Saito metric and determinant on Coxeter discriminant strata

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Overview

Root systems

Let $V = \mathbb{R}^n$, $u, \alpha \in V$ and $(\ ,\ )$ the standard bilinear form in V.

Definition

A reflection is a linear operator s_α on V defined by

$$u \mapsto s_\alpha u = u - 2\frac{(u, \alpha)}{(\alpha, \alpha)}\alpha.$$

It fixes a subspace of V of codimension 1, called a mirror (reflecting hyperplane). A **finite** group generated by reflections will be called *finite reflection group* and will be denoted by $W \subset O(V)$.

Definition

Let R be a finite set of non-zero vectors in V s.t

1. $R \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$,
2. $s_\alpha R = R$,

$\forall \alpha \in R$. The set R is called a root system with associated reflection group $W = \langle s_\alpha | \alpha \in R \rangle$.

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Definition

Let $R_+ \subset R$. We call R_+ a positive root system if

- for any $\alpha \in R$ exactly one of α, or $-\alpha$ is in R_+, and
- for any $\alpha \neq \beta \in R_+$ s.t $\alpha + \beta \in R$ then $\alpha + \beta \in R_+$.

The symmetric group

$W = S_n$, the symmetric group:

- Let ϵ_i, $i = 1, \ldots, n$ be the standard orthonormal basis in V,
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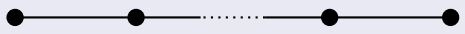
It fixes pointwise the line $L = \{\mathbb{R}\beta\}$, $\beta = \epsilon_1 + \cdots + \epsilon_n$. Hence, we usually denote W by A_{n-1}.

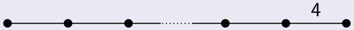
Classification-Coxeter '35

Coxeter diagrams of (irred.) finite Coxeter groups/ Classical series

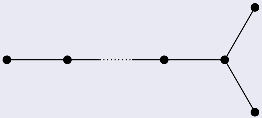
A_n, $(n \geq 1)$:



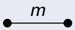
B_n, $(n \geq 2)$:



D_n, $(n \geq 4)$:



$I_2(m)$, $(m \geq 5)$:

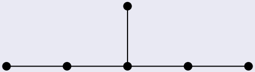


Code for graphs- B. McKay

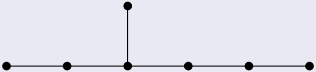
Classification-Coxeter '35

Coxeter diagrams of (irred.) finite Coxeter groups / Exceptional series

E_6:

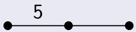


E_7:

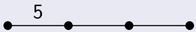


E_8:

H_3:



H_4:



F_4:

Frobenius algebras

Definition

An algebra $(\mathcal{A}, \circ, <,>)$ over $\mathbb{C}$ is called Frobenius if

- it is commutative, associative, with unity e,
- $<,>: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a non-degenerate bilinear form s.t

$$< a \circ b, c > = < a, b \circ c > \quad \forall a, b, c \in \mathcal{A}.$$

Frobenius manifolds

Definition (Dubrovin '94)

$(M, \circ, e, <,>, E)$ is a Frobenius manifold if each tangent space is a Frobenius algebra, $T_t M = \mathcal{A}_t$ varying smoothly over M s.t

- $<,>$ is a flat metric (Complex valued quadratic form),
- $\nabla e = 0$,
- the tensor $(\nabla_Z c)(X, Y, W)$ is totally symmetric for all $X, Y, Z, W \in TM$ with $c(X, Y, W) = \langle X \circ Y, W \rangle$,
- $\exists$ a linear vector field E i.e, $\nabla(\nabla E) = 0$ s.t

$$[E, e] = c_1 e, \quad \mathcal{L}_E <,>= c_2 <,>, \quad c_1, c_2 \in \mathbb{C}.$$

WDVV

In fact, a geometric reformulation of the **WDVV** equations.

