Second-order symmetry operators and separation of variables for the Dirac equation in two-dimensional spin manifolds with external fields

-o-o-

International Journal of Geometric Methods in Modern Physics Vol. 12, No. 1 (2015)

L. Fatibene[†] , R.G.McLenaghan[‡] , G.Rastelli[†]

[†] Dipartimento di Matematica "G. Peano", Universita' di Torino, Italia - [‡] University of Waterloo, Canada

Dubna, November 2015

Outline

- Spin manifolds
- Dirac equation
- Geometric SOV for Scalar Equations
- The Dirac equation in dimension two
- Multiplicative separation of eigenvalue-type systems of PDE
- Separated equations
- Decoupling and Second order symmetry operators
- The Dirac equation, D5
- Examples
- Conclusions

Separation of variables for Dirac's equation and GR

Proc. R. Soc. Lond. A. **349**, 571–575 (1976)
Printed in Great Britain

The solution of Dirac's equation in Kerr geometry

By S. Chandrasekhar, F.R.S.
The School of Natural Sciences, The Institute for Advanced Study, Princeton, New Jersey, 08540‡

(*Received* 21 *April* 1976)

Dirac's equation for the electron in Kerr geometry is separated; and the general solution is expressed as a superposition of solutions derived from a purely radial and a purely angular equation.

1. Introduction

Teukolsky's (1972) separation of the variables of the equations governing the electromagnetic, the gravitational, and the two component neutrino-field perturbations of a Kerr black hole has been central to much of the later developments. But the lack of a similar separation of the variables of Dirac's equation for the electron has been an obstacle to progress along many desired directions (particularly, for the treatment of massive fields in the context of Hawking's (1975) quantal process of the evaporation of black holes). In this short paper, we shall show that Dirac's equation can also be separated and the solution expressed in terms of certain radial and angular functions satisfying decoupled equations; in consequence problems associated with an electron in the vicinity of Kerr black holes become amenable to treatment.

Spin-manifolds

M is a $m = r + s$ dimensional manifold (connected, paracompact)
η is a symmetric tensor of signature (r, s) (Minkowski metric...)
$C(\eta)$ is the Clifford Algebra associated with η
$\{\gamma_a\}$, $a = 0, \ldots, m-1$ is a representation of $C(\eta)$ (Dirac matrices):

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}\mathbb{I}$$

$E_a = e_a^\mu \partial_\mu$ is a moving spin frame
orthonormal w.r.t. to the metric tensor on M defined by

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b,$$

g is determined by e^μ and η
e_a^μ are determined by g and η

Dirac equation with external fields

$i\gamma^\mu(\partial_\mu - iqV_\mu)\psi - V\psi = \mu\psi$ flat space - Cartesian coordinates
$i\gamma^\mu\nabla_\mu\psi - V\psi = \mu\psi$ spin manifold M
where

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\Gamma^{ab}_\mu\gamma_{[a}\gamma_{b]}\,\psi$$

is the covariant derivative of the spinor ψ
($F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ electro-magnetic force, $q =$ charge)

$$\Gamma^{ab}_\mu = e_\alpha{}^a(\Gamma^\alpha_{\beta\mu}e^{\beta b} + \partial_\mu e^{\alpha b})$$

is the spin connection
V is the matrix function of the external fields: scalar, pseudoscalar, vector potentials

Dirac equation in 2D

We choose the Dirac representation of the Clifford algebra

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad \gamma = \gamma_0\gamma_1 = \begin{pmatrix} 0 & -\eta k \\ -\eta k & 0 \end{pmatrix}.$$

where $k = \sqrt{-\eta}$ ($\eta = \pm 1$), both Riemannian and pseudo-Riemannian cases
V is a combination of a electro-magnetic term $q\gamma^\mu V_\mu$, a scalar term $V\mathbb{I}$ and a pseudoscalar term $\hat{V}\gamma$

$$\left[\begin{pmatrix} ie_0^0 & -ike_1^0 \\ ike_1^0 & -ie_0^0 \end{pmatrix} \partial_x + \begin{pmatrix} ie_0^1 & -ike_1^1 \\ ike_1^1 & -ie_0^1 \end{pmatrix} \partial_y + \tilde{C} \right] \psi = \mu\psi,$$

$$\tilde{C} = \frac{i}{2}\epsilon^{ab} e_a^\mu \Gamma_\mu^{01} \gamma_b - q e_a^\mu V_\mu \gamma^a - V\mathbb{I} - \hat{V}\gamma,$$

system of 2 first-order PDEs

Separation of variables for scalar equations

(M_n, g), $\Delta = \nabla \cdot \nabla = \nabla^2$ Laplace-Beltrami operator

Helmholtz equation

$\Delta\psi = \mu\psi$

Multiplicatively Separated (MS) solutions: $\psi = \psi_1(q^1) \cdot \ldots \cdot \psi_n(q^n)$

Theorem (Stäckel-Eisenhart...)

