RG 2008 (D.V. Shirkov Fest)

Advantages of analytic approach in description of hadronic tau decays

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– Motivations

– Overview of PT and APT descriptions

– New results

– Conclusions

D.V. Shirkov & I.L. Solovtsov **Phys. Rev. Lett.'97**
Theor. Math. Phys.'07

"Ten years of the Analytic Perturbation Theory in QCD"

The analytic approach modifies the PT expansions in such way, that new approximations reflect basic principles of the theory, such as renormalization invariance, spectrality, and causality. The APT is the next logical step in the modification of the PT by bringing into consideration the Q^2-analyticity. It is important from a point of view of a correct theoretical description, and also from the standpoint of extracting the parameter QCD from the experimental data (additional terms are 'invisible' in the PT expansions, but important numerically).

Important features of APT:

- infrared fixed point $a_{\rm an}(0) = \tilde{a}_{\rm an}(0) = 1/\beta_0$
- correct analytic properties and a self-consistent definition of analytic continuation from spacelike to timelike region
- better convergence properties and stability with respect to higher-loop corrections
- RS dependence of the results obtained is reduced drastically (three-loop level practically RS independent for the whole energy interval)
- APT $\Rightarrow$ PT at large Q^2

Basic relations

A main object in a description of the hadronic decay of the τ lepton and of many other physical processes is the correlator $\Pi(q^2)$ or the corresponding Adler function $D(Q^2)$

$$\Pi_{\mu\nu}(q^2) = i \int d^4x \, e^{iqx} \, \langle 0 \left| T V_\mu(x) V_\nu(0)^+ \right| 0 \rangle$$

$$\propto (q_\mu q_\nu - g_{\mu\nu} \, q^2) \, \Pi(q^2), \qquad V^\mu_{ij} = \bar{\psi}_j \gamma^\mu \psi_i$$

$$D(Q^2) \equiv -Q^2 \, \frac{d \, \Pi(-Q^2)}{d \, Q^2}, \qquad Q^2 = -q^2 > 0$$

[in Euclidian (spacelike) region]

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2} \; R(s) \, ,$$

where $R(s) = \mathrm{Im}\Pi(s)/\pi$. The D-function is an analytic function in the complex Q^2-plane with a cut along the negative real axis.

QCD contributions $d(Q^2)$ and $r(Q^2)$: $D \propto 1 + d\,, \quad R \propto 1 + r$

$$d(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s+Q^2)^2}\, r(s)\,, \quad r(s) = -\frac{1}{2\pi i}\int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z}\, d(\text{-}z)$$

(contour encircles the cut of $d(-z)$ on the positive real z-azis)

PT: PT expansion (after application of RG) $(a \equiv \alpha_s/\pi)$

$$d(Q^2, RS) = a(Q^2, RS)\left[1 + d_1(RS)\, a(Q^2, RS) + d_2(RS)\, a^2(Q^2, RS) + ...\right]$$

For $n_f = 3:$ $d_1^{\overline{\rm MS}} = 1.6398\,, \quad d_2^{\overline{\rm MS}} = 6.3710\,, \quad d_3^{\overline{\rm MS}} = 49.076\,.$

Baikov-Chetyrkin-Kühn – 2008

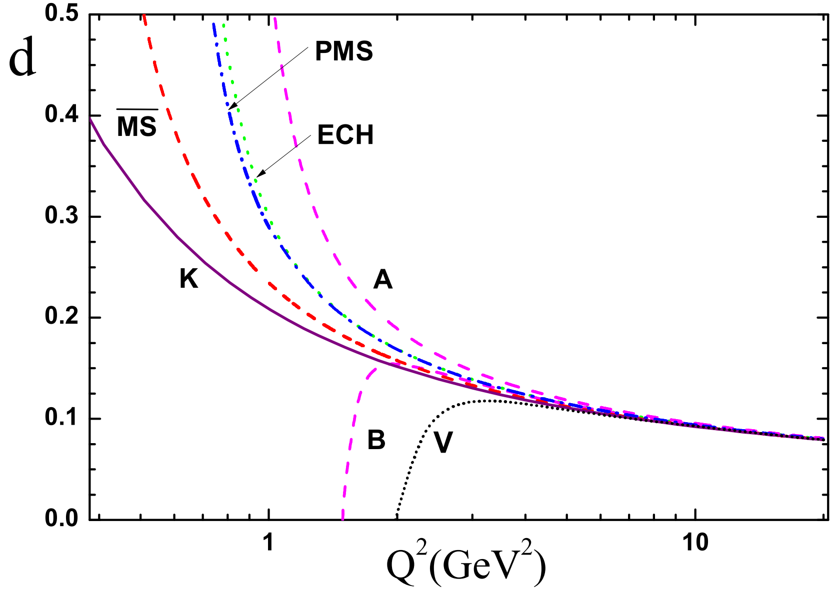
The standard PT parametrization as an expansion in inverse powers of $L \equiv \ln(Q^2/\Lambda^2)$ (PDG)

$$a(Q^2) = \frac{4}{\beta_0 L}\left\{1 - \frac{\beta_1}{\beta_0^2}\frac{\ln L}{L} + \frac{1}{L^2}\left[\frac{\beta_1^2}{\beta_0^4}\left(\ln^2 L - \ln\, L - 1\right) + \frac{\beta_2}{\beta_0^3}\right] + \cdots\right\}$$

(!) The correct analytic properties of the Adler D function are no longer valid due to unphysical singularities of the PT running coupling.

Renormalization scheme dependence for PT

The dependence of result on choice of the RS can be a significant source of theoretical uncertainty. In QCD that uncertainty is the greater, the smaller typical energy scale of a process.



