A few-body method for many-body systems

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Abstract. We used the Faddeev expansion of the potential in pairwise acting forces and the expansion of the resulting amplitudes in Potential Harmonics to obtain an integrodifferential equation valid for A boson systems. By introducing suitable transformation and taking limits $A \to \infty$ this equation is reduced into an Integro–Differential Equation suitable for handling bound states of large number of bosons. The new equation depends only on the input two-body interaction, it is quite simple, and the kernel has a simple analytic form. We employ the new equation to obtain results for $A \in (10-100)$ ^{87}Rb atoms interacting via a semi-realistic inter-atomic interactions and confined by an externally applied trapping potential $V_{\rm trap}(r)$. Our results are in excellent agreement with those previously obtained using the Potential Harmonic Expansion Method (PHEM) and the Diffusion Monte Carlo (DMC) method.

Keywords: static properties of condensates, hyperspherical methods, many-body theory
PACS: 03.75.Hh, 31.15.Ja, 24.10.Cn

INTRODUCTION

The study the A-boson bound state problem for systems up to $A = 4$ can be achieved using several methods. Among these methods those based on Faddeev-type equations and on Hyperspherical Harmonics Expansion (HHE) methods were extensively used during the last few decades to study in a rigorous way not only bosonic but fermionic systems as well. Going beyond the A=4 system, however, is not at present practical within the Faddeev scheme as the resulting equations (either in momentum or configuration space) are too complicated while in the HHE schemes, apart from the complexity of the equations, one faces also the question of convergence in the expansion especially when the interparticle forces have a short range repulsive core. Therefore alternative methods have to be used instead.

One such method is the Integro–Differential Equation Approach (IDEA) valid for A–body systems suggested by Fabre de la Ripelle and collaborators [1, 2]. It is based on the expansion of the Faddeev amplitudes in terms of Potential Harmonics (PH) [3, 4, 5] and it has been successfully applied in few–body calculations [6, 7], in realistic fermion systems [8], in unequal mass particle systems [9, 10, 11, 12] as well as in model calculations for the $A = 16$ system [13]. In all applications, the binding energies obtained are in good agreement with other results in the literature obtained by other methods.

When however the number of particles increases, the number of degrees of freedom also increases and the numerical complexity becomes intractable and one has no alternative but to use methods suitable for handling many-body systems. The typical number of atoms involved in the Bose-Einstein condensation (BEC), for example, is $10^3 - 10^6$

[14] and consequently studies of the BEC phenomenon are naturally based on quantum Monte Carlo type methods, such as, the Diffusion Monte Carlo (DMC) [15, 16], the Variational Monte Carlo (VMC) [17] and the practically exact Green Function Monte Carlo (GFMC) [18] methods.

A different approach to Monte Carlo methods is the one based directly on the PH expansion and it has been employed by Das and collaborators [19, 20, 21] to study the BEC phenomenon for ^{87}Rb atoms using repulsive inter-boson interactions. This Potential Harmonics Expansion Method (PHEM) requires the solution of a large number of differential equations which in turn requires the evaluation of Jacobi polynomials $P_K^{\alpha,\beta}(z)$ with $\alpha = (D-5)/2$, $\beta = 1/2+\ell$, D being the dimensionality of the A-boson system, $D = 3(A-1)$, ℓ is the partial wave for the system, and z is an angular variable. Furthermore, it requires the use of the so-called weight function $W(z) \equiv (1-z)^{\alpha}(1+z)^{\beta}$. It is clear that the accuracy in calculating the relevant quantities suffers with increasing A and the $W(z)$ has a spike similar to a δ-function for $z \sim -1$ which is difficult to treat numerically.

In the present work we also start by expanding the wave function for the A-body system system in to Faddeev components which in turn are expanded in terms of PH. The resulting system is then projected on the space of the pair (ij) resulting in the aforementioned IDEA equation which depend on two variables only, namely, the hyperradius r and the angular variable z while the corresponding kernel is expressed in terms of Jacobi polynomials $P_K^{\alpha,\beta}(z)$ and the weight function $W(z)$ and therefore one faces similar difficulties as in the PHEM. However, these difficulties can be removed by obtaining appropriate limits for $A \to \infty$. The new equation thus obtained, are quite simple, and the kernel depends on the much simpler Associated Laguerre polynomials $L_K^{1/2}$ which are independent of α [22]. The kernel can be even further simplified to have an analytic form, which does not depend on any polynomial, is independent from α, and only depends linearly on the number of particles A.

