The method of Mellin–Barnes representation in RG calculations

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- Introduction. Evaluating Feynman integrals
- Mellin–Barnes representation. Simple one-loop examples
- General prescriptions. Multiple Mellin–Barnes integrals
- Examples and results
- $N = 4$ SUSY YM: iterative conjecture and evaluating cusp anomalous dimension
- Further developments and open problems
- Summary

V.A. Smirnov, *Evaluating Feynman integrals* (STMP 211, Springer 2004) and *Feynman Integrals Calculus* (Springer 2006)

Introduction

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

$d = 4 - 2\epsilon$

The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0}\,, \qquad k^2 = k_0^2 - \vec{k}^2$$

Methods to evaluate Feynman integrals: analytical, numerical, semianalytical . . .

A straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An advanced strategy:

to derive, without calculation, and then apply integration by parts (IBP) identities [K.G. Chetyrkin & F.V. Tkachov'81] between the family of given Feynman integrals as recurrence relations.

A general integral of the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation→

- constructing a reduction procedure
- evaluating master integrals

Methods to evaluate master integrals:

- Feynman/alpha parameters
- Mellin–Barnes representation [N.I. Ussyukina'75 . . . , A.I. Davydychev'89 . . . , V.A. Smirnov'99, J.B Tausk'99]
- method of differential equations [A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00]

Mellin transformation, Mellin integrals as a tool for Feynman integrals: [M.C. Bergère & Y.-M.P. Lam'74]

Evaluating individual Feynman integrals:

[N.I. Ussyukina'75 . . . , A.I. Davydychev'89 . . .]

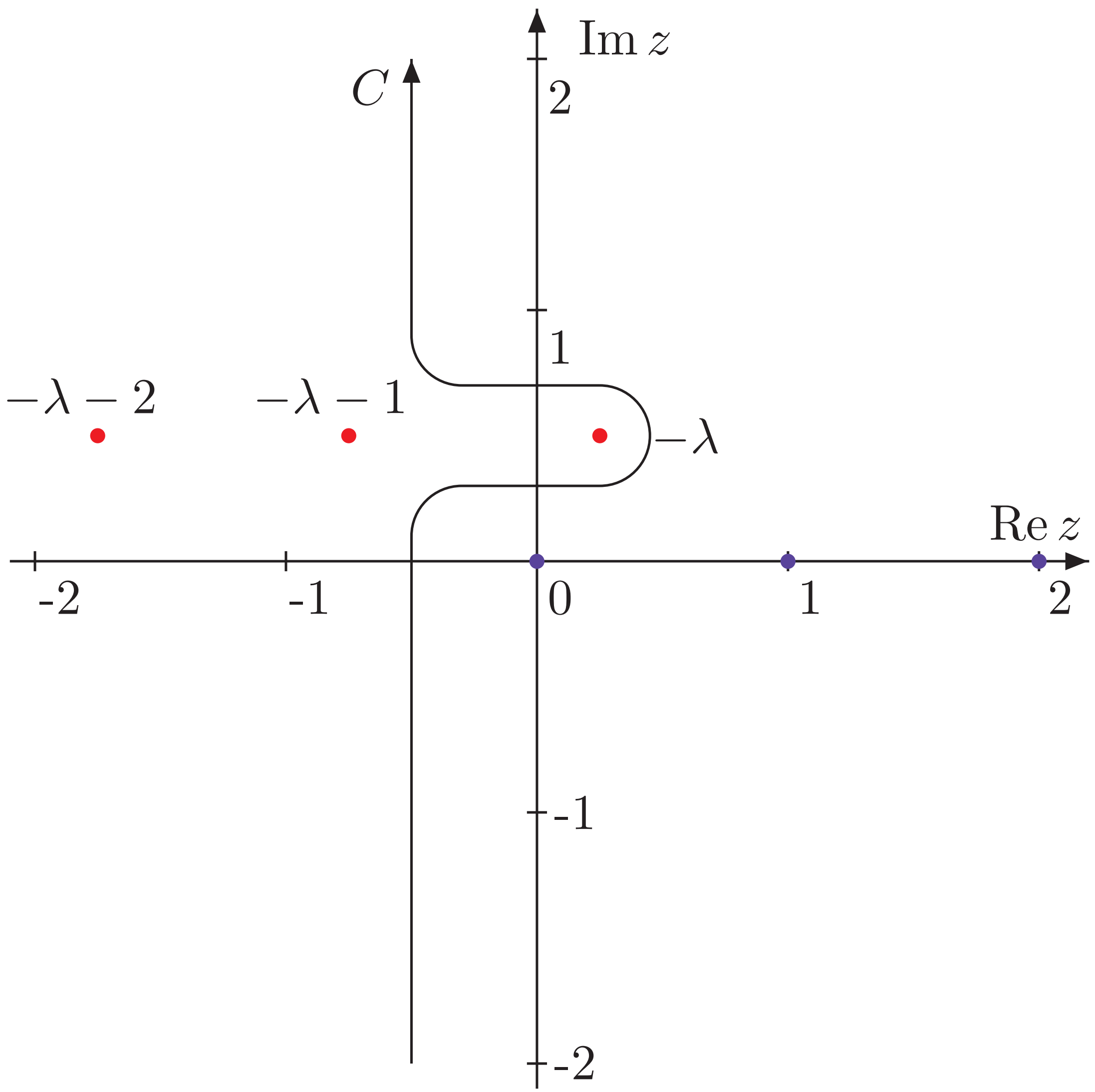
Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ) [V.A. Smirnov'99, J.B. Tausk'99]

Mellin–Barnes representation

The basic formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z) \; .$$

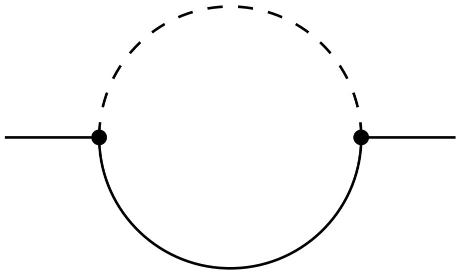
The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right


Im z
C
2
1
$-\lambda - 2$
$-\lambda - 1$
$-\lambda$
Re z
-2
-1
0
1
2
-1
-2

The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z)$$

An example



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1,a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} \,,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

$$\begin{aligned} F_\Gamma(q^2,m^2;a_1,a_2,d) &= \frac{\mathrm{i}\pi^{d/2}(-1)^{a_1+a_2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\ &\times \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1+a_2+\epsilon-2+z) \\ &\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)} \end{aligned}$$