Helmholtz equation is MS in orthogonal coordinates $\leftrightarrow$
i) there exist n (including g) pointwise independent Killing tensors (K_r) simultaneously diagonalized and in involution (i.e $\{K_r^{ij} p_i p_j, K_s^{lm} p_l p_m\} = 0$)
ii) the Ricci tensor is simultaneously diagonalized with the (K_r) (Robertson condition)

The eigenvectors (eigenforms) of the (K_r) determine the separable coordinates (Stäckel coordinates)

Symmetry operators: $\bar{K}_r = \nabla_i(K_r^{ij}\nabla_j\psi)$

$\Delta\psi = \mu\psi$ MS $\leftrightarrow [\bar{K}_r, \Delta] = 0$

Schrödinger equation $\Delta\psi + V\psi = \mu\psi$

Schrödinger MS $\leftrightarrow$ Helmholtz MS and $V = g^{ii}\phi_i(q^i) \leftrightarrow d(K_r dV) = 0$ (V Stäckel multiplier)

Symmetry operators: $\bar{K}_r = \nabla_i(K_r^{ij}\nabla_j\psi) + W_r\psi$

$\Delta\psi + V = \mu\psi$ MS $\leftrightarrow [\bar{K}_r, \Delta] = 0$

$W_r = K_r dV$.

The coordinates are independent from $\mu \leftrightarrow K_r$ are independent from μ

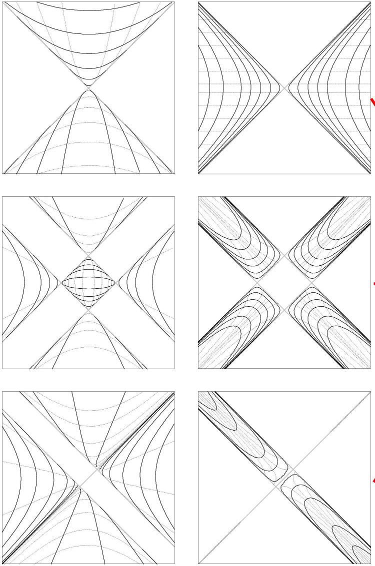
Eigenvalues of the $\bar{K}_r$ are separation constants: $\bar{K}_r\psi = \nu_r\psi$

Example: some separable variables in $\mathbb{M}^2$, real and complex

$$\begin{pmatrix} 2x & t \\ t & 0 \end{pmatrix}$$

$$\begin{pmatrix} x^2 & xt \\ xt & t^2 - 1 \end{pmatrix}$$

$$\begin{pmatrix} x^2 - 1 & xt - 1 \\ xt - 1 & t^2 - 1 \end{pmatrix}$$

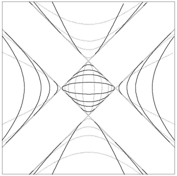


level curves of real and imaginary parts of complex variables

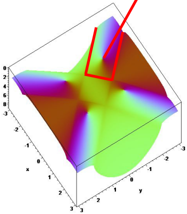
Complex (orthogonal) separable variables

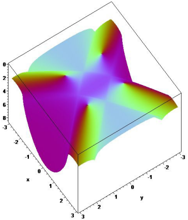
L. Degiovanni, G. R., J. Math. Phys. 48, 073519 2007

$$\begin{vmatrix} x^2 - z & xt \\ xt & t^2 - 1 + z \end{vmatrix} = 0$$



complex variable z
through this point

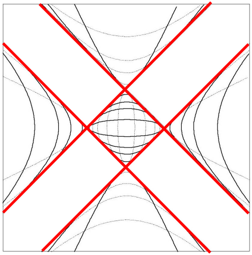




$z, \quad \bar{z}$

Example: separable coordinates in $\mathbb{M}^n$: Horizons, classification

Horizons : where eigenvectors (eigenforms) coincide. Null hypersurfaces.



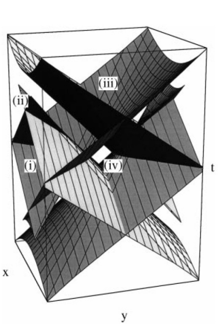


Fig. 3. (B.1.d), horizons

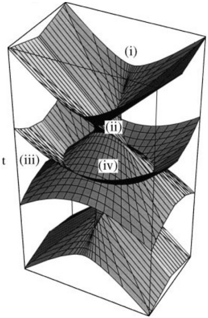


Fig. 1. (B.1.c), horizons

Examples in $\mathbb{M}^2$, $\mathbb{M}^3$ (F. Hinterleitner, Sitzungsber. Abt. II (1998) 207: 133-171)

Classification : in $\mathbb{M}^2$ 12 coordinate systems, in $\mathbb{M}^3$ many more (according to different criteria, up to 89 in Hinterleitner, less in McLenaghan et al.)

