} + \bar\rho_{\dot\alpha} y^{\dot\alpha\alpha}\,, \qquad \bar\omega^{\dot\alpha} = x^{\dot\alpha\alpha}\pi_\alpha + y^{\dot\alpha\alpha}\rho_\alpha\,,$$
$$\eta^\alpha = \bar\pi_{\dot\alpha} y^{\dot\alpha\alpha}\,, \qquad \bar\eta^{\dot\alpha} = y^{\dot\alpha\alpha}\pi_\alpha$$

and find that, up to total derivative terms, they satisfy the relation

$$p_{\alpha\dot\alpha}\dot x^{\dot\alpha\alpha} + w_{\alpha\dot\alpha}\dot y^{\dot\alpha\alpha} = \pi_\alpha\dot\omega^\alpha + \bar\pi_{\dot\alpha}\dot{\bar\omega}^{\dot\alpha} + \rho_\alpha\dot\eta^\alpha + \bar\rho_{\dot\alpha}\dot{\bar\eta}^{\dot\alpha}\,.$$

Thus, $(\omega^\alpha,\pi_\alpha)$, $(\bar\omega^{\dot\alpha},\bar\pi_{\dot\alpha})$, $(\eta^\alpha,\rho_\alpha)$, $(\bar\eta^{\dot\alpha},\bar\rho_{\dot\alpha})$ are canonically conjugated pairs of the twistorial variables which obey the following Poisson brackets:

$$\{\omega^\alpha,\pi_\beta\} = \{\eta^\alpha,\rho_\beta\} = \delta^\alpha_\beta\,, \qquad \left\{\bar\omega^{\dot\alpha},\bar\pi_{\dot\beta}\right\} = \left\{\bar\eta^{\dot\alpha},\bar\rho_{\dot\beta}\right\} = \delta^{\dot\alpha}_{\dot\beta}\,.$$

In new variables, the last constraint $p_m y^m \approx 1$ and the incidence relations lead to the twistorial constraints

$$\eta^\alpha\pi_\alpha + \bar\pi_{\dot\alpha}\bar\eta^{\dot\alpha} - 2 \approx 0\,, \qquad \eta^\alpha\pi_\alpha - \bar\pi_{\dot\alpha}\bar\eta^{\dot\alpha} \approx 0\,.$$

The sum and difference of last two constraints give equivalent constraints

$$\mathcal{F} := \eta^\alpha\pi_\alpha - 1 \approx 0\,, \qquad \bar{\mathcal{F}} := \bar\pi_{\dot\alpha}\bar\eta^{\dot\alpha} - 1 \approx 0\,.$$

Moreover, the incidence relations imply also one more twistor constraint

$$\mathcal{U} := \omega^\alpha\pi_\alpha - \bar\pi_{\dot\alpha}\bar\omega^{\dot\alpha} + \eta^\alpha\rho_\alpha - \bar\rho_{\dot\alpha}\bar\eta^{\dot\alpha} \approx 0\,.$$

This finishes the twistorial formulation of the continuous spin particle model.

The continuous spin particle in the twistorial formulation is described by eight complex variables π_α, ω^α, ρ_α, η^α, which are subjected four first class (abelian) constraints

$$\mathcal{M} \;:=\; \pi^\alpha\rho_\alpha\,\bar\rho_{\dot\alpha}\bar\pi^{\dot\alpha} - M^2 \;\approx\; 0\,,$$

$$\mathcal{F} \;:=\; \eta^\alpha\pi_\alpha - 1 \approx 0\,, \qquad \bar{\mathcal{F}} \;:=\; \bar\pi_{\dot\alpha}\bar\eta^{\dot\alpha} - 1 \approx 0\,,$$

$$\mathcal{U} \;:=\; \omega^\alpha\pi_\alpha - \bar\pi_{\dot\alpha}\bar\omega^{\dot\alpha} + \eta^\alpha\rho_\alpha - \bar\rho_{\dot\alpha}\bar\eta^{\dot\alpha} \approx 0\,.$$

The twistorial model has $n_{phys} = 8$ of physical degrees of freedom as the model in the space-time formulation.

