Interdependence between integrable cosmological models with minimal and non-minimal coupling

A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, **S.Yu. Vernov**

Skobeltsyn Institute of Nuclear Physics MSU, Moscow, Russia
Landau Institute for Theoretical Physics RAS, Moscow, Russia
Dipartimento di Fisica e Astronomia and INFN, Bologna, Italy

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Models with scalar fields are very useful to describe the observable evolution of the Universe as the dynamics of the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background with

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The confirmed discovery of the Higgs boson at the Large Hadron Collider (CERN) has initiated an intense research activity with the aim to understand the cosmological implications of this fundamental scalar field.

NON-MINIMAL COUPLING

Models with non-minimally coupled scalar fields are interesting because of their connection with the particle physics.

There are models of inflation, where the role of the inflaton is played by the Higgs field non-minimally coupled to gravity. (F.L. Bezrukov and M. Shaposhnikov, *Phys. Lett.* B **659** (2008) 703–706, arXiv:0710.3755).

The inflationary parameters obtained in this model are in good agreement with the most recent and accurate observational data (PLANCK'2015).

At the same time, the predictions of the simplest inflationary models with a minimally coupled scalar field lead to sufficiently large values of the tensor-to-scalar ratio of the density perturbations r, and are ruled out by Planck data.

Inflationary scenarios with a minimally coupled scalar field can be improved by adding a tiny non-minimal coupling of the inflaton field to gravity. This is not so artificial since the non-minimal $R\phi^2$ term is always induced by quantum corrections.

Generic quantum corrections to the action of the scalar field minimally coupled to gravity include the term, proportional to $\phi^2 R$.

N.A. Chernikov, E.A. Tagirov, Annales Poincare Phys. Theor. A **9** (1968) 109.

INTEGRABLE COSMOLOGICAL MODELS

The use of the FLRW metric essentially simplify the Einstein equations.
But, only a few cosmological models with scalar fields are integrable.
P. Fré, A. Sagnotti, A.S. Sorin,
Nucl. Phys. B **877** (2013) 1028, arXiv:1307.1910.
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Our goal is to find integrable model with non-minimal coupling using the knowledge of integrable models with minimal coupling.
To do this we use the FLRW metric with a parametric time and find the correspondence between potentials and lapse functions in the Einstein and Jordan frames.

The standard way to integrate a cosmological model is as follows:

- to use the FLRW metric with a parametric time

$$ds^2 = N^2(\tau)d\tau^2 - a^2(\tau)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$

- to guess a suitable lapse function $N(\tau)$.
- to simplify (for example, linearize) equations, introducing new depending variables.
- to get the general solution.

CONFORMAL TRANSFORMATION

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + V(\sigma)\right]. \quad (1)$$

where $U(\sigma)$ and $V(\sigma)$ are differentiable functions of the scalar field σ.
Flat Friedmann space-time with the interval

$$ds^2 = N^2(\tau)d\tau^2 - a^2(\tau)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \quad (2)$$

where $a(\tau)$ is the cosmological radius and $N(\tau)$ is the lapse function.
Let us make the conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_0}{U}\tilde{g}_{\mu\nu},$$

where U_0 is a constant, and introduce a new scalar field σ such that

$$\frac{d\tilde{\varphi}}{d\sigma} = \frac{\sqrt{U_0(U + 3U'^2)}}{U} \quad \Rightarrow \quad \tilde{\varphi} = \int \frac{\sqrt{U_0(U + 3U'^2)}}{U}d\sigma. \quad (3)$$

In this case the action (1) becomes the action for a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[U_0 R(\tilde{g}) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\varphi}_{,\mu} \tilde{\varphi}_{,\nu} + W(\tilde{\varphi}) \right], \tag{4}$$

where

$$W(\tilde{\varphi}) = \frac{U_0^2 V(\sigma(\tilde{\varphi}))}{U^2(\sigma(\tilde{\varphi}))}. \tag{5}$$