In particular,

$$F_\Gamma(2,1,4) = \frac{\mathrm{i}\pi^2}{q^2}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(1+z)\Gamma(-z)^2}{\Gamma(1-z)}$$

with $-1 < \mathrm{Re} z < 0$

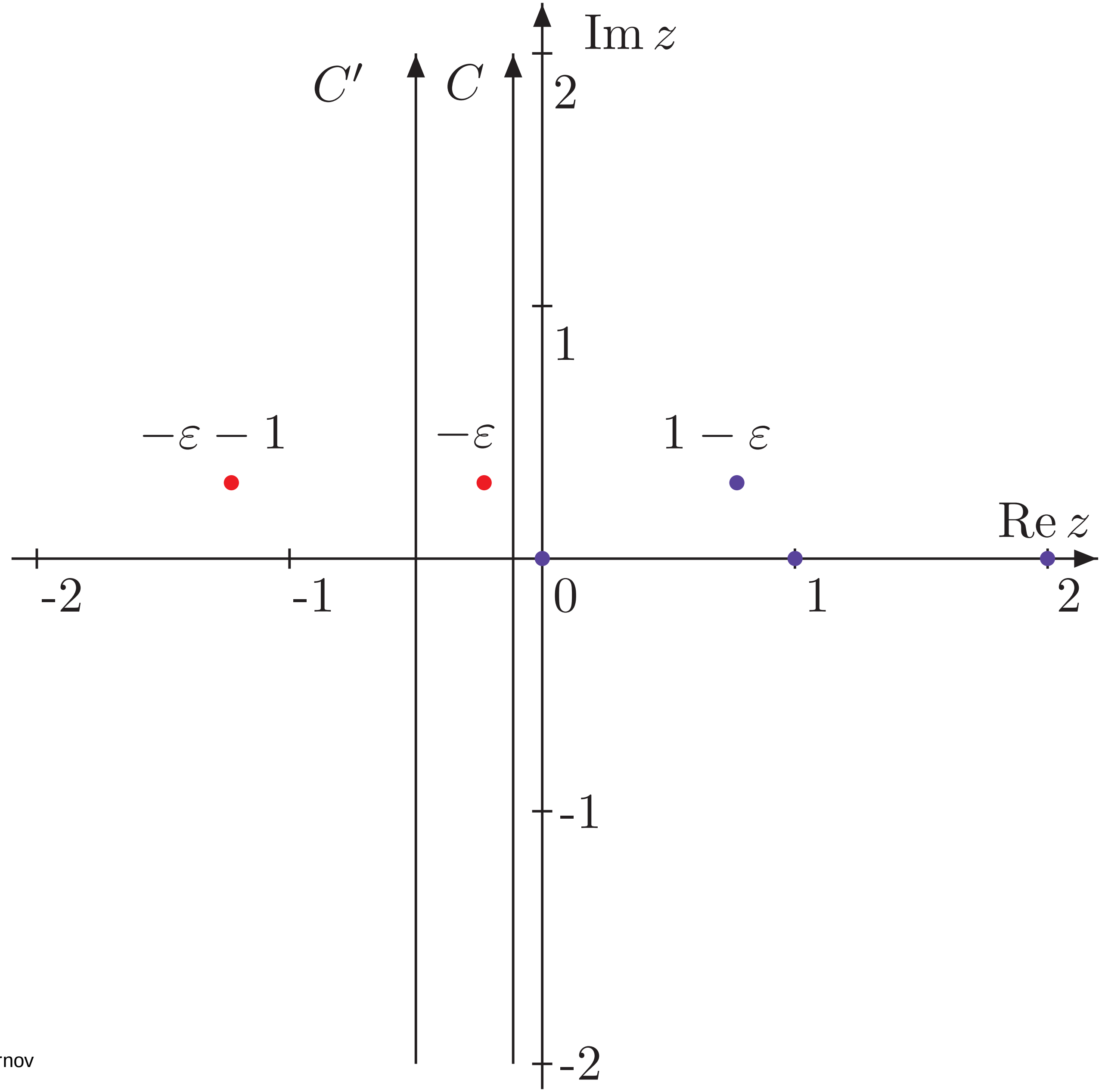
Closing the integration contour to the right and take a series of residues at the points $z = 0, 1, 2, \ldots \rightarrow$

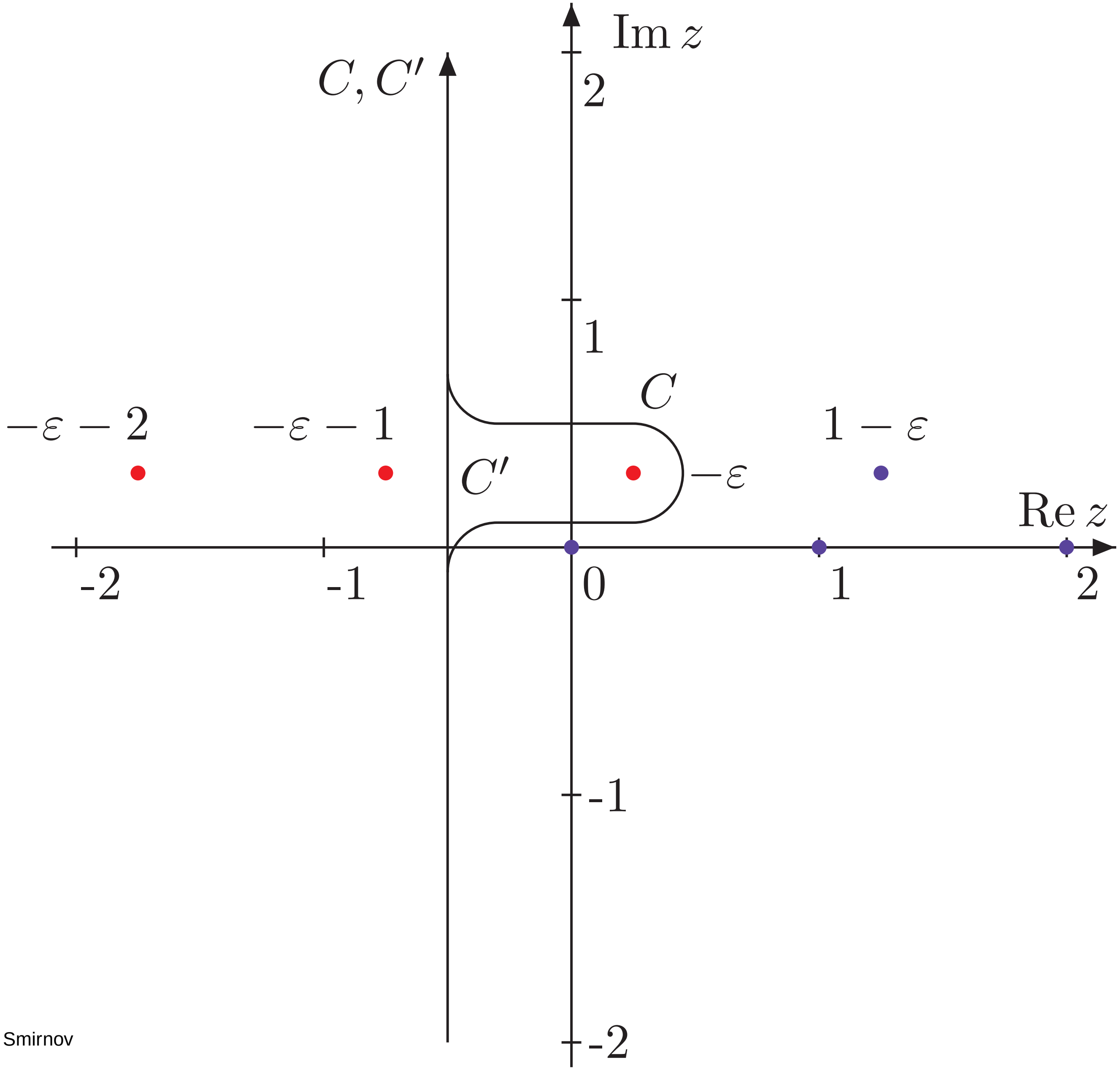
$$F_\Gamma(2,1,4) = \mathrm{i}\pi^2 \frac{\ln\left(1 - q^2/m^2\right)}{q^2}$$