In what follows, we describe, in Sect. , how one can obtain from the IDEA, the new integro-differential equation suitable for large number of particles A. We then apply it, in Sect. , to obtain results, first, for the hybrid nuclear model for ^{16}O system where the particles are assumed to interact via short range strong forces of Wigner type; second, we apply it to ^{87}Rb atoms for various A and the results obtained are compared to those of the PHEM and the DMC methods. Our conclusions are summarized in Sect. .

THE FADDEEV-HHE FORMALISM

In the IDEA formalism the A-body wave function can be written as

$$\Psi(\mathbf{x}) = H_{[L_m]}(\mathbf{x}) \sum_{i<j\leq A} F(\mathbf{r}_{ij}, r) \tag{1}$$

where $H_{[L_m]}(\mathbf{x})$ is a harmonic polynomial of minimal degree L_m for the ground state, $\mathbf{x}$ is the coordinate vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_A)$, $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, in terms of the particle coordinates $\mathbf{x}_i$ while r is the hyperradius, $r = \left[2/A \sum_{i<j\leq A} r_{ij}^2\right]^{1/2}$. The functions $F(\mathbf{r}_{ij}, r)$

are two–body amplitudes obeying the Faddeev-type equation

$$\begin{aligned}\left[T+\frac{A(A-1)}{2}V_{[L_m]}(r) \quad - \quad E\right] H_{[L_m]}(\mathbf{x})\,F(\mathbf{r}_{ij},r) \\ = \quad -\left[V(r_{ij})-V_0^{[L_m]}(r)\right] H_{[L_m]}(\mathbf{x}) \sum_{k<l\leq A} F(\mathbf{r}_{kl},r)\,. \qquad (2)\end{aligned}$$

The hypercentral potential $V_0^{[L_m]}(r)$ is the average potential $V(r_{ij})$ taken over the $[L_m]$ state on the unit hypersphere $r=1$ of surface element $\mathrm{d}\Omega$

$$V_0^{[L_m]}(r)=\frac{\int H^*_{[L_m]}(\mathbf{x})V(r_{ij})H_{[L_m]}(\mathbf{x})\,\mathrm{d}\Omega}{\int |H_{[L_m]}(\mathbf{x})|^2\;\mathrm{d}\Omega}\,. \qquad (3)$$

We note that for ground states, the pairs are in an S–state and the amplitude F is a function of the hyperradius and $r_{ij}=|\mathbf{x}_i-\mathbf{x}_j|$ only.

The amplitudes can be expanded in terms of any set of harmonic polynomials such as the Hyperspherical Harmonics (HH). Expansion in terms of HH, however, results in a system of coupled differential equations which is impractical for numerical calculations especially when hard core potentials are employed or the number of particles considered is large. In addition it gives rise to degeneracy for a given grand orbital L and thus converged solutions are difficult to obtain.

A more efficient expansion can be made in terms of Potential Harmonics (PH) $\mathscr{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})$ [3, 4, 5] which form a complete basis for expanding continuous functions depending only on the relative coordinate r_{ij}. For systems in which the pair (ij) is in an ℓ-state while the other pairs are in an S-state, these polynomials are given by

$$\mathscr{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})=N_{K,\ell}Y_{\ell m}(\omega_{ij})\left(\frac{r_{ij}}{r}\right)^{\ell}P_K^{\alpha,\beta+\ell}(2\frac{r_{ij}^2}{r^2}-1) \qquad (4)$$

Here $Y_{\ell m}(\omega_{ij})$ is the spherical harmonic, $P_K^{\alpha,\beta}(z)$ is a Jacobi polynomial, and $N_{K,\ell}$ is a normalization constant which can be obtained from

$$\int_{(r=1)}\mathscr{P}^{\ell,m*}_{2K+\ell'}(\Omega_{ij})\mathscr{P}^{\ell',m'}_{2K'+\ell'}(\Omega_{ij})\,\mathrm{d}\Omega=\delta_{KK'}\delta_{\ell\ell'}\delta_{mm'}. \qquad (5)$$