The FLRW metric (2) becomes

$$ds^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 \left(dx_1^2 + dx_2^2 + dx_3^2 \right),$$

where

$$\tilde{N} = \sqrt{\frac{U}{U_0}} N, \qquad \tilde{a} = \sqrt{\frac{U}{U_0}} a.$$

$$\frac{6U\dot{a}^2}{a^2}+\frac{6U'\dot{a}\dot{\sigma}}{a}=\frac{1}{2}\dot{\sigma}^2+N^2V. \tag{6}$$

$$\frac{4U\ddot{a}}{a}+\frac{2U\dot{a}^2}{a^2}+\frac{4U'\dot{a}\dot{\sigma}}{a}-\frac{4U\dot{a}\dot{N}}{aN}+2U''\dot{\sigma}^2+2U'\ddot{\sigma}-\frac{2U'\dot{\sigma}\dot{N}}{N}=-\frac{1}{2}\dot{\sigma}^2+N^2V. \tag{7}$$

The variation with respect to σ gives the Klein–Gordon equation:

$$\ddot{\sigma}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\sigma}-6U'\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]+6\frac{\dot{a}\dot{N}}{aN}U'+N^2V'=0. \tag{8}$$

FRIEDMANN EQUATIONS

In the Einstein frame we have the following equations:

$$6U_0\tilde{h}^2 = \frac{1}{2}\dot{\tilde{\varphi}}^2 + \tilde{N}^2 W, \tag{9}$$

$$4U_0\dot{\tilde{h}} + 6U_0\tilde{h}^2 - 4U_0\tilde{h}\frac{\dot{\tilde{N}}}{\tilde{N}} = -\frac{1}{2}\dot{\tilde{\varphi}}^2 + \tilde{N}^2 W, \tag{10}$$

$$\ddot{\tilde{\varphi}} + \left(3\tilde{h} - \frac{\dot{\tilde{N}}}{\tilde{N}}\right)\dot{\tilde{\varphi}} + \tilde{N}^2 W_{,\tilde{\varphi}} = 0, \tag{11}$$

where $\tilde{h} \equiv \dot{\tilde{a}}/\tilde{a}$.

$$N(\tau) = \sqrt{\frac{U_0}{U(\sigma[\tilde{\varphi}(\tau)])}}\,\tilde{N}(\tau).$$

THE GENERAL ALGORITHM

Let us suppose that for some potential W we know the general exact solution of the system of equations (9)–(11): $\tilde{\varphi}(\tau)$, $\tilde{a}(\tau)$, $\tilde{N}(\tau)$.
We also suppose that *the function* $\sigma(\tilde{\varphi})$ *is known explicitly*.
In this case, we can also find the general solution of the system of equations (6)–(8) with the potential

$$V(\sigma) = \frac{U^2(\sigma)W(\tilde{\varphi}(\sigma))}{U_0^2}\,, \tag{12}$$

because we know $\sigma(\tilde{\varphi}(\tau))$, so we easily obtain

$$a(\tau) = \sqrt{\frac{U_0}{U(\sigma(\tilde{\varphi}(\tau))}}\tilde{a}(\tau), \qquad N(\tau) = \sqrt{\frac{U_0}{U(\sigma(\tilde{\varphi}(\tau))}}\tilde{N}(\tau).$$

Sometimes really need only we know

$$N(\tau) = \sqrt{\frac{U_0}{U(\sigma(\tilde{\varphi}(\tau))}}\tilde{N}(\tau).$$

It is the most important information.
After this we consider only equations in the Jordan frame and linearize them.