Dirac sep. $\to$ Helmholtz sep.

$$(Dirac)^2\psi = \Delta \mathbb{I}\psi + \text{lower-order terms}$$

If on a manifold the Dirac equation has multiplicatively separated solutions in orthogonal coordinates, then also the Helmholtz equation does

The Dirac equation is orthogonally separable only in Stäckel coordinates

Dirac equation as eigenvalue-type equation preserves the analogy with Helmholtz equation

Separation constants are constants of the motion $\leftrightarrow$ are eigenvalues of symmetry operators: $[\mathbf{L}, \mathbf{D}] = 0$, $L\psi = \nu\psi$

L independent from $\mu \leftrightarrow E_i^x$ or E_i^y independent from $\mu \leftrightarrow$ coordinates independent from μ

Separation of variables theory for the Dirac's equation

- Shapovalov-Miller (1973, 1988) theory: first-order symmetry operators
- Fels and Kamran (1989): examples of separation associated only to second-order symmetry operators.
- we build eigenvalue-type operators **L** (first and second-order) using E_i^x or E_i^y so that $\mathbf{L}\psi = \nu\psi$.

Our model of separation for Dirac's equation, including second-order operators, enhances analogies with Helmholtz and Schrödinger separation

Multiplicative separation of eigenvalue-type PDE systems

Let (x, y) a (local) coordinate system on a two dimensional manifold and

$$\psi = \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{pmatrix}.$$

Let $\mathbf{D}$ the operator defined by

$$\mathbf{D} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \partial_x + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \partial_y + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where A_i, B_i and C_i are functions of (x, y), such that

$$\mathbf{D}\,\psi = \begin{pmatrix} A_1\partial_x\psi_1 + A_2\partial_x\psi_2 + B_1\partial_y\psi_1 + B_2\partial_y\psi_2 + C_1\psi_1 + C_2\psi_2 \\ A_3\partial_x\psi_1 + A_4\partial_x\psi_2 + B_3\partial_y\psi_1 + B_4\partial_y\psi_2 + C_3\psi_1 + C_4\psi_2 \end{pmatrix}.$$

Let $\mu \neq 0$ a real or complex number, we consider the equation

$$\mathbf{D}\psi - \mu\psi = 0.$$

If we assume multiplicative separation for ψ, i.e.

$$\psi_i = a_i(x)b_i(y)$$

then the equation $\mathbf{D}\psi - \mu\psi = 0$ becomes

$$\begin{cases} A_1\dot{a}_1 b_1 + A_2\dot{a}_2 b_2 + B_1 a_1 \dot{b}_1 + B_2 a_2 \dot{b}_2 + (C_1 - \mu)a_1 b_1 + C_2 a_2 b_2 = 0 \\ A_3\dot{a}_1 b_1 + A_4\dot{a}_2 b_2 + B_3 a_1 \dot{b}_1 + B_4 a_2 \dot{b}_2 + C_3 a_1 b_1 + (C_4 - \mu)a_2 b_2 = 0 \end{cases}$$

hence, we define separability of $\mathbf{D}$ as follows:

Naïve separation

The operator **D** is separate in (x, y) if there exist nonzero functions $R_i(x, y)a_r b_s$ such that the above equations can be written as:

$$\begin{cases} R_1 a_r b_s (E_1^x + E_1^y) = 0 \\ R_2 a_t b_u (E_2^x + E_2^y) = 0 \end{cases}$$

where $E_i^x(x, a_j, \dot{a}_j)$, $E_i^y(y, b_j, \dot{b}_j)$.

$$E_i^x = \nu_i = -E_i^y$$

provide the separation constants ν_i.

1- Three kinds of separation

Separation ansatz

$$\psi = \begin{pmatrix} a_1(x)b_1(y) \\ a_2(x)b_2(y) \end{pmatrix}$$

I. $a_1 \neq a_2$ and $b_1 \neq b_2$.

II. $a_1 = a_2 = a$ and $b_1 \neq b_2$ (or vice-versa).

III. $a_1 = a_2 = a$ and $b_1 = cb_2 = b$ (c constant).

Not "constrained separation": setting $a_1 = e^x$, $a_1 = x^2 - 1$,...

