Integrable Systems

Near Horizon Extremal Myers-Perry Black Holes

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Introduction

What is Myers-Perry Black Hole?

Myers-Perry (MP) Black Hole is the higher dimensional generalization of the rotating Kerr BH.

- In $d = 4$ dimensions the MP BH reduces to the Kerr BH.
- Setting all rotation parameters a_i to 0 will reduce the d dimensional MP BH to d dimensional Schwarzschild (non-rotating) BH. Now, also setting $M = 0$ yields the flat space metric.
- The form of MP metrics differs slightly for odd and even dimensions.

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What is Extremal MP Black Hole?

- Event horizon of Kerr BH is described by

$$r_H = M + \sqrt{M^2 - a^2}$$

- It follows that BHs with $a > M$ are not physical.
- r_H is real only for $a \leq M$. Black holes with $a = M$ are called **extremal** (BHs with biggest possible angular momentum $J = M^2$ for given BH mass)
- This discussion can be generalized for MP black hole

What is Near Horizon Limit?

Near Horizon Limit (NHL) is a vacuum solution of Einstein equations, which describe the space-time near the event horizon of extremal Kerr BH.

- Naturally, one can assume that NHL can be obtained by redefining the radial coordinate r in the metric of the extremal Kerr BH

$$r \longrightarrow r_H + \epsilon r_H r \qquad \text{with } \epsilon \longrightarrow 0$$

- But this redefinition gives rise to a degenerate metric. The problem can be resolved by taking additional limits

$$t \longrightarrow \frac{\alpha t}{\epsilon}, \qquad \phi_i \longrightarrow \phi_i + \frac{\beta_i t}{\epsilon}$$

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Near Horizon limit of an Extremal Myers-Perry Black Hole

NHEMP

NHEMP geometry slightly differs in odd and even dimensions. For that reason we introduce a unified description for arbitrary dimensions

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(x,\sigma) x_i x_j D\varphi^i D\varphi^j$$

where

$$N_\sigma = N + \sigma, \qquad \sigma = \begin{cases} 0 & \text{when} \quad D = 2N+1 \\ 1 & \text{when} \quad D = 2N+2 \end{cases}$$

NHEMP

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(\sigma) x_i x_j D\varphi^i D\varphi^j$$

- Latitudinal coordinates x_I and rotation parameters m_I are restricted:

$$\sum_{I=1}^{N_\sigma} \frac{x_I^2}{m_I} = 1, \qquad \sum_{I=1}^{N_\sigma} \frac{1}{m_I} = \frac{1+2\sigma}{1+\sigma}.$$

- One additional latitudinal coordinate in even dimensions
- Part of the metric is similar to AdS_2

The conformal $SO(2,1)$ symmetry

$$\{H,D\}=H,\quad \{H,K\}=2D,\quad \{D,K\}=K,\qquad \mathcal{I}=HK-D^2$$

allows us to redefine r and its canonical conjugate momentum p_r so the Hamiltonian takes formally non-relativistic form[1]

$$H=\frac{1}{2}p_R^2+\frac{2\mathcal{I}(x,p_x,p_\varphi)}{R^2}.$$

- $R=\sqrt{2K}$, $p_R=\frac{2D}{\sqrt{2K}}$ are the “radius" and its canonical conjugate momentum
- $\mathcal{I}$ is the Casimir element of $SO(2,1)$

[1] Hakobyan, Krivonos, Lechtenfeld, Nersessian. “Hidden symmetries of integrable conformal mechanical systems”. In: *Phys. Lett.* A374 (2010), pp. 801–806. arXiv: 0908.3290 [hep-th].

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Some important consequences...

- The radial part of the Hamiltonian is separated.
- We just need to study Casimir of $SO(2,1)$, which is called angular mechanics.
- The variables φ_i are cyclic. Thus their canonically conjugate momenta p_{φ_i} are first integrals (in total N for both odd and even dimensions).

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Fully non-isotropic NHEMP

Separation of variables

Here we assume that none of the rotation parameters m_I are equal to each other

- Angular mechanics is

$$\mathcal{I} = A(x)\left[\sum_{a,b=1}^{N_\sigma - 1} h^{ab}(x) p_a p_b + \sum_{i=1}^{N} \frac{p_{\varphi_i}^2}{x_i^2} + g_0(p_\varphi)\right]$$

- Separation of variables takes place in ellipsoidal coordinates

$$x_I^2 = (m_I - \lambda_I)\prod_{J \neq I}^{N_\sigma} \frac{m_I - \lambda_J}{m_I - m_J}$$