Cancelation index criterium: $C \leq C_{\max}$

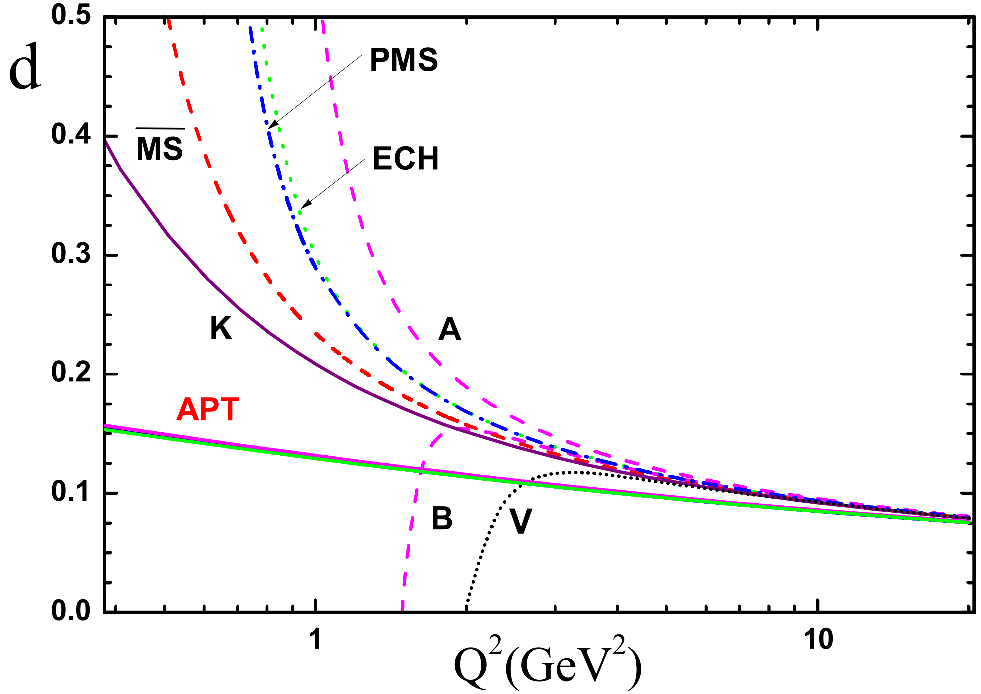
PMS – principle of minimal sensitivity $\quad C_{\rm A} \simeq C_{\rm B} \simeq C_{\rm PMS} \simeq 2$

ECH – method of effective charge

$\overline{\rm MS}$ – modified minimal subtraction scheme

K – gives a fixed point for the three-loop running coupling ($C_{\rm K} = 5.3$)

RS dependence in APT



The APT gives very stable results over a wide range of renormalization schemes.

$$\boxed{d_{\text{an}}(Q^2) = a_{\text{an}}(Q^2) + d_1 \delta_{\text{an}}^{(1)}(Q^2) + d_2\, \delta_{\text{an}}^{(2)}(Q^2)}$$

The Euclidean functions $\delta_{\rm an}^{(n)}(Q^2)$ satisfy the Källen–Lehmann representation

$$\delta_{\rm an}^{(n)}(Q^2) = \frac{1}{\pi}\int_0^\infty d\sigma\, \frac{\rho_n(\sigma)}{\sigma + Q^2},$$

with the spectral function being defined as the discontinuity of the respective power of the invariant charge across the physical cut:

$$\varrho_n(\sigma) = \mathrm{Im}\left[\, a_{\rm pt}^{n+1}(-\sigma - i\epsilon)\,\right]$$

Running coupling in the Euclidian region

$$a_{\rm an}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma + Q^2}\,\varrho_0(\sigma)$$

Leading order

$$\varrho_0(\sigma) = \frac{1}{\beta_0} \frac{\pi}{\ln^2(\sigma/\Lambda^2) + \pi^2}$$

$$a_{\rm an}(Q^2) = \frac{1}{\beta_0} \left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right] \quad \text{(spacelike)}$$

$$\tilde{a}_{\rm an}(s) = \frac{1}{\beta_0} \left[\frac{1}{2} - \frac{1}{\pi} \mathrm{arctg} \frac{\ln(s/\Lambda^2)}{\pi} \right] \quad \text{(timelike)}$$

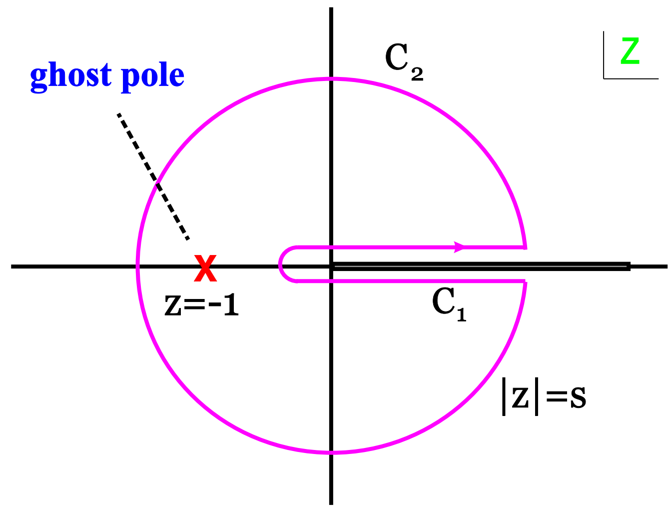
$$a_{\rm an}(Q^2) = Q^2 \int_0^\infty \frac{d\,s}{(s+Q^2)^2} \tilde{a}_{\rm an}(s),$$

$$\tilde{a}_{\rm an}(s) = -\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\,\epsilon}^{s+i\epsilon} \frac{d\,z}{z} a_{\rm an}(-z).$$

$$\tilde{a}_{\rm an}(s) = -\frac{1}{2\pi \mathrm{i}} \int\limits_{C_1} \frac{dz}{z} a_{\rm an}(z) =$$

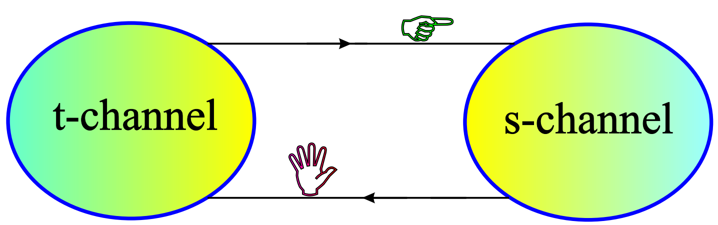
$$= \frac{1}{2\pi \mathrm{i}} \int\limits_{C_2} \frac{dz}{z}\, a_{\rm an}(z) =$$

$$= \frac{1}{\pi\beta_0}\left(\frac{\pi}{2} - \mathrm{arctg}\, \frac{\ln s/\Lambda^2}{\pi}\right)$$