The corresponding twistorial Lagrangian is

$$L_{twistor} = \pi_\alpha\dot\omega^\alpha \;+\; \bar\pi_{\dot\alpha}\dot{\bar\omega}^{\dot\alpha} \;+\; \rho_\alpha\dot\eta^\alpha \;+\; \bar\rho_{\dot\alpha}\dot{\bar\eta}^{\dot\alpha} \;+\; l\,\mathcal{M} \;+\; k\,\mathcal{U} \;+\; \ell\,\mathcal{F} \;+\; \bar\ell\,\bar{\mathcal{F}}\,.$$

Some comments about our twistorial formulation of the continuous spin particle:

• In terms of the matrix

$$\|\,\Pi_\alpha{}^b\,\| := \Big(\Pi_\alpha{}^1, \Pi_\alpha{}^2\Big) \;=\; (\pi_\alpha, \rho_\alpha)$$

$(b = 1, 2)$ the constraint $\mathcal{M} \approx 0$ is equivalent to $\det\Pi = M\exp(i\varphi)$, where φ defines the phase of the contraction $\pi^\alpha\rho_\alpha$. So the matrix $M^{-1/2}\Pi_\alpha{}^b$ at $\varphi = 0$ describes group manifold $\mathrm{SL}(2,\mathbb{C})$ parametrizing the spinor Lorentz harmonics.

• The constraints $\mathcal{F} \approx 0$, $\bar{\mathcal{F}} \approx 0$ generate local transformations $\delta\Pi = \Pi\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$, where $n = n(\tau)$ is the local infinitesimal complex parameter. Thus, configuration space of the model is described by the matrix Π defined up to local transformations $g \to gN$, where the matrices $N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for any $n \in \mathbb{C}$ form the Borel subgroup $B_+(2,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$. The coset $\mathrm{SL}(2,\mathbb{C})/B_+(2,\mathbb{C})$ is the two-dimensional complex affine plane.

Quantization of twistorial model

Before the quantization is performed, we fix some gauges and pass to an appropriate phase variable, in which the constraints have a simple form.

The constraint $\mathcal{M} \approx 0$ involves only the norm $|\pi^\alpha \rho_\alpha|$. Gauge fixing condition for this constraint is the generator of conformal transformations of twistor components:

$$\mathcal{R} := \omega^\alpha \pi_\alpha + \bar{\pi}_{\dot\alpha} \bar{\omega}^{\dot\alpha} + \eta^\alpha \rho_\alpha + \bar{\rho}_{\dot\alpha} \bar{\eta}^{\dot\alpha} \approx 0 \,.$$

The constraint $\mathcal{U} \approx 0$ is the generator of phase transformations in twistor space. As result, the condition

$$\mathcal{K} := \ln\left(\frac{\pi^\alpha \rho_\alpha}{\bar{\rho}_{\dot\alpha} \bar{\pi}^{\dot\alpha}}\right) \approx 0$$

fix gauge.

Second class constraints $\mathcal{M} \approx 0$ and $\mathcal{K} \approx 0$ are equivalent to the constraints

$$\mathcal{N} := \pi^\alpha \rho_\alpha - M \approx 0 \,, \qquad \bar{\mathcal{N}} := \bar{\rho}_{\dot\alpha} \bar{\pi}^{\dot\alpha} - M \approx 0 \,.$$

In addition, the sum and difference of the second class constraints $\mathcal{U} \approx 0$ and $\mathcal{R} \approx 0$ are

$$\mathcal{V} := \omega^\alpha \pi_\alpha + \eta^\alpha \rho_\alpha \approx 0 \,, \qquad \bar{\mathcal{V}} := \bar{\pi}_{\dot\alpha} \bar{\omega}^{\dot\alpha} + \bar{\rho}_{\dot\alpha} \bar{\eta}^{\dot\alpha} \approx 0 \,.$$

Residual two first class constraints are

$$\mathcal{F} := \eta^\alpha \pi_\alpha - 1 \approx 0 \,, \qquad \bar{\mathcal{F}} := \bar{\pi}_{\dot\alpha} \bar{\eta}^{\dot\alpha} - 1 \approx 0 \,.$$

Let us pass to the variables in which the constraints have a simple form.

We use the following expansion of the matrix $\|\Pi_\alpha{}^b\| = (\pi_\alpha, \rho_\alpha)$:

$$\|\Pi_\alpha{}^b\| = \sqrt{M} \begin{pmatrix} z_1 & 0 \\ z_2 & s/z_1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where

$$z_\alpha = \pi_\alpha/\sqrt{M}\,, \qquad s = \pi^\alpha \rho_\alpha/M\,, \qquad t = \rho_1/\pi_1$$

are four dimensionless complex variables.