Type I

$R_i = 1.$ $\rightarrow$ one at least of $a_l b_s$ in $i-th$ equation must be equal to i (μ is arbitrary). Then, separation is achieved by

$$\begin{pmatrix} \frac{1}{a_l b_m} & 0 \\ 0 & \frac{1}{a_p b_q} \end{pmatrix} D\psi$$

where one of l, m is 1 and one of p, q is 2. All the possible separation schemes are

$$\begin{cases} (1,1,1,2) & (1,1,2,1) & (1,1,2,2) \\ (1,2,1,2) & (1,2,2,1) & (1,2,2,2) \\ (2,1,1,2) & (2,1,2,1) & (2,1,2,2) \end{cases}$$

The separation conditions $(\mathbf{D}-\mu)_i\psi = a_l b_m (E_i^x + E_i^y)$ impose restrictions on functions A_i, B_i and C_i

therefore the possible separate forms of $\mathbf{D}$ are

Separation schemes

$$\mathbf{D}_1 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_2 : \begin{pmatrix} A_1(x) & 0 \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_3 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y +$$

$$+ \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix},$$

$$\mathbf{D}_4 : \begin{pmatrix} 0 & A_2(x) \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_5 : \begin{pmatrix} 0 & A_2(x) \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_6 : \begin{pmatrix} 0 & A_2(x) \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix},$$

$$\mathbf{D}_7 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_8 : \begin{pmatrix} A_1(x) & 0 \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_9 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix}.$$

Exchanging x and y some of the previous operators coincide: $\mathbf{D}_1 \equiv \mathbf{D}_2$, $\mathbf{D}_4 \equiv \mathbf{D}_8$, $\mathbf{D}_5 \equiv \mathbf{D}_7$, $\mathbf{D}_6 \equiv \mathbf{D}_9$ and $\mathbf{D}_3 \equiv \mathbf{D}_3$.
By introducing the operator

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that

$$\mathbf{J} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

$\mathbf{D}_1\psi$ is of the same form as $\mathbf{J}\mathbf{D}_9\mathbf{J}\psi$, then $\mathbf{D}_1$ and $\mathbf{D}_9$ are equivalent.
Four distinct classes: $\mathbf{D}_1$, $\mathbf{D}_3$, $\mathbf{D}_4$ and $\mathbf{D}_5$.

Separated equations associated with no operators

The class $\mathbf{D}_1$:
Let us consider $\mathbf{D}_1\psi - \mu\psi = 0$, assuming here and in the following that $\mu \neq 0$, the first component can be separated as

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 - \mu a_1 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 = -\nu_1 b_1 \end{cases}$$

or

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 - \mu b_1 = -\nu_1 b_1 \end{cases}$$

according to alternative grouping of μ with terms in x or y. The second component reads

$$\begin{cases} A_4(x)\dot{a}_2 + C_4(x)a_2 - \mu a_2 = \nu_2 a_1 \\ B_3(y)\dot{b}_1 + C_3(y)b_1 = -\nu_2 b_2 \end{cases}$$

where ν_1 and ν_2 are the separation constants.
Separated equations can be decoupled by integrating a_1 and b_1 from the first two and substituting the results in the last. The solutions a_i, b_i are in all cases given by first-order ODE's.

By using the only terms independent of μ and assuming $\nu_1 = \nu_2$ one can try to build an operator $\mathbf{L}$. However, this is impossible under the assumption of independence among a_i and b_i. The same for higher-order operators, when we can assume $\nu_1 \neq \nu_2$. It follows that no eigenvalue-operator is associated with the $\mathbf{D}_1$ separation scheme.

Separated equations associated with first-order operators

The class $\mathbf{D}_3$.
By considering equation $\mathbf{D}_3\psi - \mu\psi = 0$ we obtain the following four systems of separation equations, according to the possible different groupings of μ:

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 - \mu a_1 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 = -\nu_1 b_1 \end{cases}$$

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 - \mu b_1 = -\nu_1 b_1 \end{cases}$$

$$\begin{cases} A_4(x)\dot{a}_2 + C_{41}(x)a_2 - \mu a_2 = \nu_2 a_2 \\ B_4(y)\dot{b}_2 + C_{42}(y)b_2 = -\nu_2 b_2 \end{cases}$$

$$\begin{cases} A_4(x)\dot{a}_2 + C_{41}(x)a_2 = \nu_2 a_2 \\ B_4(y)\dot{b}_2 + C_{42}(y)b_2 - \mu b_2 = -\nu_2 b_2 \end{cases}$$

All equations are decoupled in a_i, b_i and solutions are always given by first-order ODE's.

By putting $\nu_1 = \nu_2 = \nu$ we can obtain from the previous systems the following couples of equations respectively, suitable for the construction of operators:

$$\begin{cases} (B_1\partial_y + C_{12})b_1 = -\nu b_1 \\ (B_4\partial_y + C_{42})b_2 = -\nu b_2 \end{cases}$$

$$\begin{cases} (B_1\partial_y + C_{12})b_1 = -\nu b_1 \\ (A_4\partial_x + C_{41})a_2 = \nu a_2 \end{cases}$$

$$\begin{cases} (A_1\partial_x + C_{11})a_1 = \nu a_1 \\ (B_4\partial_y + C_{42})b_2 = -\nu b_2 \end{cases}$$

$$\begin{cases} (A_1\partial_x + C_{11})a_1 = \nu a_1 \\ (A_4\partial_x + C_{41})a_2 = \nu a_2 \end{cases}$$