The APT leads to a self-consistent definition of analytic continuation

$$a_{\rm pt}(z) \neq -z\int_0^\infty \frac{d\,s}{(s-z)^2}\tilde{a}_{\rm pt}(s)$$



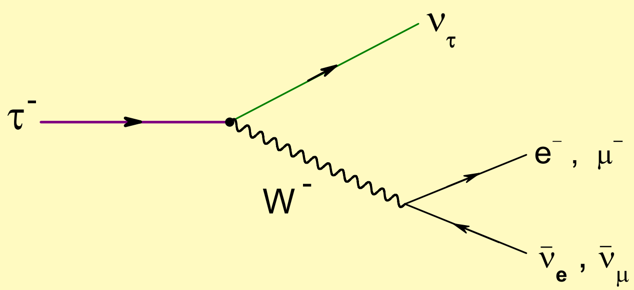
At present, there is rich high-precision experimental material obtained from hadronic decays of the τ lepton by ALEPH and OPAL Collaborations.

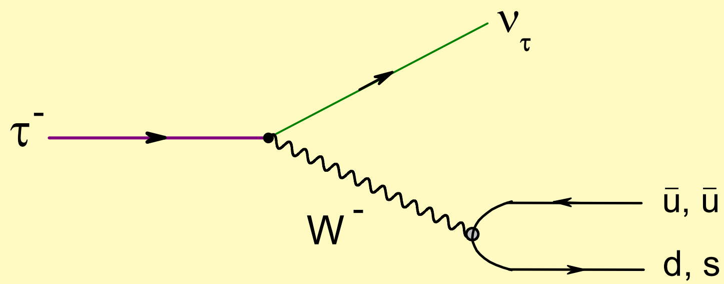
The mass of the τ lepton, $M_\tau = 1.777\,\mathrm{GeV}$, is large enough in order to produce decays with a hadronic mode. At the same time, in the context of QCD, the mass is sufficiently small to allow one to investigate perturbative and non-perturbative QCD effects.

The theoretical analysis of the hadronic decays of a heavy lepton was performed Y.S. Tsai (1971) before the experimental discovery of the τ lepton in 1975.

Hadronic τ decays provide a clean laboratory for the precise study of low-energy QCD. It initializes many theoretical developments which concentrate primarily on perturbative expansions. These approaches mainly distinguish themselves in how they deal with the fact that the perturbative series is truncated.

QCD contribution to R_τ-ratio





$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}\ \nu_\tau)}{\Gamma(\tau^- \to \ell\ \bar{\nu}_\ell\ \nu_\tau)} \cong N_C(|V_{ud}|^2 + |V_{us}|^2) \simeq 3$$

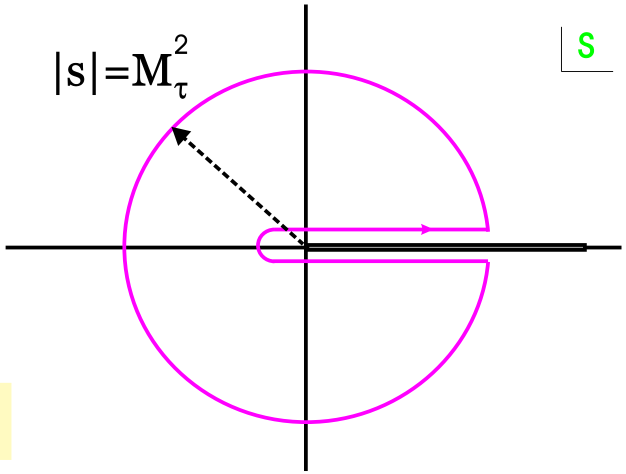
$$R_\tau^{exp} = 3.642 \pm 0.012$$

The initial theoretical expression

$$R_\tau = \frac{2}{\pi} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\Pi(s)$$

(!) cannot be directly calculated in PT due to unphysical singularities of the PT running coupling lying in the range of integration.

Possible solution (?) The initial integral is rewritten by using the Cauchy theorem in the form of a contour integral in the complex plane with the contour running around a circle with radius M_τ^2



$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right) R(s)$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) D(\text{-}z) \qquad (\,*\,)$$

$$= 3(|V_{ud}|^2+|V_{us}|^2)\, S_{\mathrm{EW}}(1+\delta_\tau) = R_\tau^{(0)}(1+\delta_\tau)$$

$|V_{ud}|$ and $|V_{us}|$ are CKM matrix elements, S_{EW} – electroweak factor

δ_τ – QCD contribution to R_τ-ratio

(!) If a calculation method maintains the correct analytic properties of the D-function, then both representations are equivalent.

PT

The PT description is based on the contour representation ($*$) and

can be developed in the following two ways.

NP effects being smaller that the PT uncertainties.

- In the Braaten's method (Phys. Rev. Lett.'88) [fixed-order PT (FOPT)] δ_τ is represented in the form of truncated power series with the expansion $a_\tau = \alpha_s(M_\tau^2)/\pi$:

known up α_s^4!
(Baikov-Chetyrkin-Kühn – 2008)

$$\delta_\tau^{\rm FOPT} = a_\tau + K_1(a_\tau)^2 + K_2(a_\tau)^3 + K_3(a_\tau)^4$$

$K_1 = 5.2023$, $K_2 = 26.366$, $K_3 = 127.079$ (in the $\overline{\rm MS}$, $n_f = 3$).