$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}$$

$$\times \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

$\Gamma(\epsilon+z)\Gamma(-z) \rightarrow$ a singularity in ϵ


Im z
C'
C
2
1
$-\varepsilon - 1$
$-\varepsilon$
$1 - \varepsilon$
Re z
-2
-1
0
1
2
-1
-2


Im z
C, C'
2
1
C
$-\varepsilon - 2$
$-\varepsilon - 1$
$1 - \varepsilon$
C'
$-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2

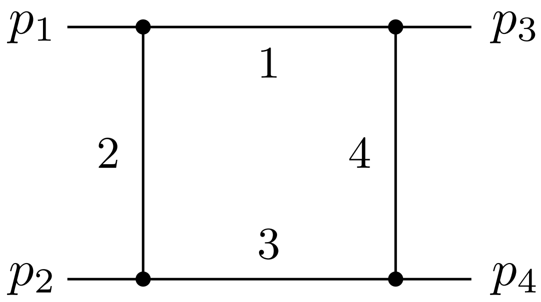
Take a residue at $z = -\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)}$$

and shift the contour:

$$\mathrm{i}\pi^2 \frac{1}{2\pi \mathrm{i}} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(z)\Gamma(-z)}{1-z}$$

An example. The massless on-shell box diagram, i.e. with $p_i^2 = 0, \; i = 1, 2, 3, 4$



$$F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)$$
$$= \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k-p_3)^2]^{a_4}} \, ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

$$
\begin{aligned}
& F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) \\
= & (-1)^a \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)} \\
\times & \int_0^1\int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2-t(1-\xi_1)(1-\xi_2)-\mathrm{i}0]^{a+\epsilon-2}}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\ ,
\end{aligned}
$$

where $a=a_1+a_2+a_3+a_4$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

Change the order of integration over z and ξ-parameters, evaluate parametric integrals in terms of gamma functions

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{(-1)^a \mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

$$\times \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{t}{s}\right)^z \Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$

$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

General prescriptions

- Derive a (multiple) MB representation for general powers of the propagators. (The number of MB integrations can be large (more than 10)).
  - Use it for checks. Reducing a line to a point $\to$ tending a_i to zero $\to$ (usually) taking some residues. A typical situation:
    $$\frac{\Gamma(a_2+z)\Gamma(-z)}{\Gamma(a_2)}, \quad a_2 \to 0$$
    Gluing of poles of different nature. Take a (minus) residue at $z_2 = 0$, then set $a_2 = 0$.
  - Unambiguous prescriptions for choosing integration contours
  - Try to have a minimal number of MB integrations.

- Resolve the singularity structure in ϵ. The goal: to represent a given MB integral as a sum of integrals where a Laurent expansion in ϵ becomes possible.
  The basic procedure:
  take residues and shift contours
  Two strategies:
  - #1 [V.A. Smirnov'99]
    E.g., the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$ generates a pole of the type $\Gamma(-\epsilon)$.
    The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$. 'Key' gamma functions

- #2 [J.B. Tausk'99, Anastasiou'05, Czakon'05].

  Choose a domain of ϵ and Rez_i, $\ldots$ Rew_i in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.
  Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.
  For every resulting residue, which involves one integration less, apply a similar procedure, etc.

  Two algorithmic descriptions [C. Anastasiou'05, M. Czakon'05]

  The Czakon's version implemented in `Mathematica` is public.

Evaluate MB integrals after expanding in ϵ.
Apply the first and the second Barnes lemmas

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)$$
$$= \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_6+z)}$$
$$= \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)}$$
$$\times \frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)}\,, \qquad \lambda_6 = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$$

multiple corollaries, e.g.,

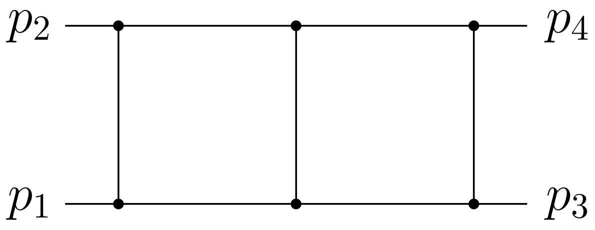
$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda_1+z)\Gamma^*(\lambda_2+z)\Gamma(-\lambda_2-z)\Gamma(\lambda_3-z)$$
$$= \Gamma(\lambda_1-\lambda_2)\Gamma(\lambda_2+\lambda_3)\left[\psi(\lambda_1-\lambda_2)-\psi(\lambda_1+\lambda_3)\right]$$

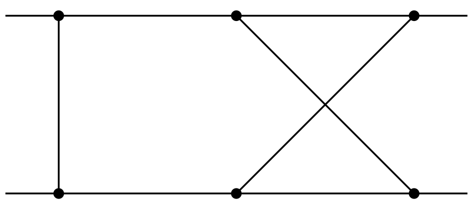
Use SUMMER to sum up series [J.A.M. Vermaseren'00]

IBP is also possible, e.g.

$$\int_C \mathrm{d}z \frac{f(z)}{z^2} = \int_C \mathrm{d}z \frac{f'(z)}{z}$$

Examples and results





Massless on-shell ($p_i^2 = 0,\ i = 1, 2, 3, 4$) double boxes:
done in 1999-2000, with multiple subsequent applications.
Master integrals calculated with the help of MB
representation [V.A. Smirnov'99, J.B Tausk'99, V.A. Smirnov & O.L. Veretin'99]