In new variables the second constraints take the form

$$s - 1 \approx 0\,, \qquad \bar{s} - 1 \approx 0\,,$$
$$-z_\alpha p_z^\alpha - 2sp_s \approx 0\,, \qquad -\bar{z}_{\dot\alpha}\bar{p}_z^{\dot\alpha} - 2\bar{s}\bar{p}_s \approx 0\,.$$

First class constraints take the form

$$p_t + 1 \approx 0\,, \qquad \bar{p}_t + 1 \approx 0\,.$$

We take account of the second class constraints by introducing the Dirac bracket for them and remove the variables s, $\bar{s}$, p_s, $\bar{p}_s$ from the phase space of the model.

We will perform quantization in the coordinate representation when the wave function is

$$\Psi = \Psi(z_\alpha, \bar{z}_{\dot\alpha}, t, \bar{t})\,.$$

The wave function of physical states is subjected to the first class constraints

$$\frac{\partial}{\partial t}\,\Psi = \frac{\partial}{\partial \bar{t}}\Psi = -i\Psi\,.$$

The solution of these equations, with taking into account the conditions of the second class constraints, is

$$\Psi \;=\; \delta(s-1)\,\delta(\bar{s}-1)\,e^{-i(t+\bar{t})}\;\tilde{\Psi}(z_\alpha, \bar{z}_{\dot\alpha})\,,$$

where $\tilde{\Psi}(z_\alpha, \bar{z}_{\dot\alpha})$ is the function on the two-dimensional complex affine plane parametrized by two complex coordinates $z_\alpha \in \overset{0}{\mathbb{C}}{}^2 = \mathbb{C}^2 \backslash (0,0)$.

Restoring the dependence of the wave function on twistor variables, we obtain the twistor wave function in the form

$$\Psi(\pi_\alpha, \bar{\pi}_{\dot\alpha}; \rho_\alpha, \bar{\rho}_{\dot\alpha}) \;=\; \delta\left(\pi^\beta \rho_\beta - M\right)\,\delta\left(\bar{\rho}_{\dot\beta}\bar{\pi}^{\dot\beta} - M\right)\,e^{-i\left(\dfrac{\rho_1}{\pi_1} + \dfrac{\bar{\rho}_1}{\bar{\pi}_1}\right)}\,\tilde{\Psi}(\pi_\alpha, \bar{\pi}_{\dot\alpha})\,,$$

where $\tilde{\Psi}(\pi_\alpha, \bar{\pi}_{\dot\alpha})$ is an arbitrary function on π_α, $\bar{\pi}_{\dot\alpha}$.

Let us analyze spin (helicity) content of the twistor wave function.

The Poincaré algebra operators in the twistor formulation have the form

$$\mathbb{M}_{\alpha\beta} = \pi_{(\alpha} \frac{\partial}{\partial \pi^{\beta)}} + \rho_{(\alpha} \frac{\partial}{\partial \rho^{\beta)}} , \qquad \bar{\mathbb{M}}_{\dot\alpha\dot\beta} = \bar\pi_{(\dot\alpha} \frac{\partial}{\partial \bar\pi^{\dot\beta)}} + \bar\rho_{(\dot\alpha} \frac{\partial}{\partial \bar\rho^{\dot\beta)}} , \qquad \mathbb{P}_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\beta} .$$

We see that $\mathbb{P}^{\alpha\dot\alpha}\mathbb{P}_{\alpha\dot\alpha} = 0$.

The Pauli-Lubanski operator $\mathbb{W}_{\alpha\dot\alpha} = \bar{\mathbb{M}}_{\dot\alpha\dot\beta}\mathbb{P}_{\alpha}^{\dot\beta} - \mathbb{M}_{\alpha\beta}\mathbb{P}_{\dot\alpha}^{\beta}$, takes the form

$$\mathbb{W}_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}\, \Lambda + \frac{1}{2}\left[\pi_\alpha \bar\rho_{\dot\alpha}\left(\bar\pi_{\dot\beta}\frac{\partial}{\partial \bar\rho_{\dot\beta}}\right) - \rho_\alpha \bar\pi_{\dot\alpha}\left(\pi_\beta \frac{\partial}{\partial \rho_\beta}\right)\right] + \frac{1}{2}\left[(\bar\pi^{\dot\beta}\bar\rho_{\dot\beta})\pi_\alpha \frac{\partial}{\partial \bar\rho^{\dot\alpha}} - (\pi^\beta \rho_\beta)\bar\pi_{\dot\alpha}\frac{\partial}{\partial \rho^\alpha}\right]$$

where

$$\Lambda = -\frac{1}{2}\left(\pi_\beta \frac{\partial}{\partial \pi_\beta} - \bar\pi_{\dot\beta}\frac{\partial}{\partial \bar\pi_{\dot\beta}}\right) .$$

