1. QUADRATIC TRIGONOMETRIC SUMS

<u>Def</u> Given $a \in (-1,1) \setminus \{0\}$ and $N \in \mathbb{N}$, define the quadratic trigonometric sum

$$S_a(N) := \sum_{n=0}^{N-1} \exp(\pi i a n^2) \in \mathbb{C}$$

Notice that $S_{-a}(N) = \overline{S_a(N)}$ and

$$S_{a+2}(N) = S_a(N)$$

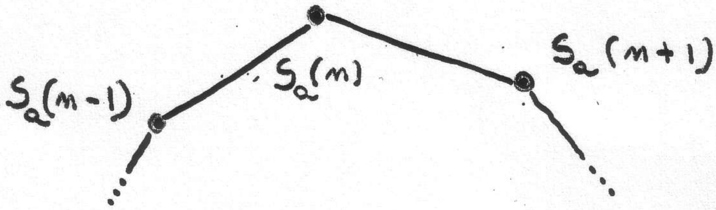
<u>References</u>

Hardy-Littlewood (1914, 23), Weyl (1914, 16), van der Corput (1923), Mordell (1926), Fiedler-Jurkat-Körner (1977), Jurkat-van Horne (1981, 82, 83), Dekking-Mendès France (1981) Mendès France (1983, 84), Deshouillers (1985), Berry-Goldberg (1988), Moore-van der Poorten (1989), Coutsias-Kazarinoff (1987, 98) Marklof (1999), Forrest (1996, 2000), Fedotov-Klopp (2005) Flaminio-Forni (2006), Fayad (2006), Greschonig-Nerurkar-Volný (2007), ...

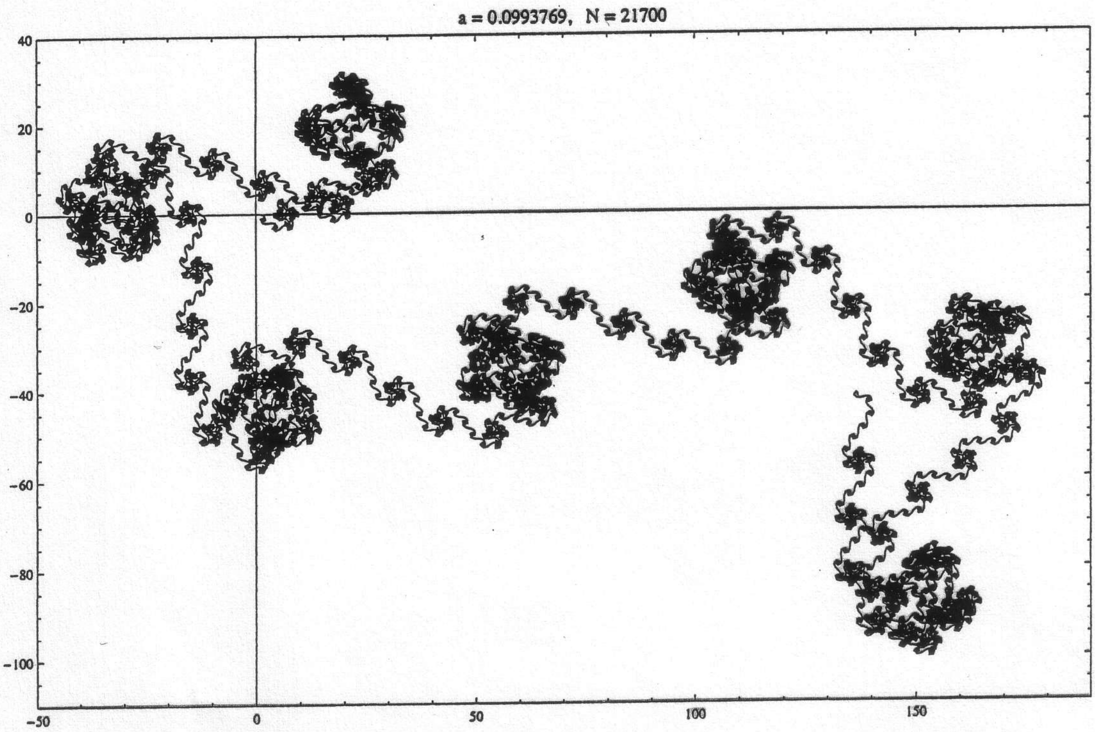
We consider the collection of points $\mathcal{S} = \{S_a(N)\}_{N \in \mathbb{N}}$

