Deformation quantization and cohomological renormalization

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[This talk is an homage to our old friend Dmitrij Vasil'evich Shirkov
from the late Moshe Flato and myself, exhibiting a small but important aspect of
the role of deformations in physics emphasized by Flato since 1971,
the year Dmitrij Vasil'evich and we spent in Sweden]

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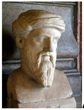
Abstract

The role of deformations in physics and mathematics lead to the deformation philosophy promoted in mathematical physics by Flato since the 70's, exemplified by deformation quantization [where quantization is realized as a deformation of the product of classical observables into a noncommutative "star-product"] and its manifold avatars. Examples show that in field theory, in the deformation quantization framework, quantizations mathematically equivalent to normal ordering (subtracting a coboundary to the cocycles leading to it) may exhibit less divergences. It is suggested that renormalization might be obtained by a further deformation of the normal product deformation, subtracting infinite cocycles from those giving normal ordering, leading to a finite result. That could be what is behind the Connes–Kreimer approach to renormalization.

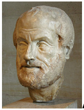
The Earth is not flat

Act 0. Antiquity (Mesopotamia, ancient Greece).

Flat disk floating in ocean, or Atlas. Similar physical assumption in (ancient) China (Φ).



Act I. Fifth century BC: Pythogoras, theoretical astrophysicist. Pythagoras is often considered as the first mathematician; he and his students believed that everything is related to mathematics. On aesthetic (and democratic?) grounds he conjectured that **all** celestial bodies are spherical.



Act II. 3rd century BC: Aristotle, phenomenologist astronomer. Travelers going south see southern constellations rise higher above the horizon, and shadow of earth on moon during the partial phase of a lunar eclipse is always circular: fits physical model of sphere for Earth.

Eratosthenes “Experiment”

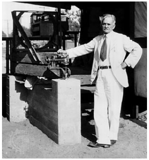
Act III. ca. 240 BC:

Eratosthenes, “experimentalist”.

Chief librarian of the Great Library in Alexandria. At summer solstice (21 June), knew that sun (practically) at vertical in Aswan and angle of $\frac{2\pi}{50}$ in Alexandria, “about” (based on estimated average daily speed of caravans of camels?) 5000 stadions “North;” assuming sun is point at ∞ (all not quite), by simple geometry got circumference of 252000 “stadions”, 1% or 16% off correct value (Egyptian or Greek stadion). Computed distance to sun as 804,000 kstadions and distance to moon as 780 kstadions, using data obtained during lunar eclipses, and measured tilt of Earth’s axis 11/83 of 2π.

In China, ca. same time, different context: measure similarly distance of earth to sun assuming earth is flat (the prevailing belief there until 17$^{\text{th}}$ century).

Relativity




Paradox coming from Michelson & Morley experiment (1887) resolved in 1905 by Einstein with special theory of relativity. Experimental need triggered theory. In modern language: Galilean geometrical symmetry group of Newtonian mechanics ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) is deformed, in Gerstenhaber's sense, to Poincaré group ($SO(3,1) \cdot \mathbb{R}^4$) of special relativity. A deformation parameter comes in, c^{-1}, c being a *new fundamental constant*, velocity of light in vacuum. Time has to be treated on same footing as space, expressed mathematically as a purely imaginary dimension. General relativity: *deform* Minkowskian space-time with nonzero pseudo-Riemannian curvature. E.g. constant curvature, de Sitter (> 0) or AdS_4 (< 0).

Flato's deformation philosophy

Physical theories have domain of applicability defined by the relevant distances, velocities, energies, etc. involved. The passage from one domain (of distances, etc.) to another doesn't happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict [Fermi quote] accepted theories. Eventually a new fundamental constant enters, the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The question is, in which category to seek for deformations? Physics is conservative: if start with e.g. category of associative or Lie algebras, tend to deform in same category. But there are important generalizations: e.g. quantum groups are deformations of (some commutative) Hopf algebras.

Epistemological remarks


Two quotes by Sir James Hopwood Jeans:

“The Great Architect of the Universe now begins to appear as a pure mathematician.”