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Counting first integrals

- $N_\sigma - 1$ first integrals follow from the angular mechanics

$$F_a(x,\sigma) = K^{bc}_{(a)}(x,\sigma)\ p_b p_c + L^{ij}_{(a)}(x,\sigma)\ p_{\varphi_i} p_{\varphi_j} + A_{(a)}(x,\sigma) m_0^2 r_H^2$$

$$\#\text{ of first integrals} \quad N_\sigma - 1$$

- This expression can be extended for F_{N_σ} which we find to be the trivial first integral m_0 (mass of the particle) generated by the inverse metric (acting as a second rank killing tensor).

$$\#\text{ of first integrals} \quad N_\sigma$$

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Counting first integrals

- We had N first integrals p_{φ_i} from the beginning.

$$\#\text{ of first integrals} \quad N_\sigma + N$$

- In addition we had 3 generators of $SO(2,1)$, each of which are in involution with the $N_\sigma + N$ first integrals described above

$$\#\text{ of first integrals} \quad N_\sigma + N + 1$$

Thus fully non-isotropic NHEMP is integrable for arbitrary higher dimensions.

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Fully isotropic NHEMP

Separation of variables

Here we assume that all of the rotation parameters m_I are equal to each other. The angular mechanics is

$$\mathcal{I}_N = \sum_{i,j=1}^{N} (\eta^2(x)\delta_{ij} - x_i x_j) p_i p_j + \sum_{i=1}^{N} \frac{\eta^2(x) p_{\varphi_i}^2}{x_i^2} + \omega(p_{\varphi_i}) \sum_{i=1}^{N} x_i^2,$$

- In odd dimensions

  $$\eta^2 = N, \qquad \omega = 0$$

  The system is a generalization of Higgs oscillator, known as Rossochatius system.

- This is not the case in even dimensions.

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Separation of variables

- Both of the systems admit separation of variables by recursively introducing spherical coordinates

$$x_{N_\sigma} = \sqrt{N_\sigma} \cos \theta_{N_\sigma - 1}, \quad x_a = \sqrt{N_\sigma} \tilde{x}_a \sin \theta_{N_\sigma - 1}, \quad \sum_{a=1}^{N_\sigma - 1} \tilde{x}_a^2 = 1,$$

- As a result we get recurrent formula for the constants of motion

$$\mathcal{I}_{odd} = F_{N-1},$$

$$\mathcal{I}_{even} = 2Np_{\theta_N}^2 + \nu \sin^2 \theta_N + \left(2N \cot^2 \theta_N + 1\right) F_{N-1},$$

where

$$F_a = p_{\theta_a}^2 + \frac{p_{\varphi_{a+1}}^2}{\cos^2 \theta_a} + \frac{F_{a-1}}{\sin^2 \theta_a}, \quad F_0 = p_{\varphi_1}^2$$

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Counting first integrals

- Spherical mechanics result in $N_\sigma - 1$ first integrals, just like in the fully non-isotropic case.
- Even dimensional case contains the odd dimensional system as a subsystem.

In addition to this ... there are hidden symmetries, which

- make the odd dimensional Rossochatius system maximally superintegrable
- make the even dimensional system superintegrable (lacking one constant of motion to be maximally superintegrable)

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Partially isotropic NHEMP

Separation of variables

Let's discuss the simplest mixed case in odd $(2N+1)$ dimensions. We have p non-equal rotation parameters and l equal rotation parameters such that $p + l = N$

$$m_1 \neq m_2 \neq \ldots \neq m_p \neq \kappa, \quad m_{p+1} = m_{p+2} = \ldots = m_N \equiv \kappa.$$

- Non of the BH rotation parameters is 0.
- Separation of variables is achieved by introducing a mixture of spherical and ellipsoidal coordinates.

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Separation of variables

First we separate the latitudinal coordinates corresponding to equal set of rotation parameters by introducing spherical coordinates for them:

$$x_{p+1} = y \prod_{i=1}^{l-1} \sin\theta_i, \qquad x_{p+a} = y \cos\theta_{a-1} \prod_{i=a}^{l-1} \sin\theta_i, \quad x_{p+l} = y \cos\theta_{l-1}.$$

- y is the radius of spherical subsystem: $\sum_{a=1}^{l} x_{p+a}^2 = y^2$
- It behaves very much like the latitudinal coordinates corresponding to the set of non-equal rotation parameters.