<u>GOAL</u>: understand the geometrical features of $\mathcal{S}$ in connection with the arithmetic properties of a.

$$S_a(N) = \sum_{n=0}^{N-1} \exp(\pi i a n^2)$$



a = 0.0993769, N = 21700



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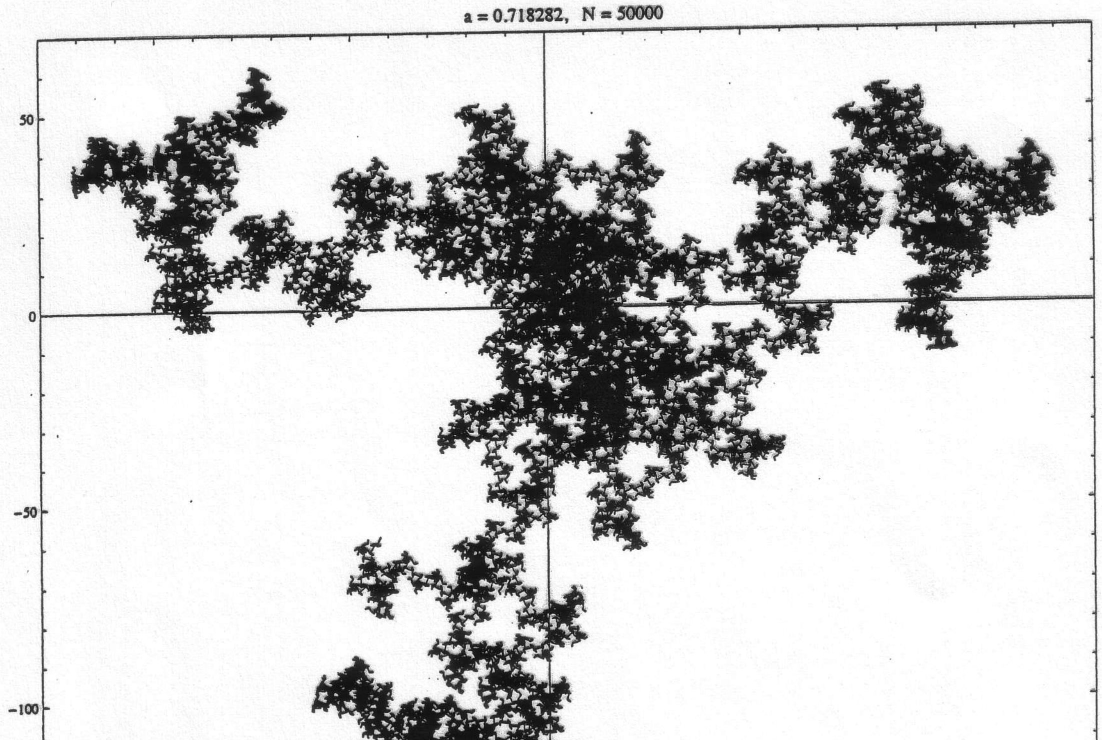
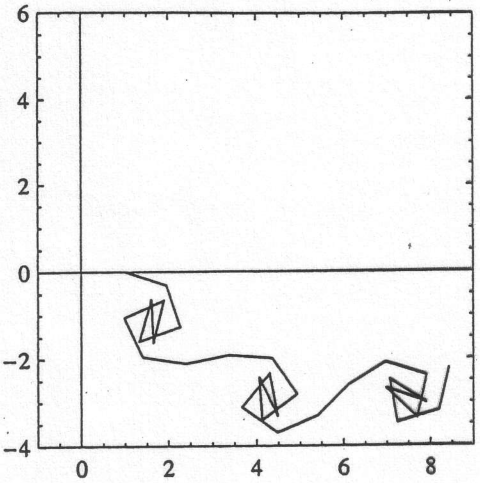