- flat metric $<,>$ implies the existence of flat coordinates $t^1, \ldots, t^n$ and one may choose $e = \partial_1$. Then

$$c_{1\alpha\beta} = \langle \partial_\alpha, \partial_\beta \rangle \equiv \eta_{\alpha\beta} \quad \text{and hence} \quad c^\alpha_{\beta\gamma}(t) = \eta^{\alpha\epsilon} c_{\epsilon\beta\gamma}(t)$$

  with $\eta^{\alpha\epsilon} = (\eta_{\alpha\epsilon})^{-1}$, are the structure constants of $\mathcal{A}_t$.
- symmetry of c and ∇c implies the local existence of a function $F = F(t^1, \ldots, t^n)$ s.t $c_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F$.
- assoc. of $\mathcal{A}_t$ then implies *WDVV* equations:

$$\boxed{c_{\alpha\beta\gamma}\eta^{\gamma\epsilon}c_{\epsilon\mu\nu} = c_{\alpha\mu\gamma}\eta^{\gamma\epsilon}c_{\epsilon\beta\nu}.}$$

- F is called **prepotential** or free energy.

Orbit spaces

- Let $V = \mathbb{C}^n$, g the W-invariant standard constant metric given by
$$g(e_i, e_j) = \delta_{ij},$$
where $e_i, i = 1, \dots, n$ is the standard basis in V and let $\{x^i\}_{i=1}^n$ be the corresponding orthogonal coordinates.

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- Let $y^1(x),\dots,y^n(x)$ be a hom. basis in the ring of invariant polynomials
$S(V^*)^W=\mathbb{C}[x^1,\dots,x^n]^W=\mathbb{C}[x]^W=\mathbb{C}[y^1,\dots,y^n]$.

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- Let $d_i = \deg y^i$, $i = 1, \ldots, n$ and fix the ordering
$$h = d_1 > \cdots \geq d_n = 2.$$

We call h the **Coxeter number** of W.

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- Let $g^{\alpha\beta}$ be the corresponding contravariant metric. g is defined on $M_W \setminus \Sigma$, $\det(g^{\alpha\beta}(y)) = 0$ on Σ.

Definition (K.Saito et. al '80, B. Dubrovin '94)

The metric $\eta^{\alpha\beta} = \mathcal{L}_e g^{\alpha\beta}$ is called the *Saito* metric, it is defined up to proportionality and it is flat, where $e = \partial_{y^1}$.

There exists a distinguished basis $t^i \in \mathbb{C}[x]^W$, $(1 \le i \le n)$ s.t η is constant and antidiagonal,

$$\eta^{\alpha\beta} = \delta^{n+1,\alpha+\beta}.$$

Such coordinates are called *Saito* polynomials. They constitute examples of polynomial twisted periods (M. Feigin, A. Silantyev '12)

Example: $W = A_n$, Saito polynomials take the form

$$t_s = \mathrm{Res}_{z=\infty} \prod_{j=1}^{n+1} (z - x_j)^{\nu}|_{\sum x_j = 0},$$

with $\nu = \frac{s}{h}$, $\quad s = 1, \dots, n$.

- $F(t)$ is defined (up to quadratic terms) by

$$g^{\alpha\beta}(t) = \frac{(d_\alpha + d_\beta - 2)}{h}\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t).$$

- the structure constants $c^{\gamma}_{\alpha\beta}(t) = \eta^{\gamma\epsilon}\partial_\alpha\partial_\beta\partial_\epsilon F(t)$ are uniquely defined.

Theorem (Dubrovin'94)

There exists a polynomial Frobenius structure on M_W with the metric $\eta =<,>$ and

- *the Euler vector field $E = \sum_{i=1}^{n} \frac{1}{h} d_i y^i \partial_{y^i}$,*
- *the identity vector field $e := \partial_{y^1}$.*

Proposition

The determinant of the covariant Saito metric η in the x coordinates is given as

$$\det \eta(x) = c \prod_{\alpha \in R_+} g(\alpha, x)^2, \quad c \in \mathbb{C}^\times.$$

Coxeter discriminant

Definition (Strachan '04)

Let M be a Frobenius manifold. A natural submanifold N of M is a submanifold $N \subset M$ s.t

- $TN \circ TN \subset TN$,
- $E_x \in TN \quad \forall x \in TN$.

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Definition

Σ is called a Coxeter discriminant. It is the image of the union of the mirrors under the natural projection map

$$\pi : V \to M_W.$$

A stratum $\pi(D) \subset \Sigma$ is the image of the intersection subspace $D = \cap_{\beta \in B} \Pi_\beta$, where $B \subset R$, $\Pi_\beta = \{x \in V | g(x, \beta) = 0\}$.

Example: There are 5 strata in A_4, of type A_3, $A_2 \times A_1$, A_2, A_1^2, A_1.