The action of the Pauli-Lubanski operator on the twistorial wave function is

$$\mathbb{W}_{\alpha\dot\alpha}\Psi = \delta\left(\pi^\beta \rho_\beta - M\right)\delta\left(\bar\rho_{\dot\beta}\bar\pi^{\dot\beta} - M\right) e^{-i\left(\frac{\rho_1}{\pi_1} + \frac{\bar\rho_1}{\bar\pi_1}\right)} D_{\alpha\dot\alpha}\,\tilde\Psi .$$

where the operator $D_{\alpha\dot\alpha}$, acting on the reduced twistor field $\tilde\Psi$, takes the form

$$D_{\alpha\dot\alpha} \;:=\; \pi_\alpha \bar\pi_{\dot\alpha}\,\Lambda + iM\left(\frac{\epsilon_{\dot\alpha 1}\pi_\alpha}{\bar\pi_1} - \frac{\epsilon_{\alpha 1}\bar\pi_{\dot\alpha}}{\pi_1}\right) .$$

Using $D^{\alpha\dot\alpha}D_{\alpha\dot\alpha}\,\tilde\Psi = -2M^2\,\tilde\Psi$ we obtain that

$$\mathbb{W}^{\alpha\dot\alpha}\mathbb{W}_{\alpha\dot\alpha}\Psi = -2M^2\Psi_{tw} = -\mu^2\Psi .$$

So the twistor field describes a massless particle of continuous spin.

The states with fixed helicities are the eigenvectors of the helicity operator $\mathbb{\Lambda} = \vec{\mathbb{J}}\,\vec{\mathbb{P}}/\mathbb{P}_0$ ($\vec{\mathbb{J}}$ is the total angular momentum). In terms of the Pauli-Lubanski vector this operator is

$$\mathbb{\Lambda} \;=\; \frac{\mathbb{W}_0}{\mathbb{P}_0} \;=\; \frac{\mathbb{W}_{\alpha\dot{\alpha}}\tilde{\sigma}_0^{\dot{\alpha}\alpha}}{\pi_\beta\bar{\pi}_{\dot{\beta}}\tilde{\sigma}_0^{\dot{\beta}\beta}}\,.$$

The above representation of the twistor field is unsuitable for helicity expansion.

For finding helicity expansion we extract from $\tilde{\Psi}$ the exponential multiplier:

$$\tilde{\Psi}(\pi_\alpha,\bar{\pi}_{\dot{\alpha}}) \;=\; e^{\frac{-iM(\pi_1\pi_2+\bar{\pi}_1\bar{\pi}_2)}{\pi_1\bar{\pi}_1(\pi_\beta\bar{\pi}_{\dot{\beta}}\tilde{\sigma}_0^{\dot{\beta}\beta})}}\,\hat{\Psi}(\pi_\alpha,\bar{\pi}_{\dot{\alpha}})\,.$$

As result, we obtain the following representation of the twistorial field

$$\Psi(\pi_\alpha,\bar{\pi}_{\dot{\alpha}};\rho_\alpha,\bar{\rho}_{\dot{\alpha}}) \;=\; \delta\left(\pi^\beta\rho_\beta - M\right)\,\delta\left(\bar{\rho}_{\dot{\beta}}\bar{\pi}^{\dot{\beta}} - M\right)\,e^{-iw_0/p_0}\,\hat{\Psi}(\pi_\alpha,\bar{\pi}_{\dot{\alpha}})\,,$$

where p_0 and w_0 have the resolved in terms of spinors representations.