The corresponding operators of the form $\mathbf{L}\psi = \nu\psi$ are then respectively

$$\mathbf{L}_1 = -\begin{pmatrix} B_1\partial_y + C_{12} & 0 \\ 0 & B_4\partial_y + C_{42} \end{pmatrix}$$

$$\mathbf{L}_2 = \begin{pmatrix} -B_1\partial_y - C_{12} & 0 \\ 0 & A_4\partial_x + C_{41} \end{pmatrix}$$

$$\mathbf{L}_3 = \begin{pmatrix} A_1\partial_x + C_{11} & 0 \\ 0 & -B_4\partial_y - C_{42} \end{pmatrix}$$

$$\mathbf{L}_4 = \begin{pmatrix} A_1\partial_x + C_{11} & 0 \\ 0 & A_4\partial_x + C_{41} \end{pmatrix}$$

An easy computation shows that all these operators commute with $\mathbf{D}_3$ and between themselves when applied to some generic ψ. The same holds for the powers of the $\mathbf{L}_i$.

Separated equations associated with second-order operators

The class $\mathbf{D}_5$.
The separation of $\mathbf{D}_5\psi - \mu\psi = 0$ is given by

$$\begin{cases} A_2(x)\dot{a}_2 + C_2(x)a_2 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_1(y)b_1 - \mu b_1 = -\nu_1 b_2 \end{cases}$$

$$\begin{cases} A_3(x)\dot{a}_1 + C_3(x)a_1 = \nu_2 a_2 \\ B_4(y)\dot{b}_2 + C_4(y)b_2 - \mu b_2 = -\nu_2 b_1 \end{cases}$$

No first-order operator independent of μ can be defined by using the previous equations.

Decoupling relations

$$\begin{cases} (A_2\partial_x + C_2)(A_3\partial_x + C_3)a_1 = \nu_1\nu_2 a_1 \\ (A_3\partial_x + C_3)(A_2\partial_x + C_2)a_2 = \nu_1\nu_2 a_2 \end{cases}$$

Decoupling operator

$$\mathbf{L}_5 = \begin{pmatrix} (A_2\partial_x + C_2)(A_3\partial_x + C_3) & 0 \\ 0 & (A_3\partial_x + C_3)(A_2\partial_x + C_2) \end{pmatrix}$$

$$\mathbf{L}_5\psi = \nu_1\nu_2\psi$$

always..

$[\mathbf{D}_5, \mathbf{L}_5] = 0.$

$\mathbf{L}_5\psi = \nu\psi$, $\nu = \nu_1\nu_2$ The product of the separation constants is a constant of the motion $\rightarrow$ the solutions should depend on ν and not on ν_1, ν_2.

$R_i \neq 1$.

$$\mathbf{D}_k = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{D}'_k$$

where the $\mathbf{D}'_k$ are the operators seen above.

$\mathbf{D} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} D_5$ separable $\leftrightarrow$

$$R_1(y), \ R_2(y)$$

$\rightarrow [\mathbf{L}_5, \mathbf{D}] = 0$.

Separation of the Dirac equation and second-order symmetry operators

The Dirac equation in 2D

$$D\psi = \left[\begin{pmatrix} ie_0^0 & -ike_1^0 \\ ike_1^0 & -ie_0^0 \end{pmatrix}\partial_x + \begin{pmatrix} ie_0^1 & -ike_1^1 \\ ike_1^1 & -ie_0^1 \end{pmatrix}\partial_y + \tilde{C}\right]\psi = \mu\psi,$$

$$\tilde{C} = \frac{i}{2}\epsilon^{ab}e_a^\mu\Gamma_\mu^{01}\gamma_b - qe_a^\mu V_\mu\gamma^a - V\mathbb{I} - \hat{V}\gamma,$$

separation scheme D5: $D =$

$$\begin{pmatrix} R_1(y) & 0 \\ 0 & R_2(y) \end{pmatrix}\left[\begin{pmatrix} 0 & A_2(x) \\ A_3(x) & 0 \end{pmatrix}\partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix}\partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ C_3(x) & C_4(y) \end{pmatrix}\right]$$

Separated equations

$$\begin{cases} A_2(x)\dot{a}_2 + C_2(x)a_2 = \nu_1 a_1 \\ B_1(y)\dot{b}_1 + C_1(y)b_1 - \mu b_1 = -\nu_1 b_2 \end{cases}$$

$$\begin{cases} A_3(x)\dot{a}_1 + C_3(x)a_1 = \nu_2 a_2 \\ B_4(y)\dot{b}_2 + C_4(y)b_2 - \mu b_2 = -\nu_2 b_1 \end{cases}$$