a = 0.0993769

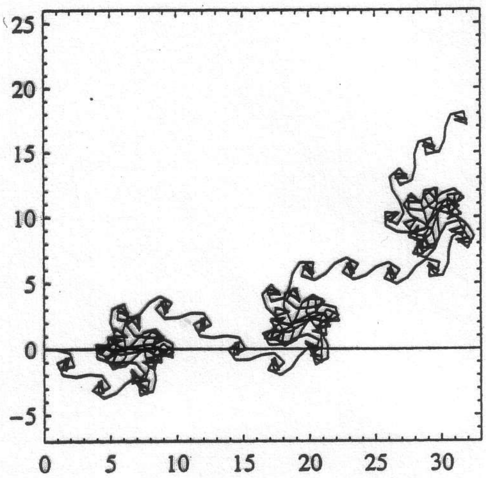
level 0

N = 30



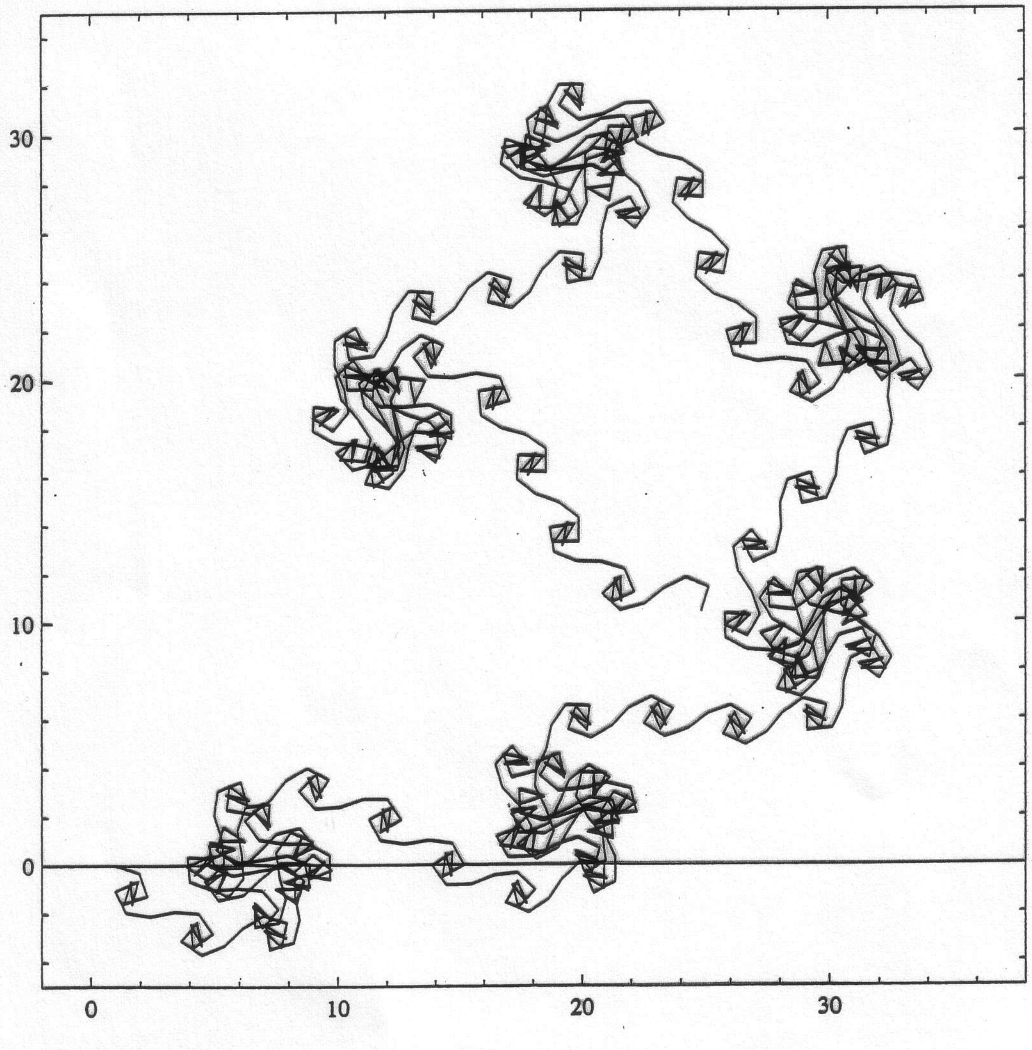
level 1