The action of the Pauli-Lubanski vector takes the form

$$\mathbb{W}_{\alpha\dot{\alpha}}\Psi = \delta\left(\pi^\beta\rho_\beta - M\right)\delta\left(\bar{\rho}_{\dot{\beta}}\bar{\pi}^{\dot{\beta}} - M\right)e^{-iw_0/p_0}\,\hat{D}_{\alpha\dot{\alpha}}\,\hat{\Psi}\,,$$

where the operator $\hat{D}_{\alpha\dot{\alpha}}$ (a rather cumbersome expression for this operator) satisfies the property $\sum\limits_{\alpha=\dot{\alpha}}\hat{D}_{\alpha\dot{\alpha}} = \mathbb{P}_0\Lambda$. As a result, the helicity operator acts on the twistorial field in the following way:

$$\mathbb{\Lambda}\,\Psi \;=\; \delta\left(\pi^\beta\rho_\beta - M\right)\delta\left(\bar{\rho}_{\dot{\beta}}\bar{\pi}^{\dot{\beta}} - M\right)e^{-iw_0/p_0}\,\Lambda\hat{\Psi}\,.$$

Thus, the eigenvalues of the helicity operator $\mathbb{\Lambda}$ is defined by the action of the operator Λ on the reduced twistorial function $\hat{\Psi}(\pi,\bar{\pi})$ living on the two-dimensional complex affine plane parametrized by π_α.

The eigenvectors of the operator $\Lambda = -\frac{1}{2}\left(\pi_\beta \frac{\partial}{\partial \pi_\beta} - \bar{\pi}_{\dot\beta} \frac{\partial}{\partial \bar{\pi}_{\dot\beta}}\right)$ are homogeneous functions.

The homogeneous components of the function $\hat{\Psi}(\pi, \bar{\pi})$ are determined by the Mellin transform

$$F^{(n_1,n_2)}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \; = \; \frac{i}{2} \int d\lambda \, d\bar{\lambda} \; \lambda^{-n_1} \, \bar{\lambda}^{-n_2} \, \hat{\Psi}(\lambda \, \pi_\alpha, \bar{\lambda} \, \bar{\pi}_{\dot\alpha}) \, ,$$

where $\chi = (n_1, n_2)$ are the pair of complex numbers whose difference is equal to an integer number. The functions $F^{(n_1,n_2)}(\pi_\alpha, \bar{\pi}_{\dot\alpha})$ are homogeneous of bi-degree $(n_1 - 1, n_2 - 1)$:

$$F^{(n_1,n_2)}(a\pi_\alpha, \bar{a}\bar{\pi}_{\dot\alpha}) \; = \; a^{n_1-1}\bar{a}^{n_2-1} F^{(n_1,n_2)}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \, .$$

which is equivalent to the fulfillment of the equations

$$\pi_\alpha \frac{\partial}{\partial \pi_\alpha} F^{(n_1,n_2)} = (n_1 - 1) F^{(n_1,n_2)} \, , \qquad \bar{\pi}_{\dot\alpha} \frac{\partial}{\partial \bar{\pi}_{\dot\alpha}} F^{(n_1,n_2)} = (n_2 - 1) F^{(n_1,n_2)} \, .$$

On such components, the helicity operator takes the values $s = -n/2$:

$$\Lambda \, F^{(n_1,n_2)} = -\frac{n}{2} \, F^{(n_1,n_2)} \, .$$

As a result this, the twistorial wave function of the continuous spin particle describes an infinite number of massless states whose helicities are equal to integer or half-integer values and run from $-\infty$ to $+\infty$:

$$\hat{\Psi}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \; = \; \sum_{n=-\infty}^{\infty} \mathcal{F}^{(n)}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \, , \qquad \mathcal{F}^{(n)}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \; = \; \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\nu \, F^{\left(\frac{n+i\nu}{2}, \, \frac{-n+i\nu}{2}\right)}(\pi_\alpha, \bar{\pi}_{\dot\alpha}) \, .$$

Precisely these twistorial fields $\mathcal{F}^{(n)}$ describe massless states with helicities $s = -n/2$.

Twistor transform for fields of continuous spin particles

Let us establish the connection of obtained twistor fields with the corresponding fields of the space-time formulation.

The twistorial variables π_α, ρ_α, on which the wave function depends, play a role of momentum variables. Therefore, we can consider the twistor wave function as a wave function in the momentum space. For this reason, up to the normalization multiplier the space-time wave function can be determined by means of the integral transformation

$$\Phi(x,y) \; = \; \int d^4\pi\, d^4\rho\, e^{ip_{\alpha\dot\alpha}x^{\dot\alpha\alpha}}\, e^{iw_{\alpha\dot\alpha}y^{\dot\alpha\alpha}}\, \Psi(\pi,\bar\pi;\rho,\bar\rho)\,,$$