- Contour-improved fixed-order PT (CIPT)

[Pivovarov (Z. Phys.'92), Le Diberder and Pich (Phys. Lett.'92)]

$$\delta_\tau^{\rm CIPT} = A^{(1)}\left(M_\tau^2\right) + d_1\, A^{(2)}\left(M_\tau^2\right) + d_2\, A^{(3)}\left(M_\tau^2\right)$$

$$A^{(n)}\left(M_\tau^2\right) = \frac{1}{2\pi i}\oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) \times\, a_{\rm pt}^{\,n}(-z)$$

- **APT** Milton-Solovtsov-S (Phys. Lett.'97)

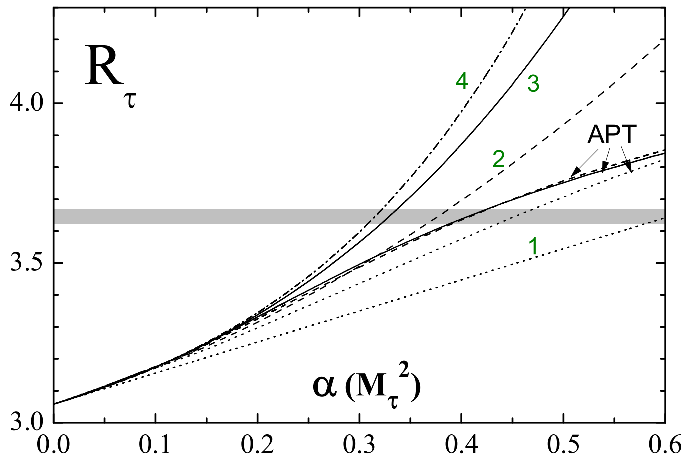
The APT description can be equivalently done either on the basis of the initial expression or on the contour representation

$$\delta_{\rm an} = \frac{1}{\pi}\int_{M_\tau^2}^{\infty}\frac{d\sigma}{\sigma}\rho(\sigma) + \frac{1}{\pi}\int_0^{M_\tau^2}\frac{d\sigma}{\sigma}\left[2\frac{\sigma}{M_\tau^2} - 2\left(\frac{\sigma}{M_\tau^2}\right)^3 + \left(\frac{\sigma}{M_\tau^2}\right)^4\right]\rho(\sigma)$$

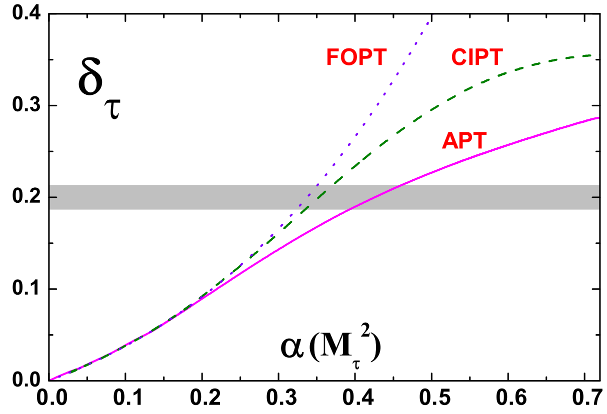
$$= \delta_{\rm an}^{(0)} + d_1\,\delta_{\rm an}^{(1)} + d_2\,\delta_{\rm an}^{(2)}\,, \qquad \varrho = \varrho_0 + d_1\,\varrho_1 + d_2\,\varrho_2$$

The additional terms, which are 'invisible' in the PT expansions turns out to be very important at a low energy scale.

Stability with respect to higher-loop corrections



Numbers – the order of FOPT.



Method			1	2	3
FOPT*)	$1 + \delta_{\rm pt}^{\rm FO}$	$=$	$1 + 0.104$	$+ 0.056$	$+$ 0.030
CIPT	$1 + \delta_{\rm pt}^{\rm CI}$	$=$	$1 + 0.148$	$+ 0.030$	$+$ 0.012
APT	$1 + \delta_{\rm an}$	$=$	$1 + 0.167$	$+ 0.021$	$+$ 0.002

*) In 4-loop : 1+0.103+0.055+0.285+ 0.014

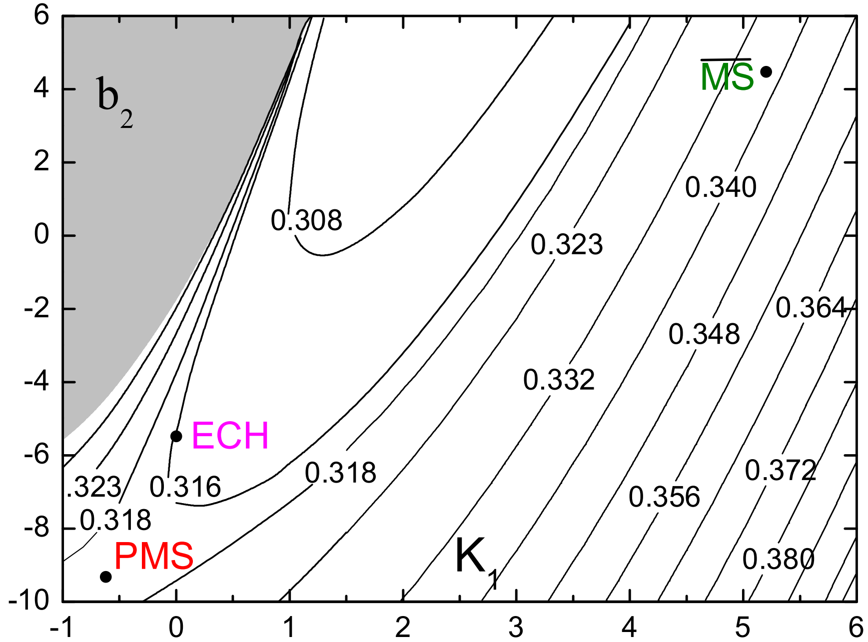
Renormalization scheme dependence

$\delta_\tau^{exp} = 0.200 \pm 0.004 \quad \Rightarrow \quad \alpha_s^{\overline{\rm MS}}(M_\tau^2) = 0.337 \pm$ 0.004