The $\mathscr{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})$ are eigenfunctions of the operator $\hat{L}^2(\Omega)$

$$\left[\hat{L}^2(\Omega)+L(L+D-2)\right]\mathscr{P}^{\ell,m}_{2K+\ell}(\Omega_{ij})=0\,,\qquad L=2K+\ell\,, \qquad (6)$$

where $L^2(\Omega)$ is given by [3]

$$\hat{L}^2(\Omega)=\frac{4}{W(z)}\frac{\partial}{\partial z}(1-z^2)W(z)\frac{\partial}{\partial z}+2\frac{\hat{\ell}^2(\omega_{ij})}{1+z}+2\frac{\hat{L}^2(\Omega_{N-1})}{1-z}\,. \qquad (7)$$

The angular variable z is defined by

$$z=\cos 2\varphi=2\frac{r_{ij}^2}{r^2}-1\,,\qquad \cos\varphi=\frac{r_{ij}}{r}\,. \qquad (8)$$

Letting

$$F(r_{ij},r) = P(z,r)/r^{\mathscr{L}_m+1}\,, \tag{9}$$

where $\mathscr{L}_m = L_m + (D-3)/2$, and projecting on the r_{ij}–space one gets the IDEA equation for an A–particle system (see, for example, [2])

$$\begin{aligned} -\frac{\hbar^2}{m}\left[\frac{\partial^2}{\partial r^2}\right. &- \left.\frac{\mathscr{L}_m(\mathscr{L}_m+1)}{r^2} + \frac{4}{r^2}T(z) + \frac{A(A-1)}{2}V_0^{[L_m]}(r) - E\right]P(z,r) \\ &= -\left[V(r_{ij}) - V_0^{[L_m]}(r)\right]\left[P(z,r) + \int_{-1}^{+1} \mathscr{F}(z,z')\,P(z',r)\,dz'\right]. \end{aligned} \tag{10}$$

where $T(z)$ is the kinetic energy operator

$$T(z) = \frac{1}{W_{[L_m]}(z)}\,\frac{\partial}{\partial z}\,(1-z^2)W_{[L_m]}(z)\,\frac{\partial}{\partial z} \tag{11}$$

and $W_{[L_m]}(z)$ is the weight function which, for bosonic systems, is given by

$$W_{[L_m]}(z) = (1-z)^{\alpha}(1+z)^{\beta} \tag{12}$$

where $\alpha = (D-5)/2 + L_m - 2\ell_m$ and $\beta = 1/2 + \ell_m$. The kernel $\mathscr{F}(z,z')$ is the projection function which is expressed in terms of the Jacobi polynomials $P_K^{\alpha,\beta}(z)$,

$$\mathscr{F}(z,z') = W_{[L_m]}(z')\sum_K \frac{(f_K^2-1)}{h_K}P_K^{\alpha,\beta}(z)P_K^{\alpha,\beta}(z')\,. \tag{13}$$

The normalization h_K is given by

$$h_K = \int_{-1}^{+1}\left(P_K^{\alpha,\beta}(z)\right)^2 W_{[L_m]}(z)dz\,, \tag{14}$$

and the constant $f_K^2 - 1$ by

$$f_K^2 - 1 = \frac{2(A-2)P_K^{\alpha,\beta}(-1/2) + [(A-2)(A-3)/2]P_K^{\alpha,\beta}(-1)}{P_K^{\alpha,\beta}(+1)}\,. \tag{15}$$

When the number of particles A is large, the calculations with the above formalism becomes time consuming and cumbersome. There are two main reasons for this, the first one being the evaluation of the Jacobi polynomials $P_K^{\alpha,\beta}$ since the value of α becomes huge and the polynomials are highly oscillatory; the second reason stems from the behavior of the weight function which for $z \to -1$ is peaked at 2^{α}.