N = 490



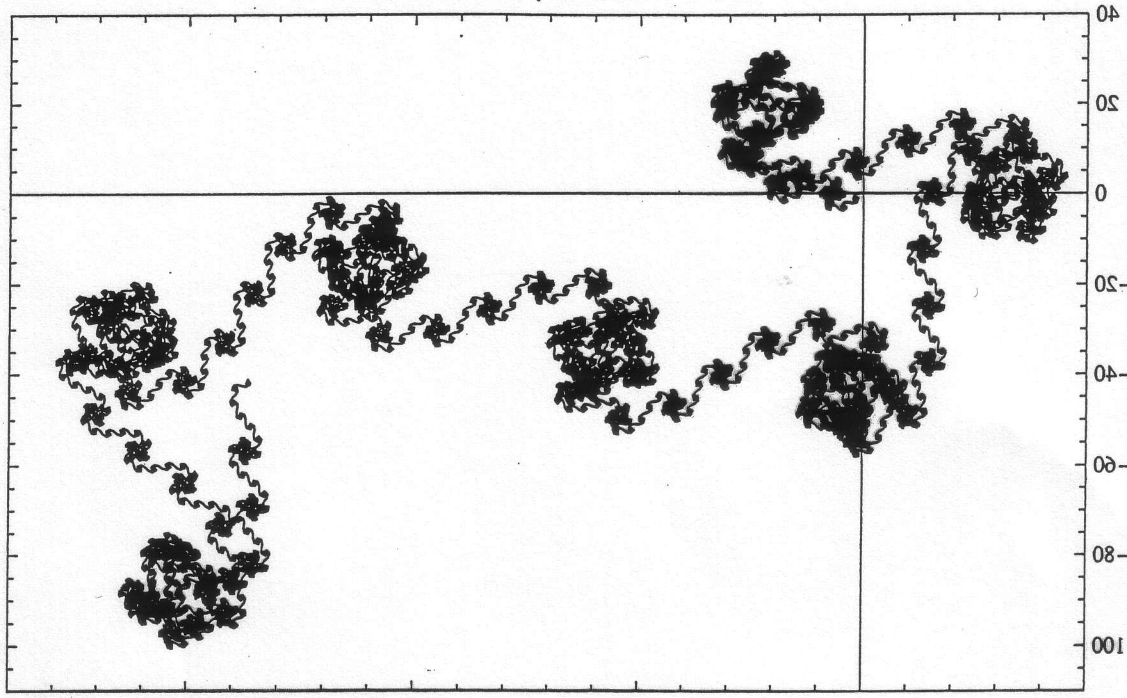
level 2

N = 990

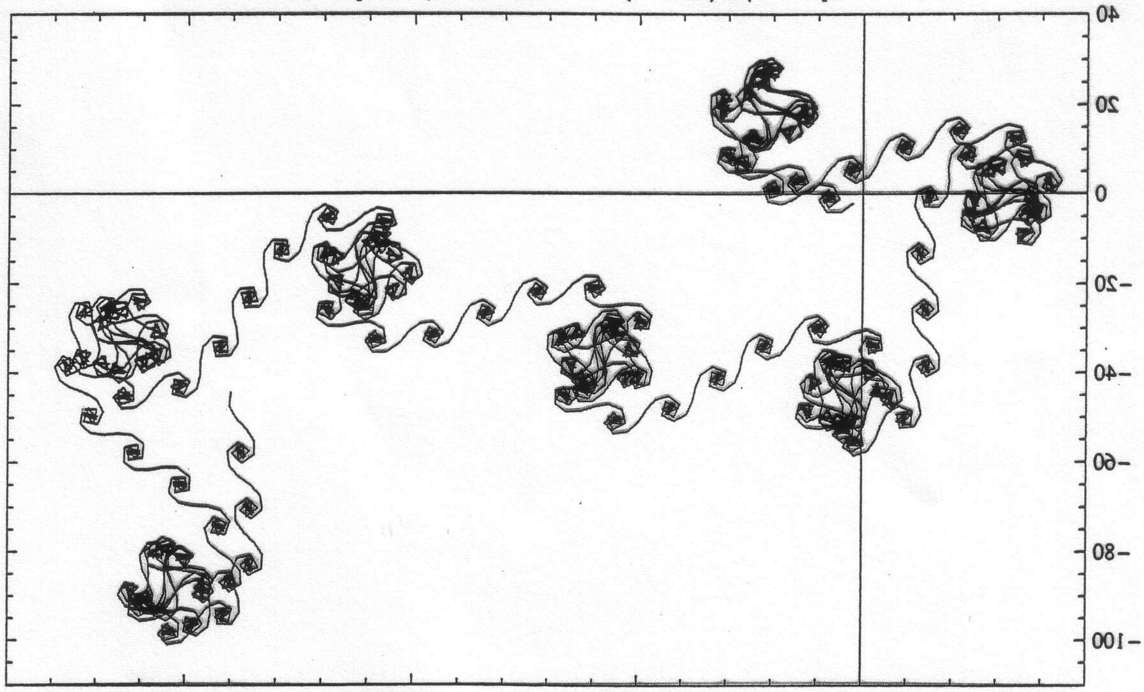


...

$a = 0.0993769, \quad N = 21700$



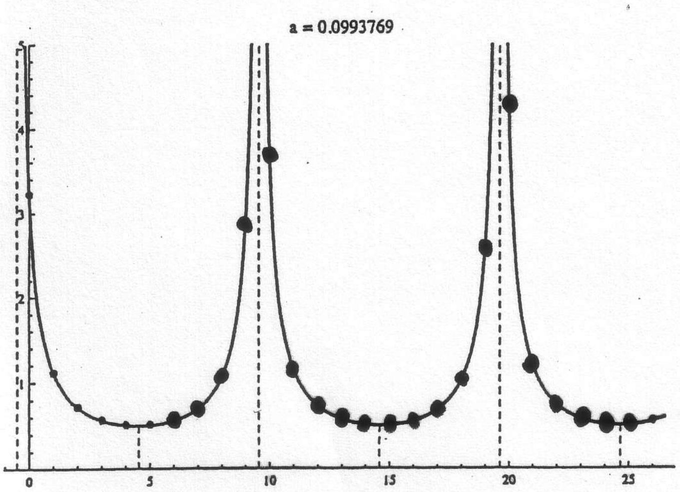
$a_1 = -1/a \pmod 2 = -0.0627048, \quad N_1 = 2156$