2% – exp, 6% – FOPT, 0.2% – APT, $\Delta\alpha_{\rm RS}^{\rm FOPT} =$ 0.021

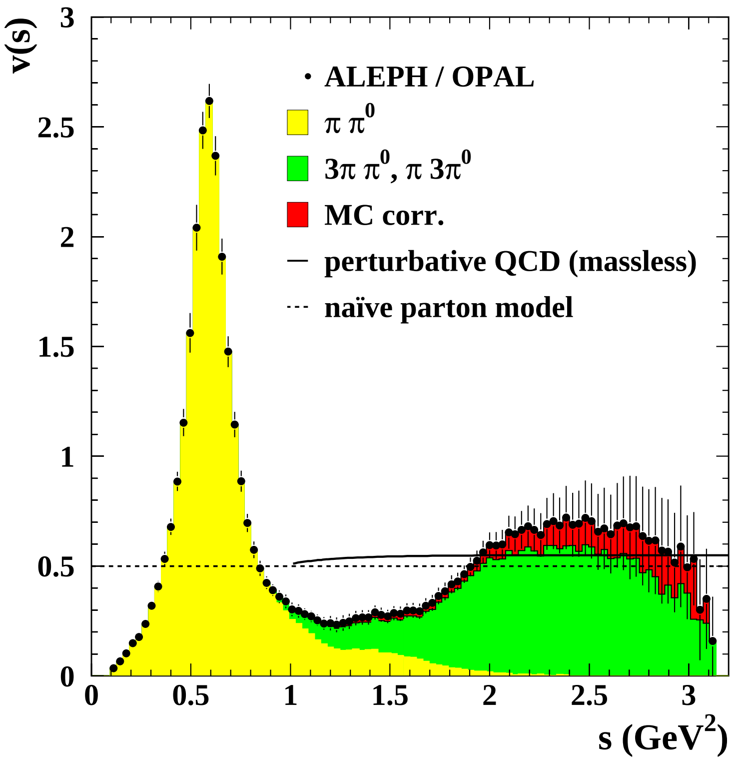
$\alpha_{\rm CIPT}^{\overline{\rm MS}} - \alpha_{\rm FOPT}^{\overline{\rm MS}} = 0.020$ 4-loop (BChK'08)

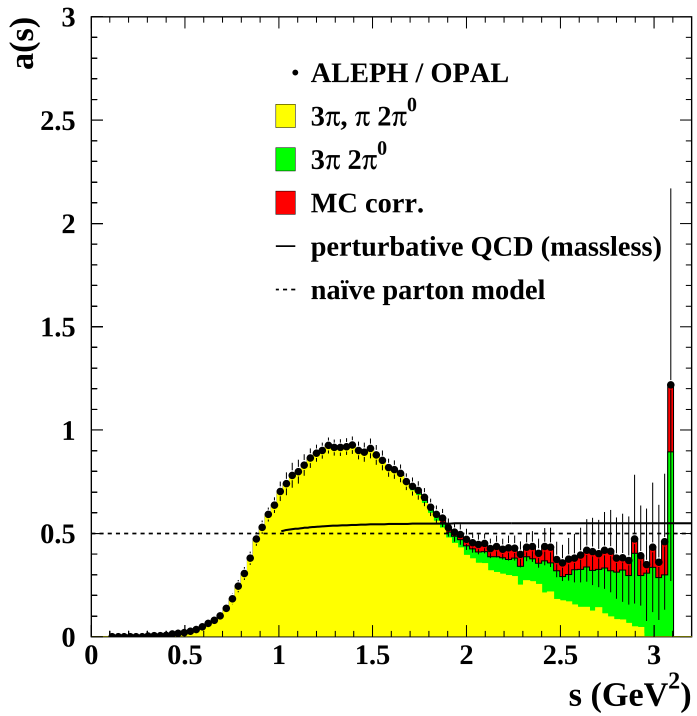
$$\delta_\tau^{\rm FOPT} = a_\tau + K_1 a_\tau{}^2 + K_2 a_\tau{}^3 , \quad b_2 = \beta_2/\beta_0$$



The APT gives the remarkable stability. (MSSYa'00)

Experimental spectral functions: V and A channels





$$v(s)/a(s) = \frac{1}{2} R_{V/A}\,, \quad (\, R \propto 1 + r\,)$$

The agreement between ALEPH/OPAL experiments is satisfying

Experimentally R_τ can be decomposed into the three contributions

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$R_{\tau,V}$ and $R_{\tau,A}$ are contributions coming from the non-strange hadronic decays associated with vector (V) and axial-vector (A) quark currents respectively, and $R_{\tau,S}$ contains strange decays (S).

$$\boxed{R_{\tau,\, V/A}^{\text{exp/theo}} = 3|V_{ud}|^2 S_{\text{EW}} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) R_{V/A}^{\text{exp/theo}}}$$

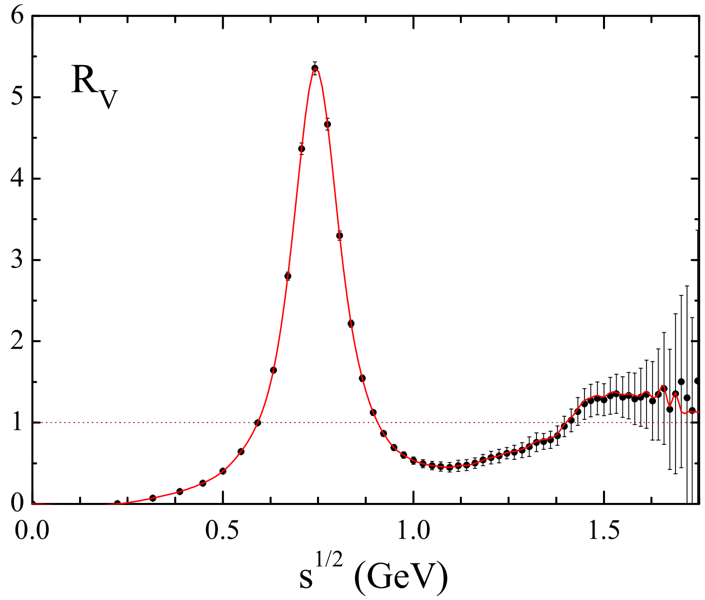
$|V_{ud}| = 0.9746 \pm 0.0006$ denotes the CKM weak mixing matrix element,