In our approach we consider first the factorization

$$r_{\rm ij} = r\zeta/\sqrt{\alpha}\,. \tag{16}$$

with $z = 2\zeta^2/\alpha - 1$. Then for $\alpha \to \infty$ we have the following limits,

$$P_K^{\alpha,\beta}(2r_{ij}^2/r^2 - 1) \underset{\alpha\to\infty}{\longrightarrow} (-1)^K L_K^{1/2}(\alpha r_{ij}^2/r^2) \equiv (-1)^K L_K^{1/2}(\zeta^2) \tag{17}$$

and

$$W(z) = C_W \frac{2^{\alpha+1/2+\ell}}{\alpha^{1/2+\ell}} \zeta^{\ell+1} \mathrm{e}^{-\zeta^2} \tag{18}$$

where C_W is the normalization constant for the weight function. For $\ell = 0$

$$\begin{aligned} h_K & \underset{\alpha\to\infty}{\longrightarrow} \int_0^{\sqrt{\alpha}} \left[L_K^{1/2}(\zeta^2)\right]^2 \mathrm{e}^{-\zeta^2} \zeta^2 \,\mathrm{d}\zeta \\ & \simeq \frac{1}{2} \int_0^{\infty} \left[L_K^{1/2}(x)\right]^2 \mathrm{e}^{-x} \sqrt{x} \,\mathrm{d}x \\ & = \frac{1}{2} \frac{\Gamma(K+3/2)}{K!} . \end{aligned} \tag{19}$$

In order to evaluate the kinetic energy $\hat{T}P(z,r)$, we consider first the factorization

$$P(\zeta, r) = \frac{\mathrm{e}^{\zeta^2/2}}{\zeta} Q(\zeta, r) . \tag{20}$$

Then

$$\hat{T}\,P = \frac{1}{W}\frac{\partial}{\partial z}(1-z^2) W \frac{\partial}{\partial z} P \tag{21}$$

$$\equiv \frac{\alpha}{4} \frac{\mathrm{e}^{\zeta^2/2}}{\zeta} \left[\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + 3 + 2\ell_m - \zeta^2 - \frac{2\ell_m}{\zeta^2} \right] Q(\zeta, r) \tag{22}$$

Therefore, Eq. (10) (we consider here the case where $L_m = 0$, $\mathscr{L}_m = \mathscr{L} \equiv (D-3)/2$, and $\ell_m = 0$) becomes

$$\begin{aligned} & \frac{\hbar^2}{m} \left[H_r + \frac{\alpha}{r^2} H_\zeta + \frac{A(A-1)}{2} V_0(r) - E \right] Q(\zeta, r) \\ = & -\left[V(r_{ij}) - V_0(r)\right] \left[Q(\zeta, r) + \int_0^{\sqrt{\alpha}} \mathscr{F}_E(z, z')\, Q(\zeta', r)\, \mathrm{d}\zeta' \right] . \end{aligned} \tag{23}$$

where

$$H_r = -\frac{\partial^2}{\partial r^2} + \frac{\mathscr{L}(\mathscr{L}+1)}{r^2} , \tag{24}$$

and

$$H_\zeta = \frac{\alpha}{4} \left[-\frac{\partial^2}{\partial \zeta^2} + \zeta^2 - 3 \right] . \tag{25}$$

The kernel $\mathscr{F}_E$ is given by

$$\mathscr{F}_E(\zeta, \zeta') = \zeta \mathrm{e}^{-\zeta^2/2} \sum_K \frac{2K!}{\Gamma(K+3/2)} (f_K^2 - 1) L_K^{1/2}(\zeta^2) L_K^{1/2}(\zeta'^2) \zeta' \mathrm{e}^{-\zeta'^2/2} , \tag{26}$$

We see that equation (23) is free from the δ-function type peak and, apart from the easily evaluable constant f_K^2-1, the kernel $\mathscr{F}_E$ does not depend on α.