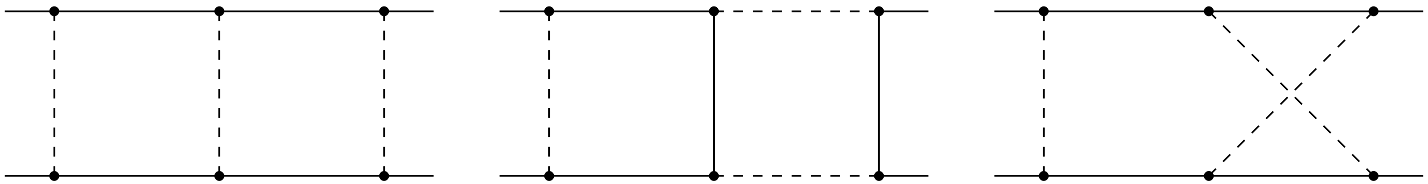
Massless double boxes with one leg off-shell, $p_1^2 = q^2 \neq 0$, $p_i^2 = 0, \; i = 2, 3, 4$:

- Reduction to master integrals [T. Gehrmann & E. Remiddi'01]
- Master integrals:
  - first results obtained by MB [V.A. Smirnov'01,02]
  - systematic evaluation by differential equations [T. Gehrmann & E. Remiddi'01]

All results are expressed in terms of two-dimensional harmonic polylogarithms which generalize harmonic polylogarithms [E. Remiddi & J.A.M. Vermaseren'00]

Applications to the process $e^+e^- \to 3$jets

Massive on-shell 2-boxes, $p_i^2 = m^2, \ i = 1, 2, 3, 4$



- first results obtained by MB

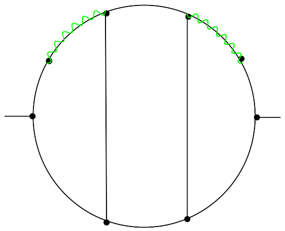
  [V.A. Smirnov'02,04; G. Heinrich & V.A. Smirnov'04]

- Reduction to master integrals by Laporta's algorithm

  [M. Czakon, J. Gluza & T. Riemann'04]

- Evaluating the master integrals by differential equations and MB [M. Czakon, J. Gluza & T. Riemann'05-08]

Evaluating Feynman integrals contributing to the three-loop static quark potential [A.V. Smirnov, V.A. Smirnov, and M. Steinhauser'08]
For example,

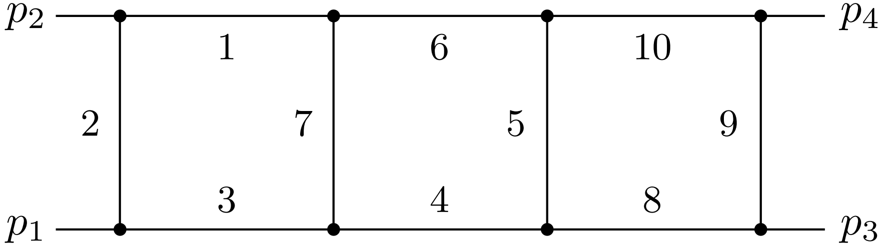


$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[\frac{56\pi^4}{135\epsilon}+\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$

Linear propagators $\frac{1}{v\cdot k+i0}$ in addition to usual massless propagators $\frac{1}{k^2+i0}$

$$v\cdot q=0$$

3 loops



The general planar triple box Feynman integral

$$T(a_1,\ldots,a_{10};s,t;\epsilon) \;=\; \int\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l\,\mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}}$$

$$\times \frac{1}{[(k+p_1+p_2)^2]^{a_3}[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}}$$

$$\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}}$$

General 7fold MB representation:

$$T(a_1,\ldots,a_{10};s,t,m^2;\epsilon) \quad = \quad \frac{\left(i\pi^{d/2}\right)^3 (-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$

$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$

$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$

$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$

$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$

$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$

$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

The master triple box:

$$
\begin{aligned}
& T(1,1,\ldots,1;s,t;\epsilon) \\
&= \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
&\times \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\times \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
&\times \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
&\times \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
&\times \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)
\end{aligned}
$$

Result

[V.A. Smirnov'03]

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}e^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}}\sum_{i=0}^{6}\frac{c_j(x,L)}{\epsilon^j}\,,$$

where $x=-t/s$, $L=\ln(s/t)$, and

$$c_6=\frac{16}{9}\,,\quad c_5=-\frac{5}{3}L\,,\quad c_4=-\frac{3}{2}\pi^2\,,$$

$$c_3=3(H_{0,0,1}(x)+LH_{0,1}(x))+\frac{3}{2}(L^2+\pi^2)H_1(x)-\frac{11}{12}\pi^2L-\frac{131}{9}\zeta_3\,,$$

$$c_2=-3\left(17H_{0,0,0,1}(x)+H_{0,0,1,1}(x)+H_{0,1,0,1}(x)+H_{1,0,0,1}(x)\right)$$

$$-L\left(37H_{0,0,1}(x)+3H_{0,1,1}(x)+3H_{1,0,1}(x)\right)-\frac{3}{2}(L^2+\pi^2)H_{1,1}(x)$$

$$-\left(\frac{23}{2}L^2+8\pi^2\right)H_{0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L-3\zeta_3\right)H_1(x)+\frac{49}{3}\zeta_3L-\frac{1411}{1080}\pi^4\,,$$

$$
\begin{aligned}
c_1 = {} & 3\left(81 H_{0,0,0,0,1}(x) + 41 H_{0,0,0,1,1}(x) + 37 H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right. \\
& +33 H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29 H_{1,0,0,0,1}(x) \\
& \left. +H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177 H_{0,0,0,1}(x) + 85 H_{0,0,1,1}(x)\right. \\
& \left. +73 H_{0,1,0,1}(x) + 3 H_{0,1,1,1}(x) + 61 H_{1,0,0,1}(x) + 3 H_{1,0,1,1}(x) + 3 H_{1,1,0,1}(x)\right) \\
& + \left(\frac{119}{2} L^2 + \frac{139}{12}\pi^2\right) H_{0,0,1}(x) + \left(\frac{47}{2} L^2 + 20\pi^2\right) H_{0,1,1}(x) \\
& + \left(\frac{35}{2} L^2 + 14\pi^2\right) H_{1,0,1}(x) + \frac{3}{2}\left(L^2 + \pi^2\right) H_{1,1,1}(x) \\
& + \left(\frac{23}{2} L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right) H_{0,1}(x) + \left(\frac{3}{2} L^3 + \pi^2 L - 3\zeta_3\right) H_{1,1}(x) \\
& + \left(\frac{9}{8} L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right) H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5\,,
\end{aligned}
$$