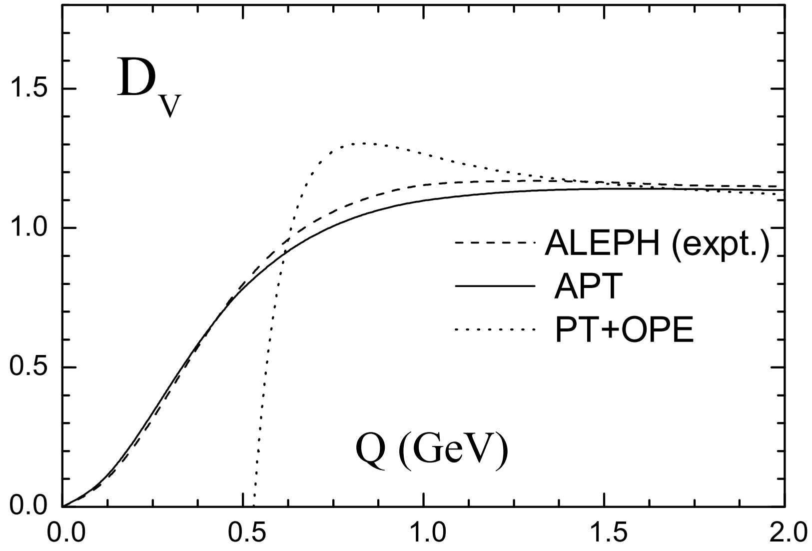
$S_{\text{EW}} = 1.0198 \pm 0.0006$ accounts for electroweak radiative corrections

Within the PT with massless quarks $R_{\tau,V}$ and $R_{\tau,A}$ coincide with each other

$$R_{\tau,V}^{\text{PT}} = R_{\tau,A}^{\text{PT}} = \frac{3}{2}|V_{ud}|^2(1 + \delta_\tau) \qquad \boxed{R_{\tau,V}^{\text{exp}} \neq R_{\tau,A}^{\text{exp}}}$$

Vector channel in τ decay





$$R^{\exp}_{\tau,V} = 1.787 \pm 0.013$$

$$R^{\rm APT}_{\tau,V} = 1.79 = R^{\exp\,,\rm centr}_{\tau,V}$$

$$D_{\tau,V/A}(Q^2) = Q^2 \int\limits_0^\infty ds\, \frac{R_{\,V/A}(s)}{(s+Q^2)^2}$$

[Milton-Solovtsov-S – 2006]

Axial-vector channel

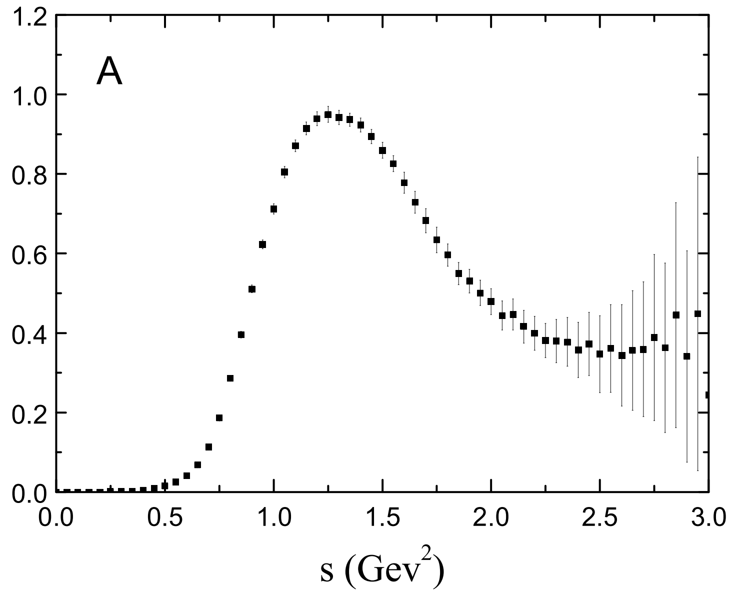
$$R^{\rm exp}_{\tau,A} = 1.695 \pm 0.013$$

$$R_{\tau,A} \Rightarrow \quad R^{(1)}_{\tau,A} + R^{(0)}_{\tau,A}$$

$$R^{\rm exp}_{\tau,\,\pi} = \; 0.612 \pm 004$$

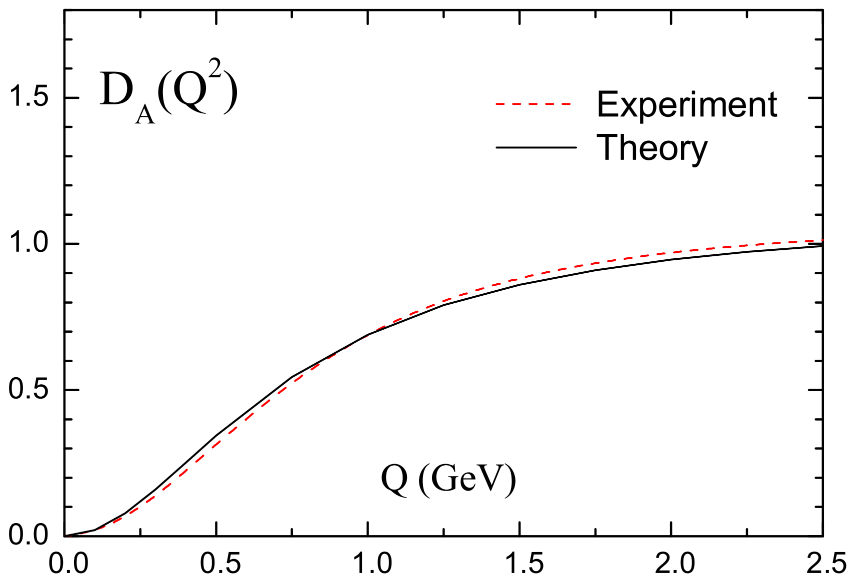
$$R^{\rm exp}_{\tau,A1} = 1.083 \pm 0.014$$

$$R^{\rm APT}_{\tau,A} = 1.087$$



The contribution to the imaginary part of the axial-vector correlator, $\mathrm{Im}\Pi^{(0)}$, is taken from the pion pole

$$R_{\tau,\,\pi} = 3|V_{ud}|^2 S_{\rm EW} \frac{8\pi^2 f_\pi^2}{M_\tau^2}\left(1 - \frac{m_\pi}{M_\tau^2}\right)^2 \to 0.612 \pm 004$$