Equation (23) can be even further simplified by noting that

$$\begin{aligned}\sum_K (f_K^2-1)\frac{K!}{\Gamma(K+3/2)}L_K^{1/2}(\zeta^2)L_K^{1/2}(\zeta'^2) \\ \underset{\alpha\to\infty}{\longrightarrow} 2(A-2)\sum_K\left(\frac{1}{4}\right)^K L_K^{1/2}(\zeta^2)L_K^{1/2}(\zeta'^2)/h_K \\ -2(A-2)\frac{1}{4}L_1^{1/2}(\zeta^2)L_1^{1/2}(\zeta'^2)/h_1 - L_1^{1/2}(\zeta^2)L_1^{1/2}(\zeta'^2)/h_1 \\ -2(A(A-2)/h_0 + [A(A-1)/2-1]/h_0 \end{aligned} \tag{27}$$

and thus by making use of the relation [23]

$$\sum_{K=0}^{\infty}\left(\frac{1}{4}\right)^K \frac{K!}{\Gamma(K+3/2)}L_K^{1/2}(\zeta^2)L_K^{1/2}(\zeta'^2) = \frac{4}{\sqrt{3\pi}}\,\mathrm{e}^{(\zeta^2+\zeta'^2)/3}\,\frac{\sinh(\frac{3}{4}\zeta\zeta')}{\zeta\zeta'}\,. \tag{28}$$

we obtain

$$\begin{aligned}\frac{\hbar^2}{m}\left\{H_r \;+\; \frac{4}{r^2}H_\zeta + \frac{A(A-1)}{2}V_0(r) - E\right\}Q(\zeta,r) \\ = \;-\left[V(r_{ij})-V_0(r)\right]\left[Q(\zeta,r)+\int_0^{\sqrt{\alpha}}\mathscr{F}_I(\zeta,\zeta')Q(\zeta',r)\,\mathrm{d}\zeta'\right]\end{aligned} \tag{29}$$

The new form of the kernel $\mathscr{F}_I$ is

$$\begin{aligned}\mathscr{F}_I(\zeta,\zeta') \;=\; \frac{2(A-2)}{\sqrt{3}}\left\{\left[A-3-\frac{2}{3}(\zeta^2-\frac{3}{2})(\zeta'^2-\frac{3}{2})\right]\zeta\zeta'\mathrm{e}^{-(\zeta^2+\zeta'^2)/2}\right. \\ \left. +\;\frac{4}{\sqrt{3}}\left[\mathrm{e}^{-[(5(\zeta-\zeta')^2+2\zeta\zeta')/6]} - \mathrm{e}^{-[(5(\zeta+\zeta')^2-2\zeta\zeta')/6]}\right]\right\}\end{aligned} \tag{30}$$

The kernel (30) has a simple form and its computation is straightforward.

In the presence of a trapping potential $V_{\mathrm{trap}}(r)$ which depends on the hyperradius only, the modifications needed are trivial and consists of replacing H_r by

$$H_r = -\frac{\partial^2}{\partial r^2} + \frac{\mathscr{L}(\mathscr{L}+1)}{r^2} + V_{\mathrm{trap}}(r) \tag{31}$$

The solution of the two-dimensional equations (23) and (29) can be readily obtained. However, the Adiabatic Approximation can also be employed. In this case we may write, as usual, $Q(\zeta,r) = Q_\lambda(\zeta,r)u_\lambda(r)$ to obtain

$$\begin{aligned}\frac{\hbar^2}{m}\left[\frac{4}{r^2}H_\zeta \;+\; U_\lambda(r)\right]Q_\lambda(\zeta,r) = -\left[V(\frac{r}{\sqrt{\alpha}}\zeta)-V_0(r)\right] \\ \times\;\left[Q_\lambda(\zeta,r)+\int_0^{\sqrt{\alpha}}\mathscr{F}_n(\zeta,\zeta')Q_\lambda(\zeta',r)\,\mathrm{d}\zeta'\right], \qquad n=E,\,I\end{aligned} \tag{32}$$

and

$$u''_\lambda(r) + \left[k^2_\lambda + V_{\rm eff}(r)\right] u_\lambda(r) = 0 \tag{33}$$

where the effective potential $V_{\rm eff}$ is given by

$$V_{\rm eff}(r) = \frac{\mathscr{L}(\mathscr{L}+1)}{r^2} + \frac{A(A-1)}{2} V_0(r) - U_\lambda(r) + V_{\rm trap}(r) \tag{34}$$

It is noted that the hypercentral potential V_0 contains effects from the higher partial waves, albeit in an approximate way, and can be omitted in which case the results are S-projected. It is further noted that the $\mathscr{L}(\mathscr{L}+1)/r^2$ or the $V_{\rm trap}(r)$ can be included in the first equation (32) without affecting the final results.