where the momenta $p_{\alpha\dot\alpha}$ and $w_{\alpha\dot\alpha}$ are composite: $p_{\alpha\dot\alpha} = \pi_\alpha\bar\pi_{\dot\alpha}$, $w_{\alpha\dot\alpha} = \pi_\alpha\bar\rho_{\dot\alpha} + \rho_\alpha\bar\pi_{\dot\alpha}$.
In the integral we perform integration over the 4-dimensional complex space with the integration measure $d^4\pi\, d^4\rho := \frac{1}{4}\, d\pi^\alpha \wedge d\pi_\alpha \wedge d\bar\pi_{\dot\alpha} \wedge d\bar\pi^{\dot\alpha} \wedge \frac{1}{4}\, d\rho^\beta \wedge d\rho_\beta \wedge d\bar\rho_{\dot\beta} \wedge d\bar\rho^{\dot\beta}$.

Using explicit form of twistor fields we obtain the space-time fields of continuous spin particles

$$\Phi(x,y) = \int d^4\pi\, d^4\rho\, e^{ip_{\alpha\dot\alpha}x^{\dot\alpha\alpha}} e^{iw_{\alpha\dot\alpha}y^{\dot\alpha\alpha}} \delta\left(\pi^\beta\rho_\beta - M\right) \delta\left(\bar\rho_{\dot\beta}\bar\pi^{\dot\beta} - M\right) e^{-i\left(\frac{\rho_1}{\pi_1} + \frac{\bar\rho_1}{\bar\pi_1}\right)} \tilde\Psi(\pi,\bar\pi)\,.$$

We can check that this integral representations for the space-time fields are the solution of the Wigner-Bargmann equations.

Thus, we have constructed the integral relationship between the space-time fields and twistor ones, which is a generalization of the Penrose field twistor transform.
The twistor function $\tilde\Psi(\pi,\bar\pi)$ plays the role of the prepotential for the space-time field Φ.

Conclusion

We constructed the new Lagrangian model which describes a relativistic massless particle of the continuous spin. This model is characterized by the following:

- We present the classical Lagrangian which describes in space-time formulation the relativistic particle corresponding to the irreducible massless representation of the Poincaré group with continuous spin.
- We construct the classical twistor Lagrangian with twistor constraints and coordinate twistor transform which give the links between phase variables of space-time formulation and twistors.
- There are found the twistor fields of the continuous spin massless particles which depend only on twistor variables and have the helicities expansion.
- There are presented field twistor transform which expresses space-time fields in terms of twistor fields by the integral transformation.
- Obtained in the such way space-time field is the exact solution of the Wigner-Bargmann equations.

Conclusion

Let us note some comments on the constructed model.

i) Mass parameter μ determining the irreducible Poincaré representations has a role similar to the mass in the massive Poincaré representations: spin (helicity) contents of the continuous spin representations are the same for different values of μ.

ii) In the limit $\mu \to 0$ our twistorial model produces the massless higher spin particle. So, in the limit $\mu \to 0$ ($M \to 0$) the constraints $\pi^\alpha \rho_\alpha = M = 0$ and c.c. imply $\pi_\alpha \sim \rho_\alpha$ and $w_{\alpha\dot\alpha} \sim p_{\alpha\dot\alpha}$. As result, the integrand in obtained field twistor transform coincides with higher spin field in unfolded formulation [M.A. Vasiliev, 2002]

$$\Psi_{hsp}(x,\pi,\bar\pi) = e^{i\pi_\alpha\bar\pi_{\dot\alpha}x^{\dot\alpha\alpha}}\,\tilde\Psi_{hsp}(\pi,\bar\pi)\,,$$

where the function $\tilde\Psi_{hsp}(\pi,\bar\pi)$ of two complex variables π_α defines an infinite tower of the massless states with arbitrary helicities.

iii) There is a mixed Shirafuji formulation of massless particles which uses both space-time vector variables and twistorial ones, with a specific term $\pi_\alpha\bar\pi_{\dot\alpha}\dot x^{\dot\alpha\alpha}$ in the Lagrangian. This formulation of the continuous spin particle will make it possible to find additional relations of this system with a higher spin particle in the unfolded formulation.

iv) Another important task is the construction of the Lagrangian field theory of continuous spin. An effective way of realizing this task is to use BRST quantization methods. Some aspects of BRST-BFV formulations of the field theories in the case of continuous spin particles were studied in [A.K.H. Bengtsson, 2013; R. Metsaev, 2018; I. Buchbinder, V. Krykhtin, H. Takata, 2018].

THANK YOU !