Model for the function $R_A(s)$ that usually used in the QCD sum rules

$$R_A^{\mathrm{had}}(s) = \frac{2\pi}{g_A^2}\, m_A^2\, \delta(s - m_A^2) + \left(1 + \frac{\alpha_s^{(0)}}{\pi}\right) \theta(s - s_0)$$

$$D_A^{\mathrm{had}}(Q^2) = \frac{2\pi}{g_A^2}\, \frac{Q^2\, m_A^2}{(Q^2 + m_A^2)^2} + \left(1 + \frac{\alpha_s^{(0)}}{\pi}\right) \frac{Q^2}{Q^2 + s_0}$$

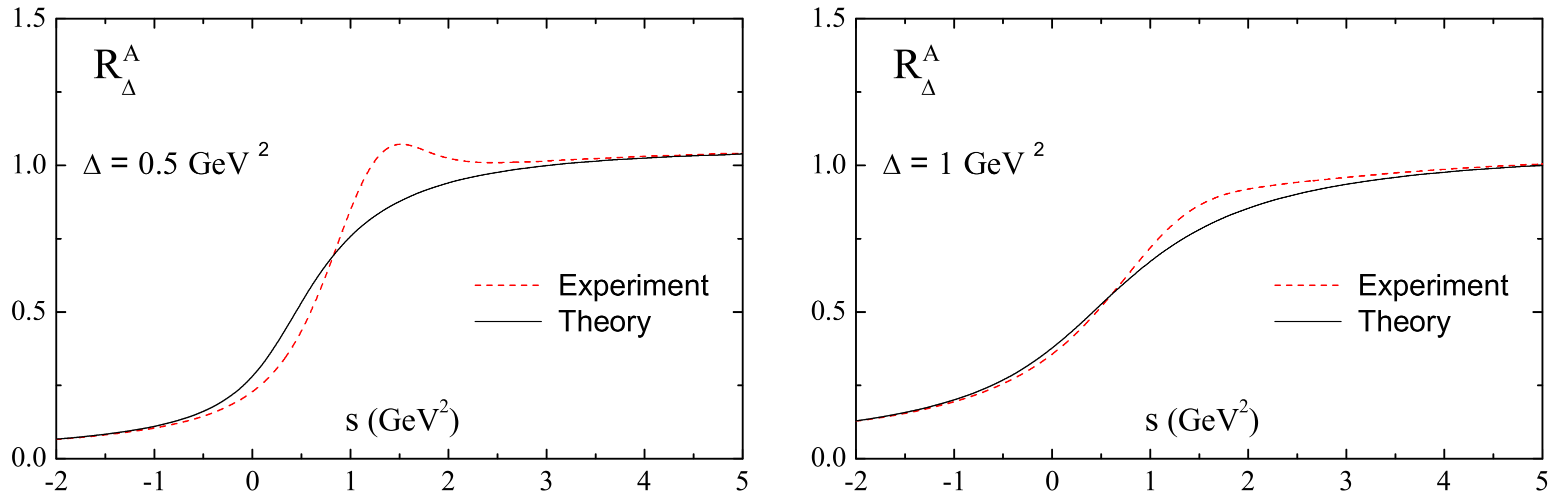
This expression reproduces well the “experimental” curve $D_A^{\mathrm{exp}}(Q^2)$.

Smeared R_Δ-function

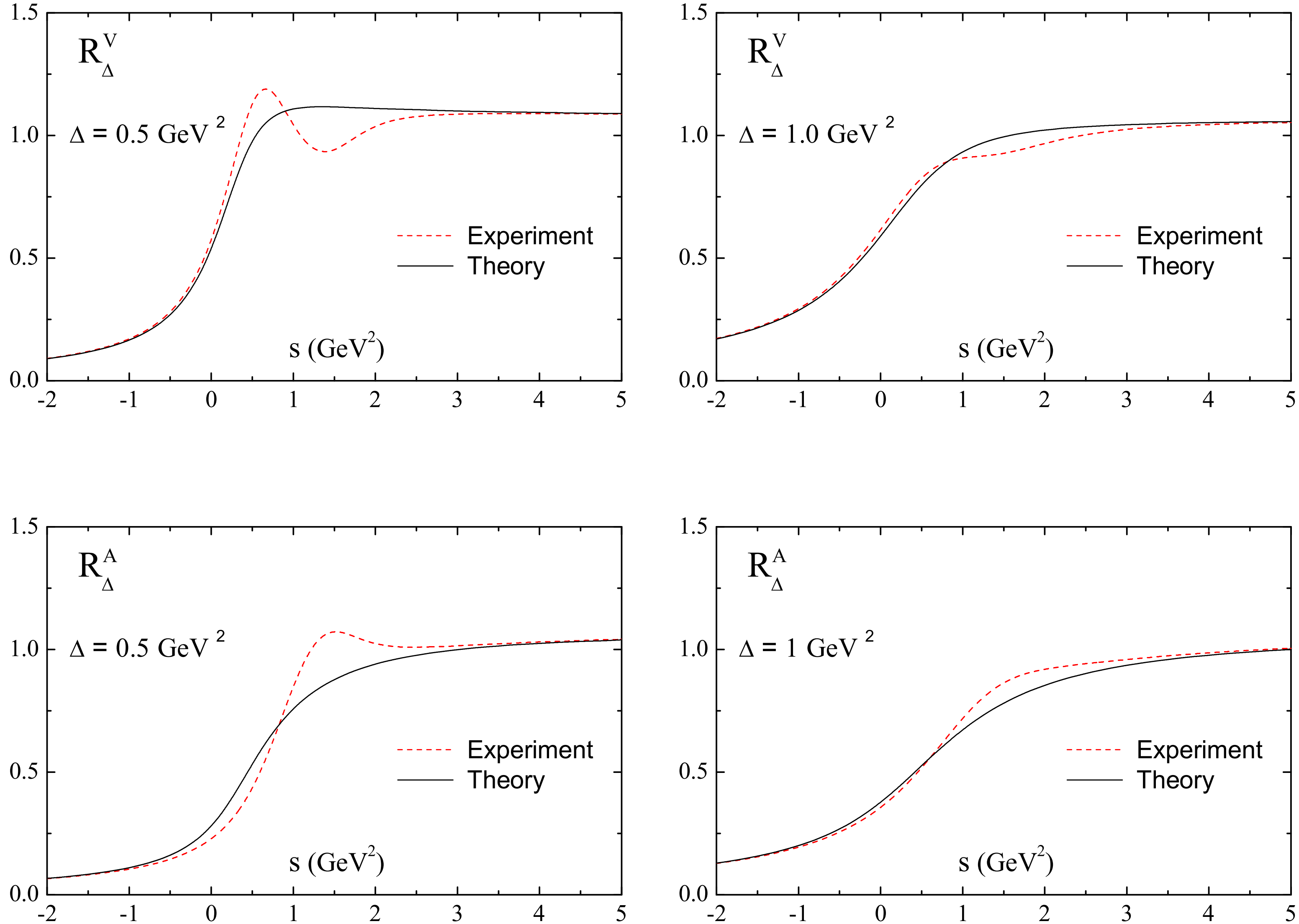
[Poggio, Quinn, Weinberg 1976]