RESULTS

We first analyze the behavior of the term $f^2_K - 1$ as $\alpha \to \infty$. In table 1 we present the results for $A = 20$ and $A = 1000$ for the two terms, $T_1 = (A-2)2P^{\alpha,1/2}_K(-1/2)/P^{\alpha,1/2}_K(1)$ and $T_2 = (A-2)(A-3)/2\ P^{\alpha,1/2}_K(-1)/P^{\alpha,1/2}_K(1)$ for $K = 0, 1, \cdots, 7$. We see that both

TABLE 1. Comparison of the two terms T_1 and T_2 of $f^2_K - 1$ (see text) for $K = 0, \cdots, 7$. for $A = 20$ and $A = 1000$.

	$A = 20$			$A = 1000$		
K	T_1	T_2	$f^2_K - 1$	T_1	T_2	$f^2_K - 1$
0	36.	153	189	1996.0000000	497503.	499499.
1	7.5	-8.5	-1	497.5000000	-498.5000000	-1.0000000
2	1.1004464	0.7589286	1.8593750	123.5018775	0.8319426	124.3338201
3	0.0729391	-0.0915948	-0.0186557	30.5345323	-0.0019425	30.5325898
4	-0.0086754	0.0137392	0.0050638	7.5185722	0.0000058	7.5185780
5	-0.0016333	-0.0024376	-0.0040709	1.8437195	-0.0000000	1.8437195
6	0.0002636	0.0004951	0.0007588	0.4502550	0.0000000	0.4502550
7	0.0000479	-0.0001125	-0.0000646	0.1095002	-0.0000000	0.1095002

terms as well as the total term $f^2_K - 1$ become very small as K increases. Consequently only few terms in the expansion (26) are required to achieved convergence. Furthermore, the behavior of the second term (only the $K = 0, 1$ are significant for large α) justifies our approximation (27).

We next present, in Fig. 1, the kernel $\mathscr{F}_I(\zeta, \zeta')$ for $A = 20$ and $A = 1000$ particles. We see that, apart from the strength, its shape and spread is not drastically changed and in both cases the kernel becomes insignificant beyond $\zeta \sim 4$.

We employed the new equation, Eq. (29), to solve first, as a model problem, the ^{16}O system where results exist in the literature. To obtain the solution, we use the Galerkin method and B-splines to reduce the problem, as usual, to an eigenvalue one. In this model nuclear problem, the particles are assumed to interact via Wigner-type forces. The results obtained using the analytic expression (30) and designated as IDEA-I, are given in Table I. Despite the fact that the $A = 16$ case corresponds to a rather small number of particles, the accuracy achieved by the new equation for strong nuclear forces is less than 1% of the exact values obtained by solving the IDEA [13] or using the Hyperspherical Harmonics

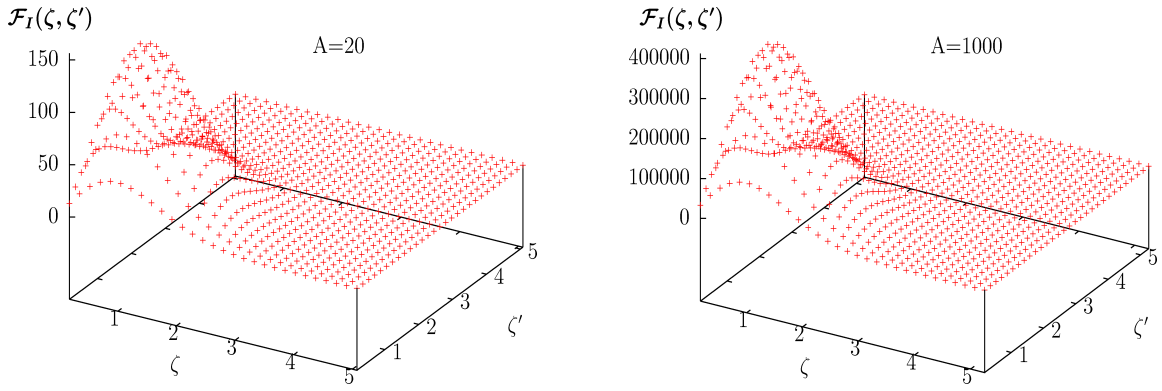

FIGURE 1. The kernel $\mathscr{F}_I(\zeta,\zeta)$ for $A=20$ and $A=1000$.