TWO EXAMPLES OF $U(\sigma)$

Let us consider the induced gravity with

$$U(\sigma)=\frac{1}{2}\xi\sigma^2. \qquad (13)$$

In this model

$$\tilde{\varphi}=\sqrt{\frac{2U_0(1+6\xi)}{\xi}}\ln\left[\frac{\sigma}{\sigma_0}\right] \quad \text{and} \quad \sigma=\sigma_0 e^{\sqrt{\frac{\xi}{2U_0(1+6\xi)}}\tilde{\varphi}}. \qquad (14)$$

We put $\xi\neq -1/6$, because at $\xi=-1/6$ we have $U+3U'^2=0$ and nontrivial solutions exist for the potential $V=V_0\sigma^4$ only.
(I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. Vernov, *Class. Quantum Grav.* **31** (2014) 065007 [arXiv:1206.2801])

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right)$$

Is means $N = 1$, so

$$6UH^2 + 6\dot{U}H = \frac{1}{2}\dot{\sigma}^2 + V\,, \tag{15}$$

$$2U\left(2\dot{H} + 3H^2\right) = -\frac{\dot{\sigma}^2}{2} - 2\ddot{U} - 4H\dot{U} + V\,, \tag{16}$$

$$\ddot{\sigma} + 3H\dot{\sigma} - 6U'\left(\dot{H} + 2H^2\right) + V' = 0\,. \tag{17}$$

From Eqs. (15)–(17) we get the following system:

$$\begin{aligned} \dot{\sigma} &= \psi, \qquad \dot{\psi} = -3H\psi - \frac{\left[(6U'' + 1)\psi^2 - 4V\right]U' + 2UV'}{2\left(3U'^2 + U\right)}, \\ \dot{H} &= -\frac{2U'' + 1}{4(3U'^2 + U)}\psi^2 + \frac{2U'H\psi}{3U'^2 + U} - \frac{6U'^2H^2}{3U'^2 + U} + \frac{U'V'}{2(3U'^2 + U)}. \end{aligned} \tag{18}$$

In the case $3U'^2 + U = 0$ system (18) is not valid!

In the case $\xi = -1/6$ we consider models with

$$U_c = U_0 - \frac{\sigma^2}{12}, \tag{19}$$

In this case

$$\tilde{\varphi} = \sqrt{3U_0}\ln\left[\frac{\sqrt{12U_0}+\sigma}{\sqrt{12U_0}-\sigma}\right] \quad \text{and} \quad \sigma = \sqrt{12U_0}\tanh\left[\frac{\tilde{\varphi}}{\sqrt{12U_0}}\right]. \tag{20}$$

Physical applications:
K. Bamba, Sh. Nojiri, S.D. Odintsov, D. Sáez-Gómez,
Phys. Lett. B **730** (2014) 136–140 [arXiv:1401.1328]
B. Boisseau, H. Giacomini, D. Polarski and A.A. Starobinsky,
JCAP **1507** (2015) 002 [arXiv:1504.07927]

Let us consider the cosmological model with a minimally coupled scalar field and the exponential potential:

$$W = W_0 e^{2\sqrt{3}\lambda\tilde{\varphi}}, \qquad (21)$$

where $\lambda \neq \pm 1$.
D.S. Salopek and J.R. Bond, *Phys. Rev.* D **42** (1990) 3936–3962.
In the induced gravity model the corresponding potential is

$$V(\sigma) = 4W_0\xi^2\sigma^4\left(\frac{\sigma}{\sigma_0}\right)^{\lambda\sqrt{\frac{6(1+6\xi)}{\xi}}} = 4W_0\xi^2\sigma^4\left(\frac{\sigma}{\sigma_0}\right)^{6\lambda\Gamma}.$$

where $\Gamma \equiv \sqrt{\frac{1+6\xi}{6\xi}}$.
In the model including the Hilbert–Einstein curvature term plus a scalar field conformally coupled to gravity

$$\mathcal{V} = W_0\left[1-\frac{\sigma^2}{3}\right]^2\left(\frac{\sqrt{3}+\sigma}{\sqrt{3}-\sigma}\right)^{3\lambda} = W_0\Theta\Upsilon^{3\lambda}, \quad \Theta \equiv \left[1-\frac{\sigma^2}{3}\right]^2, \Upsilon \equiv \frac{\sqrt{3}+\sigma}{\sqrt{3}-\sigma}.$$