$$R_\Delta(s) = \frac{1}{2\pi\, i}\left[\,\Pi(s+i\Delta) - \Pi(s-i\Delta)\,\right]$$

$$R_\Delta(s) = \frac{\Delta}{\pi}\int_0^\infty ds'\,\frac{R(s')}{(s-s')^2+\Delta^2}$$



'Light' smeared axial-vector smeared function $R^A_\Delta(s)$


R_Δ^V
Δ = 0.5 GeV ^{2}
Experiment
Theory
s (GeV^2)
R_Δ^V
Δ = 1.0 GeV ^{2}
Experiment
Theory
s (GeV^2)
R_Δ^A
Δ = 0.5 GeV ^{2}
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Theory
s (GeV^2)
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Δ = 1 GeV ^{2}
Experiment
Theory
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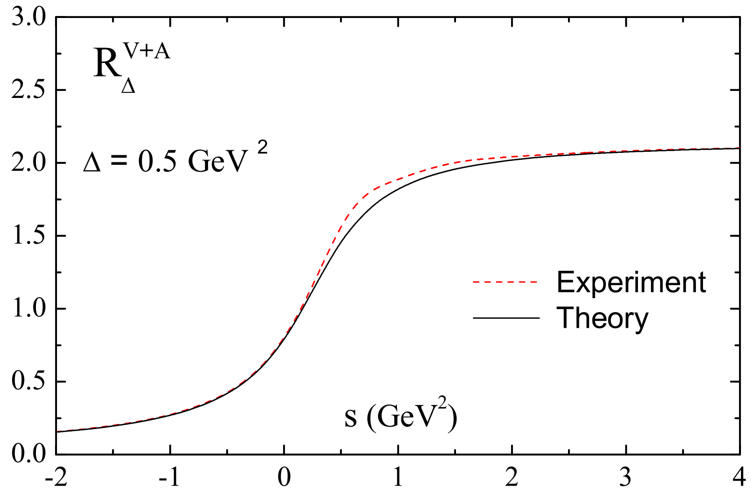
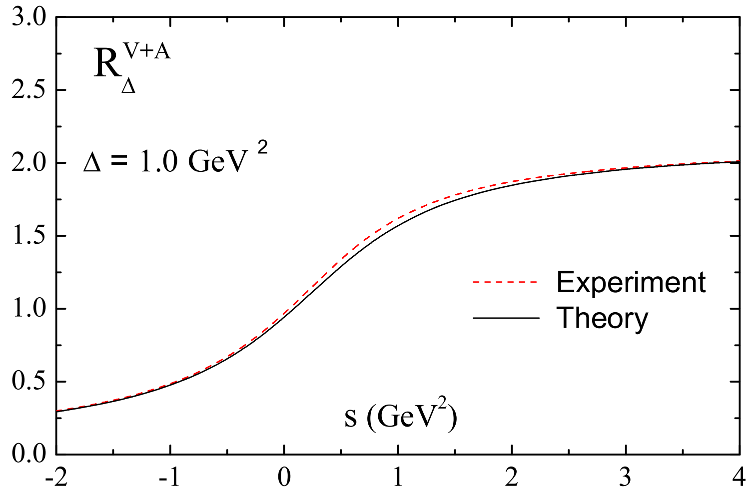


Vector + Axial-vector smeared function $R_\Delta^{\mathrm{V+A}}(s)$ for $\Delta = 0.5\ \mathrm{GeV}^2$.



Vector + Axial-vector smeared function $R_\Delta^{\mathrm{V+A}}(s)$ for $\Delta = 1.0\ \mathrm{GeV}^2$.

The method allows us to describe well the smeared functions.

Let us emphasize that in the spacelike region ($s \lesssim 0$) there is an excellent agreement between data and theory.

Conclusions

The method of analytic perturbation theory, which resolves the problem of ghost-pole type singularities and gives a self-consistent description of both spacelike and timelike regions, was applied to a description of hadronic tau decays.

We presented the arguments in favor of the APT, which has a number of practical advantages. The better convergence properties of APT approximations and stability with respect to higher-loop corrections allows one to reduce the uncertainties of theoretical predictions drastically. Three-loop level gives stable and practically RS independent results for the whole energy interval.

Note also that if the calculation method maintains the correct analytic properties of the correlator two methods are EQUIVALENT as it is in APT.

It was shown that the APT leads to good agreement with the experimental Adler function both in the vector and in the axial-vector channel down to the lowest energy scale.

There is still room for improvements in theoretical methods stimulate further intensive studies along lines associated with both the perturbative description and with nonperturbative effects.

The reliability of extracting information on nonperturbative effects is connected to the indeterminacy in the description of the perturbative component of calculations.

Calculations in the framework of APT are self-consistent and give stable results down to low energy scale.

Thank You for Your attention !