Decoupling relations

$$\begin{cases} (A_2\partial_x + C_2)(A_3\partial_x + C_3)a_1 = \nu_1\nu_2 a_1 \\ (A_3\partial_x + C_3)(A_2\partial_x + C_2)a_2 = \nu_1\nu_2 a_2 \end{cases}$$

Decoupling operator = symmetry operator

$$\mathbf{L}_5 = \begin{pmatrix} (A_2\partial_x + C_2)(A_3\partial_x + C_3) & 0 \\ 0 & (A_3\partial_x + C_3)(A_2\partial_x + C_2) \end{pmatrix}$$

- Stäckel coordinates in 2D $\rightarrow$ Liouville form $g_{11} = A(x) + B(y)$, $g_{00} = \eta g_{11}$,
- canonical Killing tensor: $K_{00} = -g_{00}B$, $K_{11} = g_{11}A$
- $\rightarrow$ Robertson condition holds
- D5 separation needs one geodesically ignorable coordinate at least [MR,CFMR], $\rightarrow A = 0$, we put $B(y) = \beta(y)^2$ (to avoid square roots)
- D5 separation $\rightarrow$ spin-frame components (e_a^μ) and V must be chosen so that

$$\mathbb{D} = \frac{ik}{\beta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\partial_x + \frac{i}{\beta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_y + \begin{pmatrix} \frac{i\beta'}{2\beta^2} + \frac{q}{\beta}V_1 - V & -\frac{kq}{\beta}V_0 + \eta k\hat{V} \\ \frac{kq}{\beta}V_0 + \eta k\hat{V} & -\frac{i\beta'}{2\beta^2} - \frac{q}{\beta}V_1 - V \end{pmatrix}$$

with

with

$$\begin{cases} qV_0 = \frac{1}{2k}(C_3(x) - C_2(x)), \\ qV_1 = \frac{1}{2}\left(C_1(y) - C_4(y) - i\frac{\beta'}{\beta}\right), \\ V = -\frac{1}{2\beta}(C_1(y) + C_4(y)) \\ \hat{V} = \frac{1}{2k\eta\beta}(C_2(x) + C_3(x)). \end{cases} \tag{1}$$

Then

In Liouville coordinates the vector potential V_μ separable in the scheme D5 is necessarily exact and the force field $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is equal to zero.

In Liouville coordinates, the scalar and pseudoscalar potentials are compatible with separation of variables in the scheme D5 only if V^2 and $\hat{V}^2$ are Stäckel multipliers, that is only if

$$d(K\ d(V^2)) = 0, \quad d(K\ d(\hat{V}^2)) = 0.$$

Decoupling symmetry operator in Liouville coordinates

$$\eta\left[\left(-\partial^2_{xx}+2iqV_0\partial_x+iq\partial_x V_0+q^2V_0^2-\beta^2\hat{V}^2\right)\mathbb{I}+i\eta\beta\partial_x\hat{V}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\psi=\nu\psi,$$

Remaining decoupling relations: not symmetry operator

$$\left[\left(\partial^2_{yy}+\left(\frac{\beta'}{\beta}-2iqV_1\right)\partial_y+\frac{\beta''}{2\beta}-\frac{1}{4}\left(\frac{\beta'}{\beta}\right)^2-q^2V_1^2-i\frac{\beta'}{\beta}qV_1-\right.\right.$$

$$\left.\left.-iq\partial_y V_1+\beta^2V^2+2\mu\beta V+\mu^2\right)\mathbb{I}+i\partial_y(\beta V)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\psi=\nu\psi.$$

In Liouville coordinates, the vector, scalar and pseudoscalar potentials are compatible with separation of variables in the scheme D5 associated with a symmetry operator if and only if they are of the form (1).

Remark: first-order terms in decoupling relations disappear if

$$qV_0 = 0, \quad qV_1 = -\frac{i\beta'}{2\beta}. \tag{2}$$

(V_μ) exact $\rightarrow$ this term can always be introduced without affecting the physics of the system (gauge invariance)

With this choice of (V_μ), for any $\beta(y)$ the decoupling relations (2) and (2) give respectively

$$\begin{cases} -\eta\left(a_1''(x)+\beta(\beta\hat{V}^2-i\eta\partial_x\hat{V})\right)a_1=\nu a_1(x),\\ -\eta\left(a_2''(x)+\beta(\beta\hat{V}^2+i\eta\partial_x\hat{V})\right)a_2=\nu a_2(x),\\ b_1''(y)+\left(i\partial_y(\beta V)+(\beta V+\mu)^2\right)b_1(y)=\nu b_1(y),\\ b_2''(y)+\left(-i\partial_y(\beta V)+(\beta V+\mu)^2\right)b_2(y)=\nu b_2(y). \end{cases} \tag{3}$$