Table: POTENTIALS OF INTEGRABLE MODELS

W (minimal coupling)	V (induced gravity)	$\mathcal{V}$ (conformal coupling)
$c_0 e^{2\sqrt{3}\lambda\tilde{\varphi}}$	$\tilde{c}_0 \sigma^{4+6\lambda\Gamma}$	$c_0 \Theta \Upsilon^{3\lambda}$
$c_0 + c_1 e^{\sqrt{3}\tilde{\varphi}} + c_2 e^{-\sqrt{3}\tilde{\varphi}}$	$\tilde{c}_0\sigma^4 + \tilde{c}_1\sigma^{4+3\Gamma} + \tilde{c}_2\sigma^{4-3\Gamma}$	$\Theta\left[c_0 + c_1\Upsilon^{\frac{3}{2}} + c_2\Upsilon^{-\frac{3}{2}}\right]$
$c_1 e^{2\sqrt{3}\lambda\tilde{\varphi}} + c_2 e^{\sqrt{3}(\lambda+1)\tilde{\varphi}}$	$\tilde{c}_1\sigma^{4+6\lambda\Gamma} + \tilde{c}_2\sigma^{4+3(\lambda+1)\Gamma}$	$\Theta\left[c_1\Upsilon^{3\lambda} + c_2\Upsilon^{\frac{3}{2}(\lambda+1)}\right]$
$c_1 e^{2\sqrt{3}\tilde{\varphi}} + c_2$	$\sigma^4\left[\tilde{c}_1\sigma^{6\Gamma} + \tilde{c}_2\right]$	$\Theta\left[c_1\Upsilon^3 + c_2\right]$
$c_0\tilde{\varphi}e^{2\sqrt{3}\tilde{\varphi}}$	$\sqrt{3}\Gamma\tilde{c}_0\sigma^{4+6\Gamma}\ln\left[\frac{\sigma}{\sigma_0}\right]$	$\frac{\sqrt{3}}{2}c_0\Theta\Upsilon^3\ln(\Upsilon)$
$c_1 e^{2\sqrt{3}\lambda\tilde{\varphi}} + c_2 e^{\frac{2\sqrt{3}}{\lambda}\tilde{\varphi}}$	$\sigma^4\left[\tilde{c}_1\sigma^{6\lambda\Gamma} + \tilde{c}_2\sigma^{6\frac{\Gamma}{\lambda}}\right]$	$\Theta\left[c_1\Upsilon^{3\lambda} + c_2\Upsilon^{\frac{3}{\lambda}}\right]$

In Table 1 we present the list of the potentials of integrable cosmological models. We put $U_0 = 1/4$. The constants $\tilde{c}_i = 4\xi^2 c_i$, $\lambda \neq \pm 1$, $\lambda \neq 0$.
P. Fré, A. Sagnotti, A.S. Sorin, arXiv:1307.1910 (minimal coupling).

Table: Lapse functions for integrable cases

	$\tilde{N}$ (minimal coupling)	N (induced gravity)	$\mathcal{N}$ (conformal coupling)
1	$\frac{\sqrt{6}}{\sqrt{c_0}}e^{-\sqrt{3}\lambda\tilde{\varphi}}$	$\frac{\sqrt{3}}{\sqrt{\xi c_0}}\sigma^{-3\lambda\Gamma-1}$	$\sqrt{\frac{18}{c_0(3-\sigma^2)}}\Upsilon^{-3\lambda/2}$
2	1	$\frac{\sqrt{2}}{\sqrt{\xi}\sigma}$	$\sqrt{\frac{3}{3-\sigma^2}}$
3	$e^{-\sqrt{3}\lambda\tilde{\varphi}}$	$\frac{1}{\sqrt{2\xi}}\sigma^{-3\Gamma\lambda-1}$	$\sqrt{\frac{3}{3-\sigma^2}}\Upsilon^{-3\lambda/2}$
4	$e^{-\sqrt{3}\tilde{\varphi}}$	$\frac{1}{\sqrt{2\xi}}\sigma^{-3\Gamma-1}$	$\sqrt{\frac{3}{3-\sigma^2}}\Upsilon^{-3/2}$
5	$\frac{e^{-2\sqrt{3}\tilde{\varphi}}}{\tilde{a}^3}$	$\frac{9(\Gamma^2-1)^2}{a^3\sigma^4}\left(\frac{\sigma}{\sigma_0}\right)^{-6\Gamma}$	$\frac{9}{a^3}\frac{(\sqrt{3}-\sigma)}{(\sqrt{3}+\sigma)^5}$
6	$\tilde{a}^3$	$\frac{\sigma^2a^3}{3(\Gamma^2-1)}$	$\left(1-\frac{\sigma^2}{3}\right)^2a^3$