$$
\begin{aligned}
c_0 = &-(951H_{0,0,0,0,0,1}(x)+819H_{0,0,0,0,1,1}(x)+699H_{0,0,0,1,0,1}(x)+195H_{0,0,0,1,1,1}(x)\\
&+547H_{0,0,1,0,0,1}(x)+231H_{0,0,1,0,1,1}(x)+159H_{0,0,1,1,0,1}(x)+3H_{0,0,1,1,1,1}(x)\\
&+363H_{0,1,0,0,0,1}(x)+267H_{0,1,0,0,1,1}(x)+195H_{0,1,0,1,0,1}(x)+3H_{0,1,0,1,1,1}(x)\\
&+123H_{0,1,1,0,0,1}(x)+3H_{0,1,1,0,1,1}(x)+3H_{0,1,1,1,0,1}(x)+147H_{1,0,0,0,0,1}(x)\\
&+303H_{1,0,0,0,1,1}(x)+231H_{1,0,0,1,0,1}(x)+3H_{1,0,0,1,1,1}(x)+159H_{1,0,1,0,0,1}(x)\\
&+3H_{1,0,1,0,1,1}(x)+3H_{1,0,1,1,0,1}(x)+87H_{1,1,0,0,0,1}(x)+3H_{1,1,0,0,1,1}(x)\\
&+3H_{1,1,0,1,0,1}(x)+3H_{1,1,1,0,0,1}(x))\\
&-L\,(729H_{0,0,0,0,1}(x)+537H_{0,0,0,1,1}(x)+445H_{0,0,1,0,1}(x)+133H_{0,0,1,1,1}(x)\\
&+321H_{0,1,0,0,1}(x)+169H_{0,1,0,1,1}(x)+97H_{0,1,1,0,1}(x)+3H_{0,1,1,1,1}(x)\\
&+165H_{1,0,0,0,1}(x)+205H_{1,0,0,1,1}(x)+133H_{1,0,1,0,1}(x)+3H_{1,0,1,1,1}(x)\\
&+61H_{1,1,0,0,1}(x)+3H_{1,1,0,1,1}(x)+3H_{1,1,1,0,1}(x))\\
&-\left(\frac{531}{2}L^2+\frac{89}{4}\pi^2\right)H_{0,0,0,1}(x)-\left(\frac{311}{2}L^2+\frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)\\
&-\left(\frac{247}{2}L^2+\frac{307}{12}\pi^2\right)H_{0,1,0,1}(x)-\left(\frac{71}{2}L^2+32\pi^2\right)H_{0,1,1,1}(x)
\end{aligned}
$$

$$
\begin{aligned}
& -\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x) \\
& -\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x) \\
& -\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L\right. \\
& \left.-120\zeta_3\right)H_{0,1,1}(x)-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L\right. \\
& \left.-3\zeta_3\right)H_{1,1,1}(x)-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x) \\
& -\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x) \\
& -\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x) \\
& +\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6\,.
\end{aligned}
$$

$\zeta_3 = \zeta(3), \zeta_5 = \zeta(5)$ and $\zeta(z)$ is the Riemann zeta function. The functions $H_{a_1,a_2,\ldots,a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x)$, with $a_i = 1, 0, -1$, are HPL [E. Remiddi & J.A.M. Vermaseren'00]

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t) \mathrm{d}t \,,$$

where $f(\pm 1; t) = 1/(1 \mp t), \quad f(0; t) = 1/t,$

$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x \,,$$

with $a_i = 1, 0, -1$.

HPL are implemented in `Mathematica` [D. Maitre'06]

NB All the terms of the result have the same degree of transcendentality!

```
(* Massless onshell triple box integral. MB representation form:

  V.A. Smirnov, Phys.Lett.B567 (2003) 193 *)

Tgeneral = (-1)^a / (Gamma[a2] Gamma[a5] Gamma[a7] Gamma[a8] Gamma[a9] Gamma[a10]
      Gamma[4 - a5 - a8 - a9 - a10 - 2 ep] (-s)^(a - 6 + 3 ep)) (t / s)^w Gamma[a2 + w]
    Gamma[-w] Gamma[z2 + z4] Gamma[z3 + z4] / (Gamma[a1 + z3 + z4] Gamma[a3 + z2 + z4])
    Gamma[2 - a1 - a2 - ep + z2] Gamma[2 - a2 - a3 - ep + z3] Gamma[a7 + w - z4] Gamma[-z5]
    Gamma[-z6] / (Gamma[4 - a1 - a2 - a3 - 2 ep + w - z4] Gamma[a6 - z5] Gamma[a4 - z6])
    Gamma[a1 + a2 + a3 - 2 + ep + z4] Gamma[w + z2 + z3 + z4 - z7]
    Gamma[2 - a5 - a9 - a10 - ep - z5 - z7] Gamma[2 - a5 - a8 - a9 - ep - z6 - z7]
    Gamma[a4 + a6 + a7 - 2 + ep + w - z4 - z5 - z6 - z7] Gamma[a9 + z7] Gamma[a5 + z5 + z6 + z7]
    Gamma[-z2 - z3 - z4] / Gamma[4 - a4 - a6 - a7 - 2 ep + z5 + z6 + z7]
    Gamma[a5 + a8 + a9 + a10 - 2 + ep + z5 + z6 + z7] Gamma[2 - a6 - a7 - ep - w - z2 + z5 + z7]
    Gamma[2 - a4 - a7 - ep - w - z3 + z6 + z7] /. a → a1 + a2 + a3 + a4 + a5 + a6 + a7 + a8 + a9 + a10;

T = Tgeneral /. {a1 → 1, a2 → 1, a3 → 1, a4 → 1, a5 → 1, a6 → 1, a7 → 1, a8 → 1, a9 → 1, a10 → 1}
```

$$\Big((-s)^{-4-3\,ep}\left(\frac{t}{s}\right)^{w}\,\text{Gamma}[-w]\,\text{Gamma}[1+w]\,\text{Gamma}[-ep+z2]\,\text{Gamma}[-ep+z3]\,\text{Gamma}[1+w-z4]$$
$$\text{Gamma}[-z2-z3-z4]\,\text{Gamma}[1+ep+z4]\,\text{Gamma}[z2+z4]\,\text{Gamma}[z3+z4]\,\text{Gamma}[-z5]$$
$$\text{Gamma}[-z6]\,\text{Gamma}[w+z2+z3+z4-z7]\,\text{Gamma}[-1-ep-z5-z7]\,\text{Gamma}[-1-ep-z6-z7]$$
$$\text{Gamma}[1+ep+w-z4-z5-z6-z7]\,\text{Gamma}[1+z7]\,\text{Gamma}[-ep-w-z2+z5+z7]$$
$$\text{Gamma}[-ep-w-z3+z6+z7]\,\text{Gamma}[1+z5+z6+z7]\,\text{Gamma}[2+ep+z5+z6+z7]\Big)\Big/$$
$$\big(\text{Gamma}[-2\,ep]\,\text{Gamma}[1-2\,ep+w-z4]\,\text{Gamma}[1+z2+z4]\,\text{Gamma}[1+z3+z4]$$
$$\text{Gamma}[1-z5]\,\text{Gamma}[1-z6]\,\text{Gamma}[1-2\,ep+z5+z6+z7]\big)$$

```
Trules = MBoptimizedRules[T, ep → 0, {}, {ep}]
```