For $V = \hat{V} = 0$ the decoupling equations of above can be easily integrated

$$\begin{cases} \psi_1 = \left(c_1 e^{\frac{\sqrt{\nu}}{\hat{k}}x} + c_2 e^{-\frac{\sqrt{\nu}}{\hat{k}}x}\right)\left(d_1 \sin\sqrt{\mu^2-\nu}\,y + d_2 \cos\sqrt{\mu^2-\nu}\,y\right), \\ \psi_2 = \left(c_3 e^{\frac{\sqrt{\nu}}{\hat{k}}x} + c_4 e^{-\frac{\sqrt{\nu}}{\hat{k}}x}\right)\left(d_3 \sin\sqrt{\mu^2-\nu}\,y + d_4 \cos\sqrt{\mu^2-\nu}\,y\right), \end{cases} \tag{4}$$

where $c_3 = i(\nu)^{-\frac{1}{2}}c_1$, $c_4 = -i(\nu)^{-\frac{1}{2}}c_2$, $d_3 = d_1\mu + id_2\sqrt{\mu^2-\nu}$,
$d_4 = d_2\mu - id_1\sqrt{\mu^2-\nu}$.
Geodesic Dirac equation.

An example of Hamilton-Jacobi and Schrödinger equations with scalar potentials separable in these coordinates on curved spaces [Ballestreros, Enciso, Herranz, Ragnisco and Riglioni, Ann. of Phys. 326 (2011)]. A generalization of the harmonic oscillator to conformally flat n-dimensional Riemannian manifolds. In Liouville coordinates on Riemannian or pseudo-Riemannian manifolds

$$H = \frac{e^{-2y}}{2(1+\lambda e^{2y})}(\eta p_x^2 + p_y^2) + \frac{\omega^2 e^{2y}}{2(1+\lambda e^{2y})},$$

where λ and ω are parameters.
C_1, C_4 can be chosen so that V coincides with the scalar potential of H and $V_1 = -\frac{i\beta'}{2\beta}$, while $C_2 = C_3 = 0$ give $V_0 = \hat{V} = 0$.

The corresponding Dirac equation with $V = \frac{\omega^2 e^{2y}}{2(1+\lambda e^{2y})}$ is MS in Liouville coordinates

Second order symmetry operators of the Dirac equation: invariant form

A *second order symmetry operator* for the Dirac equation is an operator of the form

$$K = E^{ab}\nabla_{ab} + F^a\nabla_a + G\mathbb{I}$$

which commutes with the Dirac operator D. Here $\nabla_{ab} = \frac{1}{2}\left(\nabla_a\nabla_b + \nabla_b\nabla_a\right)$ denotes the *symmetrized* second covariant derivative (expressed in the frame). The coefficients E^{ab}, F^a, G are matrix zero-order operators. By expanding the condition $[K, D] = 0$ one obtains

$$\begin{cases} E^{(ab}\gamma^{c)} - \gamma^{(c}E^{ab)} = 0 \\ F^{(a}\gamma^{b)} - \gamma^{(b}F^{a)} = \gamma^c\nabla_c E^{ab} - i\left(E^{ab}\mathbf{V} - \mathbf{V}E^{ab}\right) \\ G\gamma^a - \gamma^a G = \gamma^c\nabla_c F^a - i(F^a\mathbf{V} - \mathbf{V}F^a) - \frac{1}{4}\left(E^{ab}\gamma^c + \gamma^c E^{ab}\right)\gamma_{ef}R^{ef}{}_{bc} + \frac{1}{3}\left(E^{ef}\gamma^c - 2i\gamma^c E^{ef}\right)R^a{}_{efc} + \\ \qquad +iq\left(E^{ab}\gamma^c + \gamma^c E^{ab}\right)F_{bc} - 2iE^{ab}{na}_b\mathbf{V} \\ \gamma^c\nabla_c G = i(G\mathbf{V} - \mathbf{V}G) + \frac{1}{12}\nabla_a R^{ef}{}_{bc}\left(2E^{ab}\gamma^c + \gamma^c E^{ab}\right)\gamma_{ef} + \frac{1}{8}\left(F^a\gamma^b + \gamma^b F^a\right)\gamma_{ef}R^{ef}{}_{ab} + \\ \qquad -\frac{iq}{3}\left(2E^{ab}\gamma^c + \gamma^c E^{ab}\right)\nabla_a F_{bc} - \frac{iq}{2}\left(F^a\gamma^b + \gamma^b F^a\right)F_{ab} + iE^{ab}\nabla_{ab}\mathbf{V} + iF^a\nabla_a\mathbf{V}, \end{cases}$$