TABLE 2. Binding energies (in MeV) obtained for $A=16$ with nuclear forces and by using the kernel (30).

Potential	IDEA-I	IDEA(exact)	HHEM [27]
Volkov [24]	1643	1640	–
S3[25]	1247	1246	1235
MT-V [26]	1377	1376	1363

Expansion Method (HHEM) [27]. The slightly higher deviation from the results of the HHEM can be attributed to the slow convergence rate of the HHE expansion for the S3 [25] potential having a practically repulsive hard core and for the semi-realistic Yukawa type MT-V [26] potentials.

We turn now our attention to the case where A bosons are confined in a magnetic trap which is approximated by a spherically symmetric harmonic oscillator potential

$$V_{\rm trap}(r)=\sum_{i=1}^{A}\frac{1}{2}m\omega^2x_i^2=\frac{1}{4}m\omega r^2 \tag{35}$$

In our calculations we use oscillator units (o.u) in which the energy and length are $\hbar\omega$ and $\sqrt{\hbar/m\omega}$ respectively, where ω is the harmonic oscillator circular frequency. In these units $\hbar^2/m=1$.

As a first example we employ a Gaussian potential

$$V(r_{ij})=V_0\exp[-r_{ij}^2/r_0^2] \tag{36}$$

with $V_0=3.1985\times10^6$ o.u and $r_0=0.005$ o.u which corresponds to the Joint Institute for Laboratory Astrophysics (JILA) ^{87}Rb experiment [28] with $a_{\rm sc}=100$ bohr and trap frequency $\nu=200$ Hz. The results obtained by employing the kernel (26), designated as IDEA-E, and the kernel (30), are shown in Table 3. The ground state energy for $A=3$ differs, as expected, from the corresponding value obtained within the PHEM [19] by 25%, for $A=5$ by 3.26%. For $A=10$, however, the agreement is already within 0.2%. Going beyond $A>10$, the differences from the results of PHEM are very small and can be mainly attributed to the overall numerical inaccuracies. It should be noted here

TABLE 3. Results (in o.u) obtained with IDEA-E (Eq. (23)) and IDEA-I (Eq. (29)) using the Gaussian potential (36).

A	IDEA-E	IDEA-I	PHEM
3	6.009	6.009	4.500
5	7.758	7.758	7.505
10	15.003	15.003	15.034
15	22.501	22.501	22.567
20	30.000	30.001	30.107
25	37.501	37.501	37.654
30	45.009	45.001	45.207
35	52.509	52.501	52.768

TABLE 4. Same as 3 using the sech potential (37).

A	IDEA-I	PHEM[21]	DMC[29]
10	15.143	15.1490	15.1539
20	30.625	30.6209	30.639
50	78.701	78.8704	
100	165.038	164.907	

that the binding energy per particle is of the order of $E_b/A \sim 1.50$. It should be further noted that the IDEA-E and the IDEA-I results are, to all practical purposes, identical and therefore we shall employ from now on only the kernel (30).

As a second example we use the semi-realistic potential

$$V(r_{ij}) = V_0 \,\mathrm{sech}^2(r_{ij}/r_0) \tag{37}$$

Following Das *et al.* [21] we use $V_0 = 1.81847 \times 10^9$ o.u and $r_0 = 0.001$ o.u.. We present our results in Table 4 and compare them with those of the PHEM and of the DMC results of Blume and Greene [29].