A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, and S.Yu. V., *Class. Quant. Grav.* **31 (2014) 105003**, `arXiv:1312.3540`

Let us consider the model with a minimally coupled scalar field

$$W(\phi) = 2U_0\Lambda \cosh^4\left(\frac{\phi}{2\sqrt{3U_0}}\right) - 144U_0^2 c \sinh^4\left(\frac{\phi}{2\sqrt{3U_0}}\right). \qquad (22)$$

Make the conformal transformation and get the corresponding model with

$$U_c = U_0 - \frac{\sigma^2}{12}, \qquad V_c = 2U_0\Lambda - c\sigma^4\,, \qquad (23)$$

It is easy to show that the Ricci scalar $R_c = 4\Lambda$, so it the second integral of motion and the system is integrable one.

The integrability of the system with minimal coupled scalar field has been proved in

Bars I., Chen S. H., 2011, *Phys. Rev.* D **83** 043522 (arXiv:1004.0752)

Bars I., Chen S. H., Turok N., 2011, *Phys. Rev.* D **84** 083513 (arXiv:1105.3606)

by another way.

The existence of the integrable system with U_c and V_c has been found in

Boisseau B., Giacomini H., Polarski D., Starobinsky A.A., 2015, *J. Cosmol. Astropart. Phys.* **1507** 002 (arXiv:1504.07927).

Integrable model with a constant R

Recently the following model with nonminimal coupling has been proposed:
B. Boisseau, H. Giacomini, D. Polarski and A.A. Starobinsky,
Bouncing Universes in Scalar-Tensor Gravity Models admitting Negative Potentials, JCAP **1507** (2015) 002, arXiv:1504.07927
The function U is given by (23):

$$U = U_0 - \frac{\sigma^2}{12},$$

the potential

$$V = 2U_0\Lambda - c\sigma^4,$$

$c > 0$, $U_0 > 0$, $\Lambda > 0$.

$$R = 6(\dot{H} + 2H^2) = 4\Lambda, \qquad H = \sqrt{\frac{\Lambda}{3}}\tanh\left(\sqrt{\frac{2\Lambda}{3}}(t - t_0)\right),$$

where t is the cosmic time.
It is a bounce solution!

After conformal transformation to the Einstein frame this model coincides with one of models from the Fré–Sagnotti–Sorin paper. The potential in the Einstein frame is

$$W = C_1 \cosh(\gamma\tilde{\varphi})^{2/\gamma-2} + C_2 \sinh(\gamma\tilde{\varphi})^{2/\gamma-2}, \qquad (24)$$

at $\gamma = 1/3$.

B. Boisseau, H. Giacomini and D. Polarski, JCAP **1510** (2015) 033 (arXiv:1507.00792).

The integrable cosmological models can be interesting to describe a possible evolution of the Universe.

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The behavior of the Hubble parameter are different in different frames. For example, it can has a bounce solution in the Jordan frame only.
It is possible that the model with a non-minimal coupling is more easy integrable and/or more applicable to cosmology than the corresponding model with a minimal coupling.