“We may as well cut out the group theory. That is a subject that will never be of any use in physics.” [Discussing a syllabus in 1910.] [Physicists’ liberty with rigor vs. mathematicians’ lack of physical touch.]

Spectroscopy. In atomic and molecular physics we know the forces and their symmetries. Energy levels (spectral lines) classified by UIR (unitary irreducible representations) of $SO(3)$ or $SU(2)$, and e.g. with crystals that is refined (broken) by a finite subgroup. [Racah school, Flato’s M.Sc.] The more indirect physical measurements become, the more one has to be careful.

“Curse” of experimental sciences. Mathematical logic: if A and $A \rightarrow B$, then B. In real life, imagine model or theory A. If $A \rightarrow B$ and “B is nice” (e.g. verified & more), then A! [Inspired by Kolmogorov quote.] (It ain’t necessarily so.)

Philosophy?

Mathematics and physics are two communities separated by a common language. In mathematics one starts with axioms and uses logical deduction therefrom to obtain results that are absolute truth in that framework. In physics one has to make approximations, depending on the domain of applicability.
As in other areas, a *quantitative* change produces a *qualitative change*. *(So we should deform, not extrapolate!)* Engels (i.a.) developed that point and gave a series of examples in Science to illustrate the transformation of quantitative change into qualitative change *at critical points* (see `http://www.marxists.de/science/mcgareng/engels1.htm`).
That is also a problem in psychoanalysis that was tackled using Thom's catastrophe theory. Robert M. Galatzer-Levy, *Qualitative Change from Quantitative Change: Mathematical Catastrophe Theory in Relation to Psychoanalysis*, J. Amer. Psychoanal. Assn., **26** (1978), 921–935.

Deformation theory is an algebraic mathematical way to deal with that "catastrophic" situation, most relevant to physics.

Classical Mechanics and around

What do we quantize?

Non trivial phase spaces $\rightarrow$ Symplectic and Poisson manifolds.

Symplectic manifold: Differentiable manifold M with nondegenerate closed 2-form ω on M. Necessarily $\dim M = 2n$. Locally: $\omega = \omega_{ij} dx^i \wedge dx^j$; $\omega_{ij} = -\omega_{ji}$; $\det \omega_{ij} \neq 0$; $Alt(\partial_i \omega_{jk}) = 0$. And one can find coordinates (q_i, p_i) so that ω is constant: $\omega = \sum_{i=1}^{i=n} dq^i \wedge dp^i$.
Define $\pi^{ij} = \omega_{ij}^{-1}$, then $\{F, G\} = \pi^{ij} \partial_i F \partial_j G$ is a Poisson bracket, i.e. the bracket $\{\cdot,\cdot\} \colon C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is a skewsymmetric ($\{F, G\} = -\{G, F\}$) bilinear map satisfying:

- Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$
- Leibniz rule: $\{FG, H\} = \{F, H\}G + F\{G, H\}$

Examples: 1) $\mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{i=n} dq^i \wedge dp^i$;
2) Cotangent bundle T^*N, $\omega = d\alpha$, where α is the canonical one-form on T^*N (Locally, $\alpha = -p_i dq^i$)

Poisson manifolds

Poisson manifold: Differentiable manifold M, and skewsymmetric contravariant 2-tensor (not necessarily nondegenerate) $\pi = \sum_{i,j} \pi^{ij} \partial_i \wedge \partial_j$ (locally) such that $\{F, G\} = i(\pi)(dF \wedge dG) = \sum_{i,j} \pi^{ij} \partial_i F \wedge \partial_j G$ is a Poisson bracket.

Examples:

1) Symplectic manifolds ($d\omega = 0 = [\pi, \pi] \equiv$ Jacobi identity)
2) Lie algebra with structure constants C_{ij}^k and $\pi^{ij} = \sum_k x^k C_{ij}^k$.
3) $\pi = X \wedge Y$, where (X, Y) are two commuting vector fields on M.

Facts : Every Poisson manifold is "foliated" by symplectic manifolds. If π is nondegenerate, then $\omega_{ij} = (\pi^{ij})^{-1}$ is a symplectic form.