A second order symmetry operator (up to terms proportional to D, D^2) has the form:

$$\begin{cases} E^{ab} = e^{ab}\mathbb{I} \\ F^a := (\zeta^a + \nabla_b e^{ab})\mathbb{I} + \alpha\gamma^a + \left(\frac{1}{3}\epsilon_{cb}\nabla^c e^{ab}\right)\gamma \\ G := (g' + i\alpha V)\mathbb{I} + \left(ie_a^b\nabla_b V - i\eta e^{cb}\nabla_b\hat{V}\epsilon_{ac}\right)\gamma^a + \\ \qquad + \left(\frac{1}{4}\left(\nabla^c\zeta^a - 2iqF\epsilon_b^c e^{ab}\right)\epsilon_{ca} + i\alpha\hat{V}\right)\gamma \end{cases}$$

$$\nabla^{(c}e^{ab)} = 0$$

$$\nabla_a\alpha = \omega_a$$

$$\nabla^{(c}\zeta^{a)} = \frac{1}{2}\Lambda^{ca}$$

$$\nabla_a g' = \Lambda_a$$

where we set

$$\begin{cases} \Lambda^{ca} = 4iqF\epsilon_b^{(c}e^{a)b} \quad (\to \Lambda = 0) \\ \Lambda_c = iqF\epsilon_{ca}\zeta^a - \frac{1}{4}\nabla_a\left(Re_c^a\right) + e_c^d\nabla_d\left(V^2 + \eta\hat{V}^2\right) \\ \omega_c = 2ie_c^d\nabla_d V \end{cases}$$

and where the following integrability conditions are satisfied

$$
\begin{cases}
\epsilon^{bc}\nabla_b\omega_c = 0 \qquad \rightarrow \epsilon^{ac}\nabla_c(e_a^b\nabla_b V) = 0 \\
\epsilon^{dc}\nabla_d\Lambda_c = 0 \qquad \rightarrow iq\zeta^a\nabla_a F = 2\epsilon^{dc}e_c^e\nabla_e V\nabla_d V - \frac{1}{4}\epsilon^{dc}\nabla_d\nabla_a\left(Re_c^a\right) + \eta\epsilon^{dc}\nabla_d\left(e_c^b\nabla_b(\hat{V})^2\right) \\
\zeta^a\nabla_a R = \nabla_c\left(\nabla^b\Lambda^{da}\right)\epsilon_d^c\epsilon_{ab} \\
\zeta^a\nabla_a V = -\eta\left(\frac{2}{3}\nabla_a e^{bc}\nabla_c\hat{V}\epsilon_b^a + e^{bc}\nabla_{ac}\hat{V}\epsilon_b^a\right) = -\eta(\epsilon^a_{.b}\nabla_a(e^{bc}\nabla_c\hat{V}) - \frac{1}{3}\epsilon^a_{.b}\nabla_a e^{bc}\nabla_c\hat{V}) \\
\zeta^a\nabla_a\hat{V} = \frac{2}{3}\epsilon_{cb}\nabla^c e^{ab}\nabla_a V + \epsilon_b^c\, e^{ab}\nabla_{ca}V = \epsilon_{cb}\nabla^c(e^{ab}\nabla_a V) - \frac{1}{3}\epsilon_{cb}\nabla^c e^{ab}\nabla_a V = -\frac{1}{3}\epsilon_{cb}\nabla^c e^{ab}\nabla_a V
\end{cases}
$$

Second-order operators associated to Liouville coordinates

The second-order symmetry operator associated with the separable Liouville coordinates is determined by the following conditions

1. e is the canonical Killing tensor associated with the Liouville coordinates $e_{00} = -\eta\beta^4$, $e_{10} = e_{01} = e_{11} = 0$
2. α is zero
3. F is zero
4. ζ is the zero vector
5. the function g' is given, up to additive constants, by

$$g' = \frac{1}{4}\left((2k\eta\beta\hat{V})^2 + \left(\frac{\beta'}{\beta}\right)^2\right)$$

$$\begin{cases} E^{ab} = e^{ab}\mathbb{I} \\ F^a := \nabla_b e^{ab}\mathbb{I} + \left(\frac{1}{3}\epsilon_{cb}\nabla^c e^{ab}\right)\gamma \\ G := g'\mathbb{I} + \left(ie^b_a\nabla_b V - i\eta e^{cb}\nabla_b\hat{V}\epsilon_{ac}\right)\gamma^a \end{cases}$$

Conclusions

The operators of above, even if not the most general possible, are a probe to investigate the properties of the separation of variables of the Dirac equation.
We can see how coordinates and spin frames are related to each other, what kind of metrics are generated by the separation conditions and how they are related to Hamilton-Jacobi separation of variables.
We can find explicitely the transformation from separable to pseudo-Cartesian coordinates (Horwood-McLenaghan).
Although the physical Dirac equation is in dimension four, separation in two dimension can occur after reduction by symmetries (Kerr solution).

Future directions

- μ fixed and symmetry operators depending on μ (Fixed-energy separation).
- 3D and 4D
- separation in other representations of the Clifford algebra