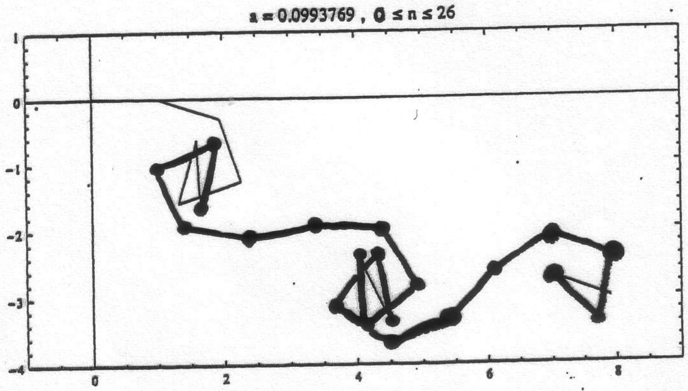


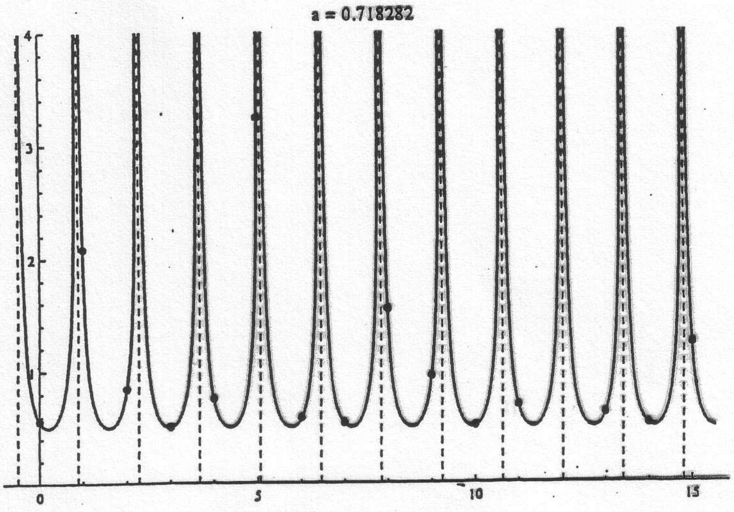
The geometric structure at level 0 comes from the integer sampling of the function

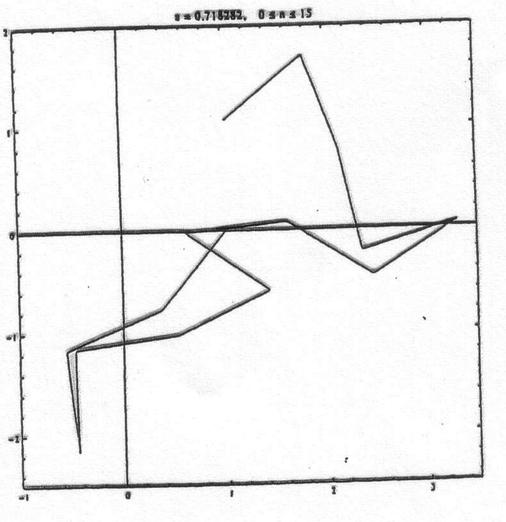
$$\nu \longmapsto \rho(\nu) = \frac{1}{2}\left|\csc\left(\frac{\pi a(2\nu+1)}{2}\right)\right|$$

- $|a|$ small $\Rightarrow$ spiral of "length" $\sim \frac{1}{|a|}$
- $|a| \sim 1$ $\Rightarrow$ no spirals









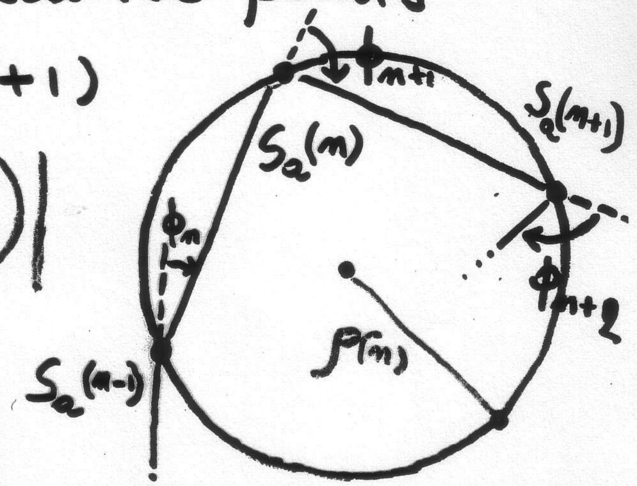
How do we see the geometric structure at higher levels?

How to study the geometrical structure?

Def. $\rho(n)$ is the local discrete radius of curvature, i.e. the radius of the circle passing through the 3 consecutive points