A Classical System is a Poisson manifold (M, π) with a distinguished smooth function, the Hamiltonian $H\colon M \to \mathbb{R}$.

Quantization in physics

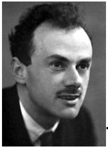


Planck and black body radiation [ca. 1900]. Bohr atom [1913]. Louis de Broglie [1924]: "wave mechanics" (waves and particles are two manifestations of the same physical reality).


Traditional quantization (Schrödinger, Heisenberg) of classical system $(\mathbb{R}^{2n}, \{\cdot,\cdot\}, H)$: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ where acts "quantized" Hamiltonian **H**, energy levels $\mathbf{H}\psi = \lambda\psi$, and von Neumann representation of CCR.
Define $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar\frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$. Then (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta}I \quad (\alpha, \beta = 1, ..., n)$.

Orderings, Weyl, Wigner





The couple $(\hat{q}, \hat{p})$ quantizes the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering, e.g. (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (Hermann Weyl [1927] with weight $\varpi = 1$):

$$H \mapsto \mathbf{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar) \varpi(\xi, \eta) d^n\xi d^n\eta.$$

E. Wigner [1932] inverse $H = (2\pi\hbar)^{-n} \mathrm{Tr}[\Omega_1(H) \exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]$.

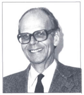
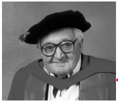
Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Can extend e.g. to distributions.

Constrained systems e.g. constraints $f_j(p, q) = 0$ ($\Rightarrow$ also algebraic varieties and manifolds with corners): Dirac formalism [1950].

Dirac quote

"... One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines. A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables? In both cases we can do addition, multiplication, division..." **Dirac**, *The relation of Classical to Quantum Mechanics* (2nd Can. Math. Congress, Vancouver 1949). U Toronto Press (1951) pp 10-31.

Classical $\leftrightarrow$ Quantum correspondence

The correspondence $H \mapsto \Omega(H)$ is not an algebra homomorphism, neither for ordinary product of functions nor for the Poisson bracket P ("Van Hove theorem"). Take two functions u_1 and u_2, then (Groenewold [1946], Moyal [1949]):
$\Omega_1^{-1}(\Omega_1(u_1)\Omega_1(u_2)) = u_1 u_2 + \frac{i\hbar}{2}\{u_1, u_2\} + O(\hbar^2)$, and similarly for bracket.
More precisely Ω_1 maps into product and bracket of operators (resp.):
$u_1 *_M u_2 = \exp(tP)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{t^r}{r!} P^r(u_1, u_2)$ (with $2t = i\hbar$),
$M(u_1, u_2) = t^{-1}\sinh(tP)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{t^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$
We recognize formulas for deformations of algebras.

Deformation quantization: forget the correspondence principle Ω and work in an *autonomous* manner with "functions" on phase spaces.

The framework

Poisson manifold (M, π), deformations of product of functions.

Inspired by deformation philosophy, based on Gerstenhaber's deformation theory [Flato, Lichnerowicz, Sternheimer; and Vey; mid 70's] [Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, LMP '77 & Ann. Phys. '78]

- $\mathcal{A}_t = C^\infty(M)[[t]]$, formal series in t with coefficients in $C^\infty(M) = A$. Elements: $f_0 + tf_1 + t^2 f_2 + \cdots$ (t formal parameter, not fixed scalar.)
- Star product $\star_t : \mathcal{A}_t \times \mathcal{A}_t \to \mathcal{A}_t$; $f \star_t g = fg + \sum_{r \geq 1} t^r C_r(f, g)$

- C_r are bidifferential operators null on constants: $(1 \star_t f = f \star_t 1 = f)$.
- $\star_t$ is associative and $C_1(f, g) - C_1(g, f) = 2\{f, g\}$, so that $[f, g]_t \equiv \frac{1}{2t}(f \star_t g - g \star_t f) = \{f, g\} + O(t)$ is Lie algebra deformation.