Let us make the conformal transformation of the integrable model with the potential W, given by (24).
We get integrable models with U_c and

$$V_c(\sigma) = \frac{c_1}{144 U_0^2 2^{\frac{2(1-\beta)}{\beta}}} \frac{\left[(\sqrt{12U_0}+\sigma)^{3\beta} + (\sqrt{12U_0}-\sigma)^{3\beta}\right]^{\frac{2(1-\beta)}{\beta}}}{(12U_0-\sigma^2)^{1-3\beta}} - \\ + \frac{c_2}{144 U_0^2 2^{\frac{2(1-\beta)}{\beta}}} \frac{\left[(\sqrt{12U_0}+\sigma)^{3\beta} - (\sqrt{12U_0}-\sigma)^{3\beta}\right]^{\frac{2(1-\beta)}{\beta}}}{(12U_0-\sigma^2)^{1-3\beta}}. \tag{25}$$

We also can get integrable models with other function U.
The corresponding induced gravity model with

$$U(\sigma) = \frac{1}{2}\xi\sigma^2.$$

has the following potential:

$$V_{ind} = \frac{9\gamma^2\tilde{\sigma}^4}{4}\left[(b+c)\left(\left[\frac{\tilde{\sigma}}{\tilde{\sigma}_0}\right]^{4\Gamma} + \left[\frac{\tilde{\sigma}}{\tilde{\sigma}_0}\right]^{-4\Gamma} + 6\right) + \right.$$
$$\left. + 4(b-c)\left(\left[\frac{\tilde{\sigma}}{\tilde{\sigma}_0}\right]^{2\Gamma} + \left[\frac{\tilde{\sigma}}{\tilde{\sigma}_0}\right]^{-2\Gamma}\right)\right].$$

A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi, S.Yu. Vernov, arXiv:1509.00590.

The minisuperspace Lagrangian, generating the Friedmann equations (9), (10) and the Klein-Gordon equation (11) is

$$L = \frac{6\left(\dot{\tilde{a}}\right)^2 \tilde{a} U_0}{\tilde{N}} - \frac{\tilde{a}^3 \dot{\phi}^2}{2\tilde{N}} + \tilde{N} W \tilde{a}^3. \tag{26}$$

If one considers the potential (24) and chooses the lapse functions

$$\tilde{N} = \frac{4U_0}{3\beta^2} \tilde{a}^{3-6\beta},$$

then the Lagrangian has the following form:

$$L = \frac{9}{2}\beta^2 \dot{\tilde{a}}^2 \tilde{a}^{6\beta-2} - \frac{3\beta^2 \tilde{a}^{6\beta} \dot{\phi}^2}{8U_0} + \frac{4U_0}{3\beta^2} W \tilde{a}^{6-6\beta}. \tag{27}$$

Let us introduce new variables x and y defined as

$$\tilde{a}^{6\beta} = xy, \qquad \exp\left(\frac{6\beta}{\sqrt{12U_0}}\phi\right) = \frac{x}{y}, \tag{28}$$

then the Lagrangian (27) takes the form

$$L = \frac{1}{2}\dot{x}\dot{y} + \frac{4U_0}{3\beta^2}(xy)^{\frac{1-\beta}{\beta}} W\,. \tag{29}$$

For the potential W, given by (24), we finally obtain, on introducing another couple of independent variables

$$\xi = \frac{x+y}{2}, \quad \eta = \frac{x-y}{2} \tag{30}$$

the following simple expression for the Lagrangian:

$$L = \frac{\dot{\xi}^2 - \dot{\eta}^2}{2} + \frac{4U_0}{3\beta^2}\left(c_1\xi^{\frac{2(1-\beta)}{\beta}} + c_2\eta^{\frac{2(1-\beta)}{\beta}}\right). \tag{31}$$