- **Discriminant strata** are shown to be natural submanifolds of the Frobenius manifold M_W (Strachan '04; Feigin, Veselov '07, AFS'17)

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- The Saito metric on M_W induces a metric on $\pi(D)$ which is naturally given as the restriction of η to the stratum. Let us denote it by η_D.

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- The Saito metric on M_W induces a metric on $\pi(D)$ which is naturally given as the restriction of η to the stratum. Let us denote it by η_D.
- The linear coordinates x^i give rise to coordinates on the stratum D and on $\pi(D)$. These are flat coordinates for the restricted metric g on the stratum D. We denote this metric by g_D.

We are interested in answering the following:

Question

How does $\det \eta_D$ look in the flat coordinates of g_D on discriminant strata?

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Let us fix some notation:

- Let $L = \{\gamma_1, \dots, \gamma_k\} \subset R$, $1 \leq k \leq n$ and consider $D = \cap_{\gamma \in L} \Pi_\gamma$ s.t $\dim D = n - k$.
- For any $\beta \in R \setminus \langle L \rangle$, $\widehat{L} = L \cup \{\beta\}$, define $U_\beta = \langle \widehat{L} \rangle \cap R$.

The set U_β is a root system and admits the decomposition

$$U_\beta = \bigsqcup_{i=1}^{p} R_i, \tag{1}$$

where $\{R_i\}_{i=1}^{p}$ are irreducible root systems.

Theorem

The determinant of η_D on D is proportional to the product of linear factors

$$\prod_{l \in A} g_D(l, x)^{m_l}, \quad m_l \in \mathbb{N},$$

where A is a collection of non-proportional vectors on D. Furthermore, each $l \in A$ has the form β_D for some $\beta \in R \setminus \langle L \rangle$, where β_D is the orthogonal projection of β on D.

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Coxeter group, $W = A_4$

Example

Consider a stratum of type A_2 in A_4, let $D = \{x_1 = x_2 = x_3\}$. Coordinates on D are chosen as: $\xi_0 = x_1 = x_2 = x_3$, $\xi_1 = x_4$, $\xi_2 = x_5$. Then,

$$\det \eta_D = c(\xi_0 - \xi_1)^4(\xi_0 - \xi_2)^4(\xi_1 - \xi_2)^2.$$

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Q: How does this match the statement of the Theorem?

1. $\beta = e_3 - e_4$, $U_\beta = \langle e_1 - e_2, e_2 - e_3, \beta\rangle \cap R \cong A_3$, $h(A_3) = 4$,
2. $\beta = e_4 - e_5$, $U_\beta = \langle e_1 - e_2, e_2 - e_3, \beta\rangle \cap R \cong A_2 \sqcup A_1$, $\beta \in A_1$, $h(A_1) = 2$,
3. $\beta = e_3 - e_5$, $U_\beta = \langle e_1 - e_2, e_2 - e_3, \beta\rangle \cap R \cong A_3$, $h(A_3) = 4$,
4. No other factors in $\det \eta_D$, e.g $(e_2 - e_4)_D = (e_3 - e_4)_D$ etc.

Strata in type A_N

An arbitrary l-dimensional stratum $D \subset V$ has the form ($k \leq l$):

$$x_1 = \ldots = x_{m_0} = \xi_0,$$
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Coordinates on D: $\xi_0, \ldots, \xi_l$, where $\xi_i = x_i$, $i = k+1, \ldots, l$.
Then,

$$\boxed{\det \eta_D = c \prod_{0 \leqslant i < j \leqslant l} (\xi_i - \xi_j)^{m_i + m_j}}$$

where $c = (-1)^{\sum_{i=1}^{l} i m_i} (N+1)^{-N} \prod_{a=1}^{l} m_a^2 \prod_{a=0}^{l} m_a^{m_a - 1}$.

Proof of the Theorem ?

For **classical** series we use Landau-Ginzburg superpotential description of the Frobenius structures on the discriminant strata. In type A this superpotential on the stratum D is

$$\lambda(p) = \prod_{i=0}^{n} (p - \xi_i)^{m_i}, \quad m_i \in \mathbb{N}.$$

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The Saito metric and multiplication are:

$$\eta(\partial_i, \partial_j) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda'(p)} dp$$

$$\eta(\partial_i \circ \partial_j, \partial_k) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\lambda)\partial_j(\lambda)\partial_k(\lambda)}{\lambda'(p)} dp,$$

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For **exceptional** series, proof relies heavily on the geometry of root systems.

Thank you for your attention!