Basic paradigm. Moyal product on $\mathbb{R}^{2n}$ with the canonical Poisson bracket P: $F \star_M G = \exp\left(\frac{i\hbar}{2} P\right)(f, g) \equiv FG + \sum_{k \geq 1} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k P^k(F, G)$.

Applications and Equivalence

Equation of motion (time τ): $\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar}(H \star_M F - F \star_M H)$
Link with Weyl's rule of quantization: $\Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G)$

Equivalence of two star-products $\star_1$ and $\star_2$.

- Formal series of differential operators $T(f) = f + \sum_{r \geq 1} t^r T_r(f)$.
- $T(f \star_1 g) = T(f) \star_2 T(g)$.

For symplectic manifolds, equivalence classes of star-products are parametrized by the 2nd de Rham cohomology space $H^2_{dR}(M)$: $\{\star_t\}/\sim = H^2_{dR}(M)[[t]]$ (Nest-Tsygan [1995] and others). In particular, $H^2_{dR}(\mathbb{R}^{2n})$ is trivial, all deformations are equivalent.
Kontsevich: $\{$Equivalence classes of star-products$\} \equiv \{$equivalence classes of formal Poisson tensors $\pi_t = \pi + t\pi_1 + \cdots\}$.

Remarks:

- The choice of a star-product fixes a quantization rule.
- Operator orderings can be implemented by good choices of T (or ϖ).
- On $\mathbb{R}^{2n}$, all star-products are equivalent to Moyal product (cf. von Neumann uniqueness theorem on projective UIR of CCR).

Existence and Classification

Let (M, π) be a Poisson manifold. $f \tilde{\star} g = fg + t\{f, g\}$ does not define an associative product. But $(f \tilde{\star} g) \tilde{\star} h - f \tilde{\star} (g \tilde{\star} h) = O(t^2)$.

Is it always possible to modify $\tilde{\star}$ in order to get an associative product?

Existence, symplectic case:
– DeWilde-Lecomte [1982]: Glue local Moyal products.
– Omori-Maeda-Yoshioka [1991]: Weyl bundle and glueing.
– Fedosov [1985,1994]: Construct a flat abelian connection on the Weyl bundle over the symplectic manifold.

General Poisson manifold M with Poisson bracket P:
Solved by Kontsevich [1997, LMP 2003]. “Explicit” local formula: $(f, g) \mapsto f \star g = \sum_{n \geq 0} t^n \sum_{\Gamma \in G_{n,2}} w(\Gamma) B_\Gamma(f, g)$, defines a differential star-product on $(\mathbb{R}^d, P)$; globalizable to M. Here $G_{n,2}$ is a set of graphs Γ, $w(\Gamma)$ some weight defined by Γ and $B_\Gamma(f, g)$ some bidifferential operators.

Particular case of Formality Theorem. Operadic approach

This is Quantization

A star-product provides an *autonomous* quantization of a manifold M. BFFLS '78: Quantization is a deformation of the composition law of observables of a classical system: $(A, \cdot) \to (A[[t]], \star_t)$, $A = C^\infty(M)$.

Star-product $\star$ ($t = \frac{i}{2}\hbar$) on Poisson manifold M and Hamiltonian H; introduce the star-exponential: $\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{\tau}{i\hbar})^r H^{\star r}$.

Corresponds to the unitary evolution operator, is a singular object i.e. belongs not to the quantized algebra $(A[[t]], \star)$ but to $(A[[t, t^{-1}]], \star)$. Singularity at origin of its trace, Harish Chandra character for UIR of semi-simple Lie groups.

Spectrum and states are given by a spectral (Fourier-Stieltjes in the time τ) decomposition of the star-exponential.

Paradigm: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, Moyal product on $\mathbb{R}^{2\ell}$.

$\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \left(\cos(\frac{\tau}{2})\right)^{-1} \exp\left(\frac{2H}{i\hbar}\tan(\frac{\tau}{2})\right) = \sum_{n=0}^\infty \exp\left(-i(n + \frac{\ell}{2})\tau\right)\pi_n^\ell$.