The corresponding Euler-Lagrange equations are

$$\ddot{\xi} - c_1 \frac{4U_0}{3\beta^2}\left(\frac{2(1-\beta)}{\beta}\right)\xi^{\frac{2}{\beta}-3} = 0, \qquad \ddot{\eta} + c_2 \frac{4U_0}{3\beta^2}\left(\frac{2(1-\beta)}{\beta}\right)\eta^{\frac{2}{\beta}-3} = 0.$$

Their first integrals are

$$\frac{\dot{\xi}^2}{2} - c_1 \frac{4U_0}{3\beta^2}\xi^{\frac{2(1-\beta)}{\beta}} = E_1, \qquad \frac{\dot{\eta}^2}{2} + c_2 \frac{4U_0}{3\beta^2}\eta^{\frac{2(1-\beta)}{\beta}} = E_2. \qquad (32)$$

These equations have solutions by quadrature. For $\beta = 1/3$ one obtains the solutions in terms of elliptic functions
(I. Bars and S-H. Chen, Phys. Rev. D 83 (2011) 043522).
It immediately follows from the Friedmann equations that $E_1 = E_2$.

We can now find the cosmological variables

$$\tilde{a} = \left(\xi^2 - \eta^2\right)^{1/(6\beta)}, \quad \tilde{N} = \frac{4U_0}{3\beta^2}\left(\xi^2 - \eta^2\right)^{(1-2\beta)/(2\beta)}, \quad \phi = \frac{\sqrt{3U_0}}{3\beta}\ln\left[\frac{\xi + \eta}{\xi - \eta}\right].$$

We can now find the cosmological variables

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One can also write down the expressions for the solutions of the corresponding models with non-minimal coupling:

$$\begin{aligned} a_c &= \frac{1}{2}\left((\xi+\eta)^{1/(3\beta)} + (\xi-\eta)^{1/(3\beta)}\right), \\ N_c &= \frac{2U_0}{3\beta^2}\left((\xi+\eta)^{1/(3\beta)} + (\xi-\eta)^{1/(3\beta)}\right)\left(\xi^2-\eta^2\right)^{2(1-3\beta)/(3\beta)}, \\ \sigma_c &= \sqrt{12U_0}\frac{(\xi+\eta)^{1/(3\beta)} - (\xi-\eta)^{1/(3\beta)}}{(\xi+\eta)^{1/(3\beta)} + (\xi-\eta)^{1/(3\beta)}} \end{aligned}$$

for the model with U_c and V_c.
If $\beta = 1/3$, then

$$a_c = \xi. \tag{33}$$

BOUNCE SOLUTIONS

The strong curvature singularity arising in the past of our Universe in FLRW models may not be avoided in generic solutions of the Einstein gravity with minimally coupled scalar fields.

The bounce solution corresponds to expanding universe after a contraction.

$H < 0$ before bounce, $H = 0$ at the bounce, and $H > 0$ after the bounce.

In the FLRW metric

$$w(t) \equiv \frac{p}{\varrho} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \tag{34}$$

where H is the Hubble parameter, p is the pressure and ϱ of an ideal cosmic fluid.

For a bounce solution there exist t_0, such that $H(t_B) = 0$ and $\dot{H}(t_B) > 0$.

So, just after the bounce $w(t_B) < -1$ and **the NEC is violated**.

A minimal coupling field σ should be a phantom or Galileons one to get a bounce solution.

A nonminimal coupling field σ can be the standard scalar field.

The model can has bounce at $t = t_0$ if $V(\sigma(t_0) < 0)$.
If $h = 0$, then

$$\frac{1}{2}\dot{\sigma}^2 + N^2 V = 0,$$

$$4U\dot{h} + \frac{1}{6}\dot{\sigma}^2 - \frac{1}{3}\sigma\ddot{\sigma} - N^2 V = 0,$$

$$\ddot{\sigma} + \sigma\dot{h} + N^2 V' = 0.$$

We obtain

$$\dot{h} = \frac{N^2}{12U_0}\left(4V - \sigma V'\right).$$

Therefore, bounce conditions are

$$4V - \sigma V' > 0, \qquad V < 0. \tag{35}$$

at the bounce point.
We always assume $U > 0$.