$$\left\{\left\{ep\to-\frac{7807}{9216}\right\},\right.$$
$$\left.\left\{w\to-\frac{1045}{3072},\ z2\to-\frac{725}{3072},\ z3\to-\frac{667}{1536},\ z4\to\frac{1355}{3072},\ z5\to-\frac{343}{1024},\ z6\to-\frac{7}{128},\ z7\to-\frac{611}{1024}\right\}\right\}$$

```
Tcont = MBcontinue[T, ep → 0, Trules];

Level 1

Taking +residue in z2 = ep

Taking +residue in z3 = ep

Taking -residue in z7 = -1 - ep - z5

Taking -residue in z7 = -1 - ep - z6

Taking +residue in z7 = ep + w + z2 - z5

Level 2

Integral {1}

Taking -residue in z4 = -ep - z3

Taking -residue in z7 = -1 - ep - z5

.......
```

```
Integral {5, 4, 5, 2, 1, 1, 3}

281 integral(s) found

    Tselect = MBpreselect[MBmerge[Tcont], {ep, 0, -4}];

    Texp = MBexpand[Tselect, -s^3 (-t)^(1 + 3 ep) Exp[3 ep EulerGamma], {ep, 0, -4}];

    (* application of the first Barnes lemma *)

    MBmerge[Texp /. MBint[i_, {a_, {z_ -> x_}}] :> Barnes1[MBint[i, {a, {z -> x}}], z]] /.
      {Log[t / s] -> -Log[s / t], Log[-s] -> Log[s / t] + Log[-t]}
```

$$\Big\{\text{MBint}\Big[\frac{1}{18\,\text{ep}^6}\Big(32 - 27\,\text{ep}^2\pi^2 + 66\,\text{ep}\,\text{Log}\Big[\frac{s}{t}\Big] + 54\,\text{ep}^2\,\text{Log}\Big[\frac{s}{t}\Big]^2 + 6\,\text{ep}\Big(16 + 33\,\text{ep}\,\text{Log}\Big[\frac{s}{t}\Big]\Big)\text{Log}[-t] + 144\,\text{ep}^2\,\text{Log}[-t]^2 + 144\,\text{ep}^2\Big(\text{Log}\Big[\frac{s}{t}\Big] + \text{Log}[-t]\Big)^2 - 6\,\text{ep}\Big(\text{Log}\Big[\frac{s}{t}\Big] + \text{Log}[-t]\Big)\Big(16 + 33\,\text{ep}\,\text{Log}\Big[\frac{s}{t}\Big] + 48\,\text{ep}\,\text{Log}[-t]\Big)\Big), \{\{\text{ep}\to 0\}, \{\}\}\Big]\Big\}$$

```
    Collect[%[[1, 1]], ep, Simplify]
```

$$\frac{16}{9\,\text{ep}^6} - \frac{3\pi^2}{2\,\text{ep}^4} - \frac{5\,\text{Log}\left[\frac{s}{t}\right]}{3\,\text{ep}^5}$$

```
    (* numerical integration down to the finite part *)

    Tselect = MBpreselect[MBmerge[Tcont], {ep, 0, 0}];

    Texp = MBexpand[Tselect, -s^3 (-t)^(1 + 3 ep) Exp[3 ep EulerGamma], {ep, 0, 0}];

    MBintegrate[Texp, {s -> 2 + I 10^-10, t -> -3},
      PrecisionGoal -> 3, Complex -> True] // AbsoluteTiming

Shifting contours...

Performing 28 1-dimensional integrations...1...2...3...4...5...6...7...8...9...10...11...12...1

Higher-dimensional integrals

Preparing MBpart1ep0 (dim 5)

Preparing MBpart2ep0 (dim 5)

Preparing MBpart3ep0 (dim 4)

    ... ... ... ... ..
```

```
Running MBpart68ep-3

Running MBpart69ep-3
```

$$\Big\{2202.244952\ \text{Second},$$

$$\Big\{(139.367 - 1242.01\, i) + \frac{1.77778}{ep^6} + \frac{0.675775 + 5.23599\, i}{ep^5} - \frac{14.8044 + 3.88578 \times 10^{-15}\, i}{ep^4} - \frac{10.3629 - 8.69308\, i}{ep^3} - \frac{124.152 - 94.1375\, i}{ep^2} - \frac{397.277 + 258.9\, i}{ep},$$

$$\Big\{0.652588 + \frac{0.000179653}{ep^3} + \frac{0.0140157}{ep^2} + \frac{0.213969}{ep},$$

$$1.71002 + \frac{3.66527 \times 10^{-15}}{ep^3} + \frac{0.0161577}{ep^2} + \frac{0.343745}{ep}\Big\}\Big\}\Big\}$$

(* the exact result gives *)

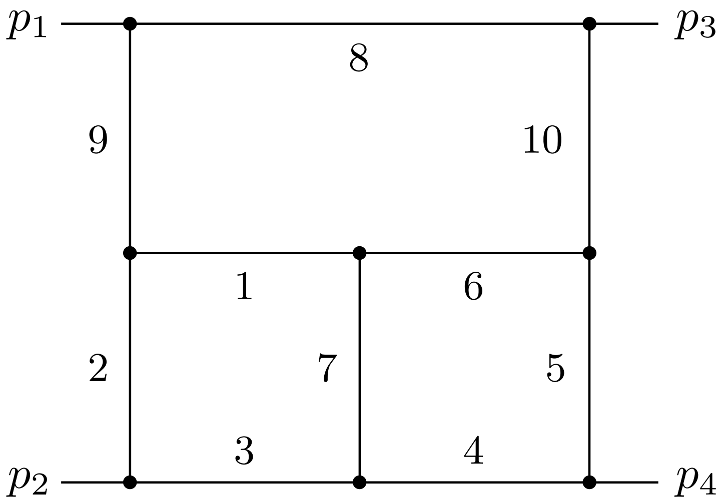
Texact

$$(139.347 - 1241.76\, i) + \frac{1.77778}{ep^6} + \frac{0.675775 + 5.23599\, i}{ep^5} - \frac{14.8044}{ep^4} - \frac{10.3629 - 8.69308\, i}{ep^3} - \frac{124.152 - 94.1372\, i}{ep^2} - \frac{397.271 + 258.899\, i}{ep}$$

Studying cross order relations in $N = 4$ supersymmetric gauge theories.
Iteration relations in two loops

[C. Anastasiou, L.J. Dixon, Z. Bern & D.A. Kosower'03,04]

To check such relations in three loops one more diagram was needed: the 'tennis court' graph with numerator $(l_1 + l_3)^2$