Here ($\ell = 1$ but similar formulas for $\ell \geq 1$, L_n is Laguerre polynomial of degree n)

$\pi_n^1(q,p) = 2\exp\left(\frac{-2H(q,p)}{\hbar}\right)(-1)^n L_n\left(\frac{4H(q,p)}{\hbar}\right)$.

Complements

The Gaussian function $\pi_0(q,p) = 2\exp\left(\frac{-2H(q,p)}{\hbar}\right)$ describes the vacuum state. As expected the energy levels of H are $E_n = \hbar(n + \frac{\ell}{2})$: $H \star \pi_n = E_n \pi_n$; $\pi_m \star \pi_n = \delta_{mn}\pi_n$; $\sum_n \pi_n = 1$. With normal ordering, $E_n = n\hbar$: $E_0 \to \infty$ for $\ell \to \infty$ in Moyal ordering but $E_0 \equiv 0$ in normal ordering, preferred in Field Theory.

• Other standard examples of QM can be quantized in an autonomous manner by choosing adapted star-products: angular momentum with spectrum $n(n + (\ell - 2))\hbar^2$ for the Casimir element of $\mathfrak{so}(\ell)$; hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

• Feynman Path Integral (PI) is, for Moyal, Fourier transform in p of star-exponential; equal to it (up to multiplicative factor) for normal ordering) [Dito'90]. Cattaneo-Felder [2k]: Kontsevich star product as PI.

Overview

The deformation quantization of a given classical field theory consists in giving a proper definition for a star-product on the infinite-dimensional manifold of initial data for the classical field equation and constructing with it, as rigorously as possible, whatever physical expressions are needed.

As in other approaches to field theory, here also one faces serious divergence difficulties as soon as one is considering interacting fields theory, and even at the free field level if one wants a mathematically rigorous theory.

But the philosophy in dealing with the divergences is significantly different and one is in position to take advantage of the cohomological features of deformation theory to perform what can be called *cohomological renormalization*.

In the same way as we quantize by deforming the (commutative) product of observables to an $\hbar$-dependent star product, keeping the classical observables unchanged, the idea is to renormalize by deforming the normal star-product to another, coupling constant dependent, quantization.

Poisson structure and field equations

Poisson structures are known on infinite-dimensional manifolds since a long time; there is an extensive literature on this subject. A typical structure, for our purpose, is a weak symplectic structure such as that defined in 1974 by Segal and by Kostant on the space of solutions of a classical field equation like $\Box\Phi = F(\Phi)$, $\Box$ = d'Alembertian. Now for scalar-valued functionals Ψ over such a space, i.e., over the phase space of initial conditions $\varphi(x) = \Phi(x, 0)$ and $\pi(x) = \frac{\partial}{\partial t}\Phi(x, 0)$, a Poisson bracket can be defined by

$$P(\Psi_1, \Psi_2) = \int \left(\frac{\delta\Psi_1}{\delta\varphi}\frac{\delta\Psi_2}{\delta\pi} - \frac{\delta\Psi_1}{\delta\pi}\frac{\delta\Psi_2}{\delta\varphi} \right) dx \qquad (1)$$

δ being the functional derivative. But while one can give a precise mathematical meaning to (1) by specifying an appropriate algebra of functionals, the formal extension to powers of P, needed to define the Moyal bracket, is highly divergent, already for P^2.

This is no surprise to physicists who know that the correct approach to field theory starts with normal ordering, and that there are infinitely many inequivalent representations of the canonical commutation relations, even if in recent physical literature some are working formally with Moyal product.