For the case $\beta = 1/3$**,** the potential (25) can be written as

$$V = V_1 + V_2\sigma^4. \tag{36}$$

We obtain the conditions

$$V_1 > 0, \qquad V_2 < 0.$$

Also, $U_c(\sigma_B) > 0$.
B. Boisseau, H. Giacomini, D. Polarski and A.A. Starobinsky, JCAP **1507** (2015) 002, arXiv:1504.07927.
For the case $\beta = 1$**,** the potential (25) can be written as

$$V = V_0 \left(U_0 - \frac{\sigma^2}{12} \right)^2. \tag{37}$$

We obtain

$$V_0 < 0, \qquad 12U_0 - \sigma^2 < 0\,. \tag{38}$$

This expression is positive if $\sigma^2 > 12U_0$, so, $U_c < 0$.

Another interesting case is $\beta = 2/3$. In this case the potential can be written as

$$V = V_0(12U_0 - \sigma^2)(\sigma^2 + 12U_0 + 2V_1\sigma). \tag{39}$$

We shall limit ourselves by consideration of a few particular values of the constant V_1.

If

$$V_1 = -\sqrt{12U_0},$$

then the potential V can be written down as

$$V = V_0(\sigma^2 - 12U_0)(\sigma - \sqrt{12U_0})^2,$$

In this case two conditions of the existence of bounce are compatible if

$$V_0 < 0, \quad \sigma > \sqrt{12U_0} \;\; \text{or} \;\; -2\sqrt{12U_0} < \sigma < -\sqrt{12U_0}.$$

In the case $V_1 = \sqrt{12U_0}$, the potential is

$$V = V_0(\sigma^2 - 12U_0)(\sigma + \sqrt{12U_0})^2.$$

The conditions of the existence of bounces are compatible if

$$V_0 < 0, \quad \sqrt{12U_0} < \sigma < 2\sqrt{12U_0}.$$

We see that in these cases the bounce corresponds to a such σ_B that $U(\sigma_B) < 0$.

Let

$$c_2 = -\frac{289\sqrt{2}}{48}c_1, \qquad \Rightarrow \qquad V_1 = -\frac{289}{24}\sqrt{6U_0}.$$

At $U_0 = 1$, the conditions (35) are satisfied if $0.20412 < \sigma < 0.27178$.
We get a bounce point at $U(\sigma_B) > 0$.

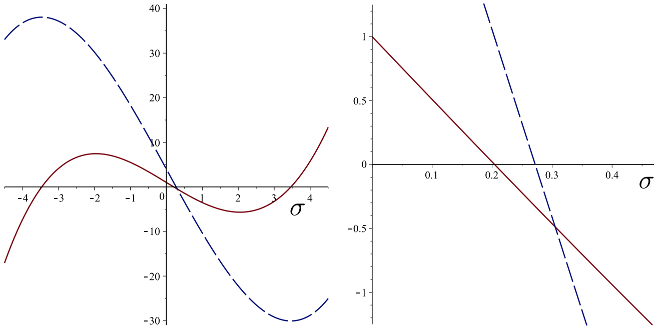

Figure: The potential V_c (solid red line) and the expression $4V_c - \sigma V_c'$ (dash blue line) at $U_0 = 1$, $c_1 = 1$, $\beta = 2/3$, $V_1 = -289\sqrt{6}/24$. The fine structure, presented on the right picture, shows that a bounce is possible.

CONCLUSIONS

- Cosmological models with non-minimally coupling scalar fields has been considered.
- We show how to get integrable models with non-minimal coupling using the suitable parametric time.
- Sometimes model with non-minimal coupling can be more easy (more explicit) integrable, than the corresponding model with a minimal coupled scalar field.
- Models with nonminimal coupling maybe useful to get the bounce solution without pathology.