[Z. Bern, L.J. Dixon & V.A. Smirnov'05]

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{1+3\epsilon}t^2}$$

$$\times \frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty} \mathrm{d}w\,\mathrm{d}z_1 \prod_{j=2}^{7}\mathrm{d}z_j\Gamma(-z_j)\left(\frac{t}{s}\right)^w \Gamma(1+3\epsilon+w)$$

$$\times \frac{\Gamma(-3\epsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\epsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_1+z_2+z_3)}$$

$$\times \frac{\Gamma(-1-\epsilon-z_1-z_2-z_4)\Gamma(2+\epsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\epsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\epsilon+z_4+z_5+z_6+z_7)}$$

$$\times\Gamma(-\epsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)\Gamma(-1+w-z_5-z_6)$$

$$\times\Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)$$

$$\times\Gamma(-\epsilon+z_1+z_2-z_5-z_6-z_7)\Gamma(1-\epsilon-w+z_4+z_5+z_6+z_7)$$

$$\times\Gamma(1+\epsilon-z_1-z_2-z_3+z_5+z_6+z_7)$$

Result:

$$W(s,t;1,\ldots,1,-1,\epsilon) = -\frac{\left(\mathrm{i}\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{(-s)^{1+3\epsilon}\, t^2} \sum_{i=0}^{6} \frac{c_j}{\epsilon^j} \,,$$

where

$$\begin{aligned}
c_6 &= \frac{16}{9}\,, \quad c_5 = -\frac{13}{6}\ln x\,, \quad c_4 = -\frac{19}{12}\pi^2 + \frac{1}{2}\ln^2 x \\
c_3 &= \frac{5}{2}\left[\mathrm{Li}_3(-x) - \ln x\,\mathrm{Li}_2(-x)\right] + \frac{7}{12}\ln^3 x - \frac{5}{4}\ln^2 x \ln(1+x) \\
&\quad + \frac{157}{72}\pi^2 \ln x - \frac{5}{4}\pi^2 \ln(1+x) - \frac{241}{18}\zeta(3) \ \ldots
\end{aligned}$$

[C. Anastasiou, Z. Bern, L.J. Dixon & D.A. Kosower'03; Z. Bern, L.J. Dixon & D.A. Kosower'04]:
for the planar MHV four-point amplitude in $N = 4$ SUSY YM in two loops, one has

$$M_4^{(2)}(\epsilon) = \frac{1}{2}\left(M_4^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_4^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon)\,,$$

where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)\,, \quad C^{(2)} = -\frac{1}{2}\zeta_2^2$$

[Z. Bern, L.J. Dixon & V.A. Smirnov'05]:

taking into account the results for the ladder triple box and the tennis court diagram up to ϵ^0, for planar double box up to ϵ^2, and for the box up to ϵ^4, we obtain, in three loops,

$$
\begin{aligned}
M_4^{(3)}(\epsilon) &= -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) \\
&\quad + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + O(\epsilon)\,,
\end{aligned}
$$

where

$$
f^{(3)}(\epsilon) = \frac{11}{2}\,\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,,
$$

$$
C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,.
$$

An exponentiation of the planar MHV n-point amplitudes in $N = 4$ SUSY YM at L loops:

$$\begin{aligned}\mathcal{M}_n &\equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \\ &= \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right].\end{aligned}$$

where

$$a \equiv \frac{N_c \alpha_s}{2\pi} (4\pi e^{-\gamma})^{\epsilon},$$

$M_n^{(1)}(l\epsilon)$ is the all-orders-in-ϵ one-loop amplitude (with $\epsilon \to l\epsilon$), and

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \,.$$

The constants $f_k^{(l)}$ and $C^{(l)}$ are independent of the number of legs n.

The $E_n^{(l)}(\epsilon)$ are non-iterating $O(\epsilon)$ contributions to the l-loop amplitudes (with $E_n^{(l)}(0) = 0$).

By definition, the all-orders-in-ϵ one-loop amplitude is absorbed into $M_n^{(1)}(\epsilon)$:

$$f^{(1)}(\epsilon) = 1 \,, \qquad C^{(1)} = 0 \,, \qquad E_n^{(1)}(\epsilon) = 0 \,.$$

$1/\epsilon^2$ pole of the four-point amplitude $\rightarrow$
soft anomalous dimension at 3 loops in $N = 4$ SUSY YM

[A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko & V.N. Velizhanin'04]

$\leftrightarrow$

leading-transcendentality part of three-loop soft anomalous dimension in QCD [S. Moch, J.A.M. Vermaseren & A. Vogt'04]

$\leftrightarrow$

$j \rightarrow \infty$ formulae [M. Staudacher]

The iterative structure of five-point N = 4 SYSY YM amplitudes in two loops:

[F. Cachazo, M. Spradlin & A. Volovich'06]

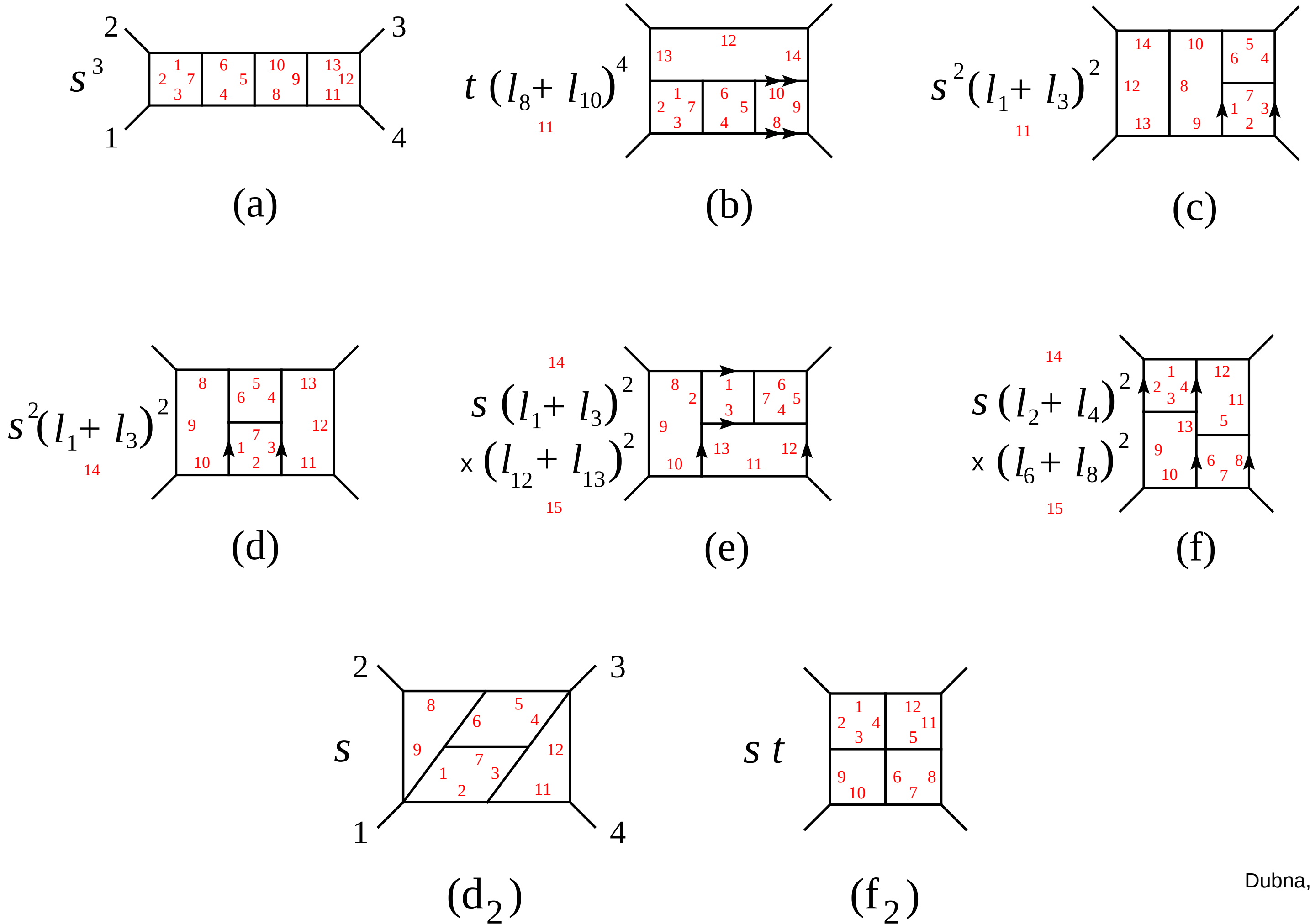
[Z. Bern, M. Czakon, D.A. Kosower, R. Roiban & V.A. Smirnov'06]