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Separation of variables

After introducing this coordinate transformation into the initial metric, we can derive the angular mechanics

$$\mathcal{I} = A\left[\sum_{a,b=1}^{p} h^{ab} p_a p_b + \sum_{a=1}^{p+1} \frac{g_a^2}{y_a^2} + g_0\right],$$

where

$$y_a = (x_1, \ldots, x_p, y), \qquad \widetilde{m}_a = (m_1, \ldots, m_p, \kappa)$$
$$g_a^2 = (p_{\varphi_1}^2, \ldots, p_{\varphi_p}^2, \mathcal{I}_l), \qquad \mathcal{I}_l = F_{l-1}$$

- The form exactly corresponds with the fully non-isotropic NHEMP angular mechanics in p dimensions and can be separated in ellipsoidal coordinates.
- This system contain isotropic system as a subsystem

Counting first integrals

If $l = 1$ the spherical subsystem is trivial and does not produce new integrals of motion. This is the fully non-isotropic integrable case.
If $l \geq 2$ then

- The $l - 1$ dimensional spherical subsystem is maximally superintegrable

$$\# \text{ of first integrals} \quad 2(l-1) - 1$$

- The non-isotropic system contains p integrals of motion

$$\# \text{ of first integrals} \quad p + 2(l-1) - 1 = (N-1) + l - 2$$

In the fully isotropic ($p = 0,\ l = N$), the angular mechanics is maximally superintegrable with $2N - 3$ first integrals

Counting first integrals

This discussions can be generalized to the even dimensions, where we have an additional latitudinal coordinate (so $p + l = N + 1$) and a rotation parameter $m_{N+1} = 1$.

Why is fully isotropic NHEMP in even dimensions not maximally superintegrable

- Non of the other rotation parameters except m_{N+1} can be 1. So we always have $p \geq 1$
- As a result l can never be $N + 1$ and we will lack one constant of motion from being maximally superintegrable.

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NH limit of Extremal Vanishing Horizon MP BH (NHEVHMP)

NHEVHMP

NHEVHMP is obtained from the extremal MP metric by taking one of the rotation parameters equal to 0 and obtaining the NH limit. This results into a well defined solution of vacuum Einstein equations.

$$\frac{ds^2}{r_0^2} = F_0(x) ds^2_{AdS_3} + \sum_{a}^{N-1} dx_a^2 + \sum_{a,b}^{N-1} \tilde{\gamma}_{ab}(x)\, x_a x_b d\varphi_a d\varphi_b,$$

- Notice $ds^2_{AdS_3}$ term in the metric.
- The isometry contains $SO(2,1) \times SO(2,1)$ part.

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NHEVHMP

- Although we have two conformal groups, they give rise to the same Casimir element. Thus we have a single angular mechanics and no additional constants of motion compared to non-EVH case.
- The rest of the discussion is the same for fully isotropic, fully non-isotropic and generic cases.

Thank you!

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Equations

NHEMP geometry

$$\frac{ds^2}{r_H^2} = A(x;\sigma)\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \sum_{I=1}^{N_\sigma} dx_I dx_I + \sum_{i,j=1}^{N} \tilde{\gamma}_{ij}(x,\sigma) x_i x_j D\varphi^i D\varphi^j$$

where

$$N_\sigma = N + \sigma, \qquad \sigma = \begin{cases} 0 & \text{when} \quad D = 2N+1 \\ 1 & \text{when} \quad D = 2N+2 \end{cases},$$

$$A(x) = \frac{\sum_{I=1}^{N_\sigma} x_I^2/m_I^2}{\frac{\sigma}{1+\sigma} + 4\sum_{i<j}^{N} \frac{1}{m_i}\frac{1}{m_j}},$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{\sum_I^{N_\sigma} x_I^2/m_I^2} \frac{\sqrt{m_i - 1}x_i}{m_i} \frac{\sqrt{m_j - 1}x_j}{m_j}$$

First integrals of fully-non isotropic NHEMP

$$F_a(x,\sigma) = K^{bc}_{(a)}(x,\sigma)\ p_b p_c + L^{ij}_{(a)}(x,\sigma)\ p_{\varphi_i} p_{\varphi_j} + A_{(a)}(x,\sigma) m_0^2 r_H^2$$

where

$$K^{bc}_{(a)} = \left(\sum_{\alpha=0}^{N_\sigma - a - 1} (-1)^{N_\sigma + \alpha - a} A_\alpha m_b^{N_\sigma - \alpha - a} + x_b^2 \sum_{\alpha=1}^{N_\sigma - a - 1} (-1)^\alpha M^{\neq b}_{N_\sigma - \alpha - a - 1} m_b^\alpha \right) \delta^{bc} + M^{\neq b,c}_{N_\sigma - a - 1} x_b x_c$$

$$L^{ij}_{(a)} = \left((1 - \delta^1_a) \sum_{\alpha=1}^{N_\sigma - a} (-1)^{N_\sigma + \alpha} A_{\alpha - 1} m_i^{N_\sigma - a - \alpha + 1} - \delta^1_a A_{N_\sigma - 1} \right) \frac{\delta^{ij}}{x_i^2}$$

$$+ (-1)^{a-1} A_{N_\sigma - a} \frac{\sqrt{m_i - 1}}{m_i} \frac{\sqrt{m_j - 1}}{m_j}$$