We endeavored to carry out calculations for up to $A = 100$ where a very good agreement is achieved in all cases for $A \geq 10$ with both the PHEM [21] DMC [29] methods. Going beyond $A = 100$ requires more refine calculations and rather an exact solution of Eq. (29), the reason being that the extreme adiabatic approximation give rise to a multitude of eigenpotentials $U_\lambda(r)$ very close to each other and the results, albeit not differing much, depend nevertheless on which eigenpotential $U_\lambda(r)$ is used. This is shown in Fig. 2 where two effective potentials, Eq. (34), corresponding to $\lambda = 1$ and $\lambda = 20$ are plotted for the case $A = 500$. This multitude of eigenpotentials close to each other does not appear in the case where forces having an attractive well are used.

CONCLUSIONS

Our conclusions can be summarized as follows:
i) Using the transformation $r_{\rm ij} = r\zeta/\sqrt{\alpha}$ and using the asymptotic form of the Jacobi

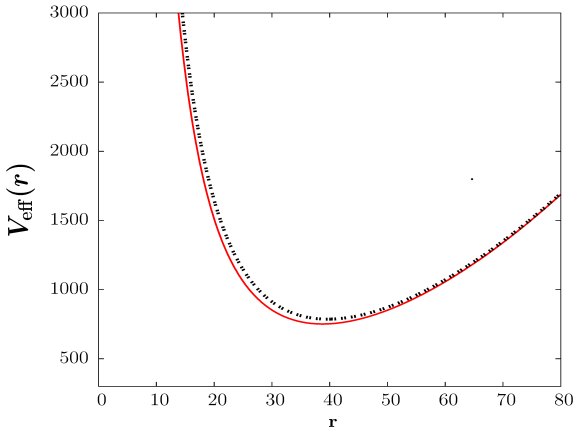

FIGURE 2. Two eigenpotentials $V_{\rm eff}(r)$ corresponding to $\lambda = 1$ and $\lambda = 20$ for $A = 500$.

polynomials $P_K^{\alpha,\beta}(z)$ which for large A are approximated by the Laguerre polynomials $L_K^{1/2}(\zeta)$ that do not depend on A, we obtained an integro-differential equation describing bound states of large number of bosons. This transformation simplifies the kinetic energy term, the weight function, and the corresponding projection function. As a result the new integro-differential equation with a fully analytic and simple kernel can be easily applied to A-body bosonic systems.

ii) The IDEA formalism is similar to the PHEM of Ref. [3] employed by Das and collaborators [19, 20, 21]. In the PHEM one has to solve a large number of differential equations which in the IDEA are transformed, with the help of Potential Harmonics, into a single integro-differential equation. (Technical details on this transformation can be found in Refs. [2, 5, 12].) Therefore, our equation for large A can also be considered as a simplified version not only of the IDEA method but also of the traditionally used PHEM.

iii) We tested the new equation by calculating the ground state binding energy of the model nuclear problem for the ^{16}O system where the short range nuclear force was of Wigner type. The good agreement achieved, with the three different type forces having a soft core, a hard core, and of Yukawa type, as compared to the results obtained using the IDEA and the HHEM methods implies that the new equation can be safely used to calculate binding energies of large number ($A \geq 10$) particles interacting via strong forces.

iv) Application of our scheme to Bose-Einstein condensates consisting of A-atoms trapped by an external field, we obtained results which are in excellent agreement with those of PHEM and the Diffusion Monte Carlo (DMC) method, at least up to $A = 100$. Going beyond this number requires improved numerical methods or a direct solution of the equation as a two-variable integro-differential equation without resorting to the EAA approximation which give rise to a plethora of eigenpotentials that are very close to each other.

v) When A increases, the centrifugal part $\mathscr{L}(\mathscr{L}+1)/r^2$ becomes extremely large and extends outwards while the inter-atomic potential is constant and restricted to smaller distances. Therefore the main contribution in the effective potential stems from the

centrifugal and the trapping potentials which generate a harmonic oscillator–type well which moves outwards as the number of particles A increases.

vi) The overall good results obtained, indicate that the derived equation can be used in studies of bound A-boson systems as an alternative to competing methods such as the variational and hyperspherical harmonics methods. Our approximations should become better with increasing A *i.e* for $\alpha \rightarrow \infty$.

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