The idea of cohomological renormalization in deformation quantization

Starting with some star-product $\star$ (e.g. similar to the normal star-product on a manifold of initial data), we would like to interpret various divergences appearing in the theory in terms of coboundaries (or cocycles) for the relevant Hochschild cohomology. If we suspect that a term in a cochain of the product $\star$ is responsible for the appearance of divergences, applying an iterative procedure of equivalence, we can try to eliminate it, or at least get a lesser divergence, by subtracting at the relevant order a divergent coboundary; we would then get a better theory with a new star-product, "equivalent" to the original one.
Furthermore, since in this case we expect to have at each order an infinity of non equivalent star-products, we can try to subtract a cocycle and then pass to a nonequivalent star-product whose lower order cochains are identical to those of the original one. We would then make an analysis of the divergences up to order $\hbar^r$, identify a divergent cocycle, remove it, and continue the procedure (at the same or hopefully a higher order).
Along the way one should preserve the usual properties of a quantum field theory (Poincaré covariance, locality, etc.) and the construction of adapted star-products should be done accordingly. The complete implementation of this program should lead to a cohomological approach to renormalization theory.

It seems (e.g. looking at the formulas in Connes 2005 lectures at Collège de France)

that the Connes–Kreimer rigorous renormalization procedure could fit in this pattern.

Normal star-product and quantized fields

Let Φ be a (classical) free massive scalar field with initial data (φ, π) in the Schwartz space $\mathcal{S}$. Replace them by their Fourier modes $(\bar{a}, a)$, also in $\mathcal{S}$ seen as a real vector space. After quantization $(\bar{a}, a)$ become the usual creation and annihilation operators. The normal star-product $\star_N$ can be written $(F \star_N G)(\bar{a}, a) = \int_{\mathcal{S}' \oplus \mathcal{S}'} d\mu(\bar{\xi}, \xi) F(\bar{a}, a + \xi) G(\bar{a} + \bar{\xi}, a)$ where μ is the Gaussian measure on $\mathcal{S}' \oplus \mathcal{S}'$ and F, G are holomorphic functions with semi-regular kernels.

Creation and annihilation operators being operator-valued distributions, we take $(\bar{a}, a) \in \mathcal{S}' \oplus \mathcal{S}'$ (the distribution aspect is present in the definition of the cochains of the star-product). Fermionic fields can also be cast in that framework.
For the above normal product one can formally consider interacting fields. The star-exponential of the Hamiltonian turns out to be, up to a multiplicative well-defined function, equal to Feynman's path integral. For free fields, we have a mathematically meaningful equality between the star-exponential and the path integrals as both of them are defined by a Gaussian measure, hence well-defined. In the interacting fields case, giving a rigorous meaning to either of them would give a meaning to the other.

Work in that direction (free scalar fields, Klein–Gordon equation etc.) is done by Dito since the 90's, including an example of cancellation of some infinities in $\lambda\varphi_2^4$-theory via a λ-dependent star-product formally equivalent to normal.

A toy model of cohomological renormalization

Take a $\lambda\varphi_2^4$ interacting Hamiltonian $H[\varphi,\pi] = H_0[\varphi,\pi] + \lambda V[\varphi]$ with $H_0 = \frac{1}{2}\int_{\mathbb{R}}(\pi^2(x) + |\nabla\varphi(x)|^2 + m^2\varphi^2(x))dx$, $V[\varphi] = \int_{\mathbb{R}}\varphi^4(x)dx$ or its equivalent form with $(\bar{a}, a)$. Singular terms appear in the $\star_N$-powers of H, not surprising since (Glimm–Jaffe) one needs an infinite renormalization of H in order to give a meaning to the operator expression of H.

We would like to leave H unchanged and define a new $\star$-product such that no singular terms occur in the $\star$-powers of H and, ultimately, that the $\star$-exponential of H is well defined. Dito (LMP 1993) constructed a $\star$-product equivalent to normal which gives a meaning to $H \star F(H)$, F an arbitrary polynomial function of H. The equivalence operator T, $T(F \star G) = TF \star_N TG$, is given by an expression $T(F) = \exp \hbar\lambda \int dkf(k)[\frac{\delta^2 F}{\delta a(k)\delta a(k)} - \frac{\delta^2 F}{\delta \bar{a}(k)\delta \bar{a}(k)}$ where f is a function adjusted in such a way to generate a counterterm for $C_4(H,H)$, the only singular term in $H \star_N H$ leading to an infinite constant. It however does not give divergenceless expressions for the $\star$-powers of H with $n \geq 3$ because these are not polynomials in H.