Violation of the iterative conjecture for 6 external gluons. Can it be cured?

[L.F. Alday & J. Maldacena'07; J.M. Drummond, J. Henn, G.P. Korchemsky & E. Sokatchev'07; J. Bartels, L.N. Lipatov & A. Sabio Vera'08; F. Cachazo, M. Spradlin & A. Volovich'08; M. Spradlin A. Volovich & C. Wen'08]

The four-loop cusp (soft) anomalous dimension

[Z. Bern, M. Czakon, L. Dixon, D.A. Kosower, & V.A. Smirnov'06]



The poles from $1/\epsilon^8$ to $1/\epsilon^4$ were evaluated analytically, the $1/\epsilon^3$ and $1/\epsilon^2$ numerically.
The $1/\epsilon^2$ part $\rightarrow$ the cusp anomalous dimension

$$\gamma_K = 4 f_0 = 4 \sum_{l=1}^{\infty} f_0^{(l)} \hat{a}^l$$

$$= 4 \left[\hat{a} - \frac{\pi^2}{6} \hat{a}^2 + \frac{11}{180} \pi^4 \hat{a}^3 - \left(\frac{73}{2520} \pi^6 - (1+r)\zeta_3^2 \right) \hat{a}^4 + \cdots \right]$$

where $\hat{a} \equiv \frac{g^2 N_c}{8\pi^2} = \frac{N_c \alpha_s}{2\pi}$ is the expansion parameter

$r = -2.03 \rightarrow r = -2$

$r = -2.00002$ [F. Cachazo, M. Spradlin& A. Volovich’06]

Further developments and open problems

MHV amplitudes in $N = 2$ SQCD and in $N = 4$ SYM

[E.W.N. Glover, V.V. Khoze & C. Williams '08]

In $N = 2$ SQCD, there may be also an iterative structure.

Subleading-color contributions to gluon-gluon scattering in N=4 SYM [S.G. Naculich, H. Nastase, & H.J. Schnitzer '08]

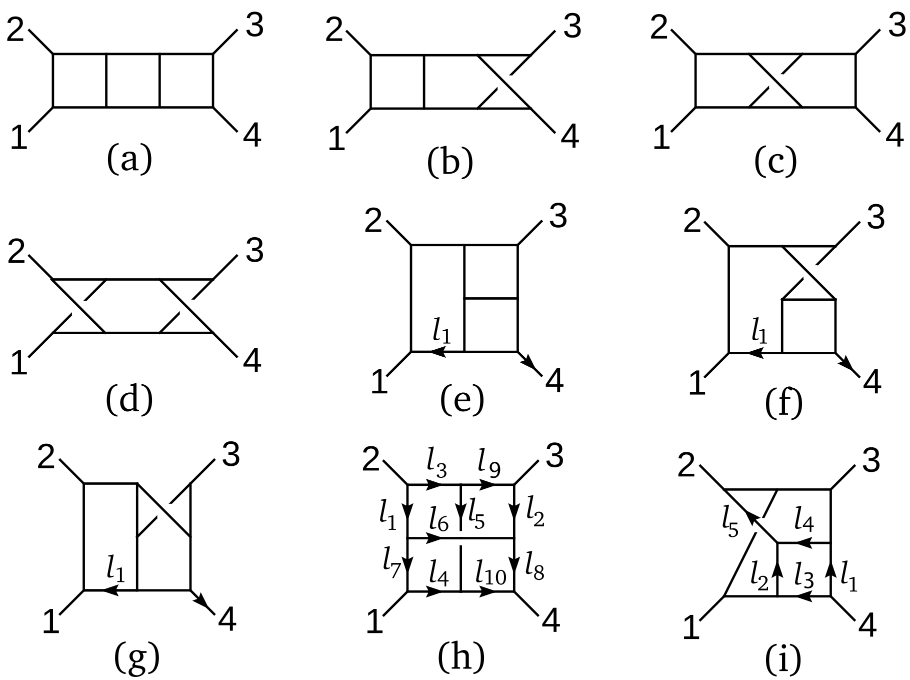
AMBRE — a package to derive MB representations for planar diagrams automatically by the loop-by-loop strategy.

[J. Gluza, K. Kajda & T. Riemann'07]

MB representation for non-planar diagrams?

The loop-by-loop strategy meets problems.

Next order in $1/N_c$



Updating `MB.m` [D. Kosower'07]

Presumably, an automation of Strategy #1 is also possible.

Inverse Feynman parameters? [T. Gehrmann, G. Heinrich, T. Huber & C. Studerus'06; G. Heinrich, T. Huber & D. Maitre'08]

Combination with PSLQ

Summary

- The method of MB representation is a powerful method. In particular, it is very flexible in resolving the singularity structure in ϵ. Implementation in computer algebra packages.
- Evaluating every Feynman integral of a given family, without reduction to master integrals, appears to be a possible alternative.