$$S_a(n-1),\ S_a(n),\ S_a(n+1)$$

$$\rho(n) = \frac{1}{2}\left|\csc\left(\frac{\pi a(2n+1)}{2}\right)\right|$$



$$\phi_n = \pi a n^2$$

Notice: $t \mapsto |\csc(t)|$ π-periodic

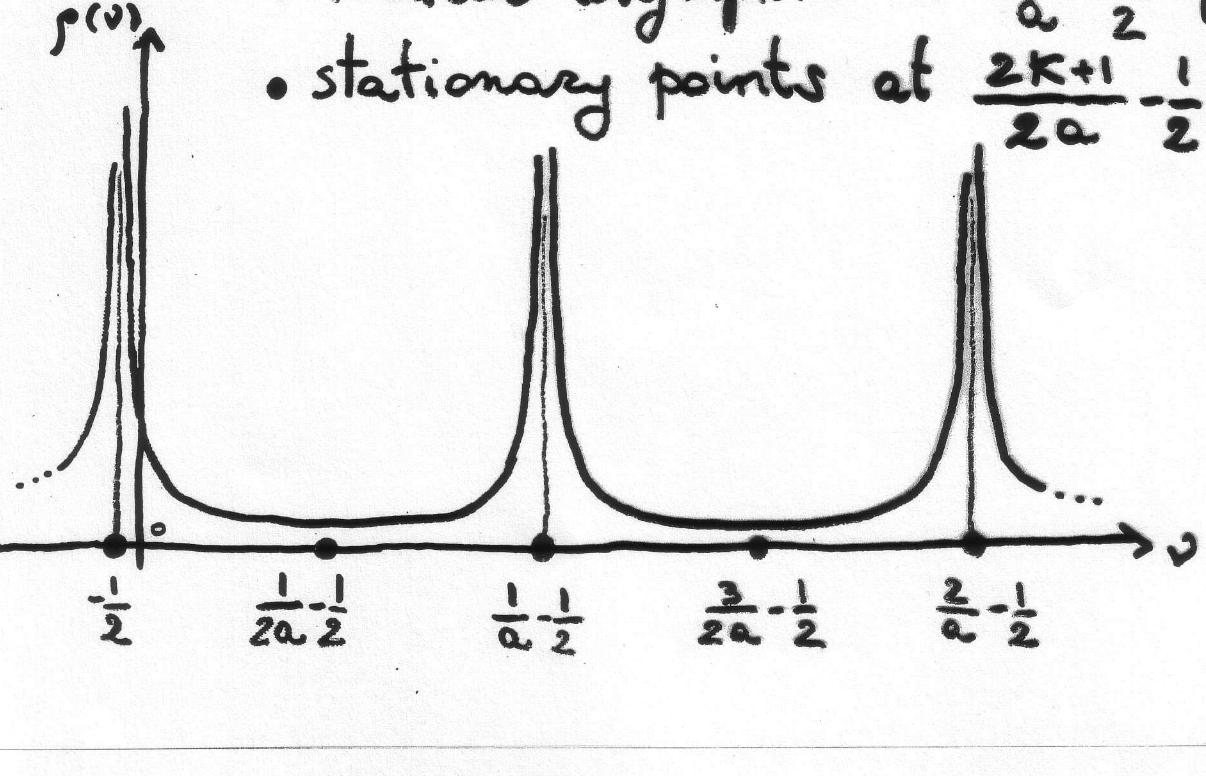
$$\Downarrow$$

$$\nu \mapsto \frac{1}{2}\left|\csc\left(\frac{\pi a(2\nu+1)}{2}\right)\right|$$

is $\frac{1}{a}$-periodic.

Moreover: $\nu \mapsto \rho(\nu)$ has

- vertical asymptote at $\frac{k}{a} - \frac{1}{2}$ $(k \in \mathbb{Z})$
- stationary points at $\frac{2k+1}{2a} - \frac{1}{2}$ $(k \in \mathbb{Z})$



Approximate Renormalization Formula

Using the Poisson summation formula & the stationary phase method we get

$$S_a(N) \sim e^{-\frac{\pi}{4}i} |a|^{-\frac{1}{2}} S_{a_1}(N_1),$$

$a \in (-1,1) \setminus \{0\}$, $\quad a_1 = -\frac{1}{a} \pmod 2$, $N_1 = \lfloor |a| \cdot N \rfloor$.

More precisely, we have the estimate (A.R.F.)

$$\left| S_a(N) - e^{-\frac{\pi}{4}i} |a|^{-\frac{1}{2}} S_{a_1}(N_1) \right| \leq C_1 |a|^{-\frac{1}{2}} + C_2$$

where C_1, C_2 are universal constants.

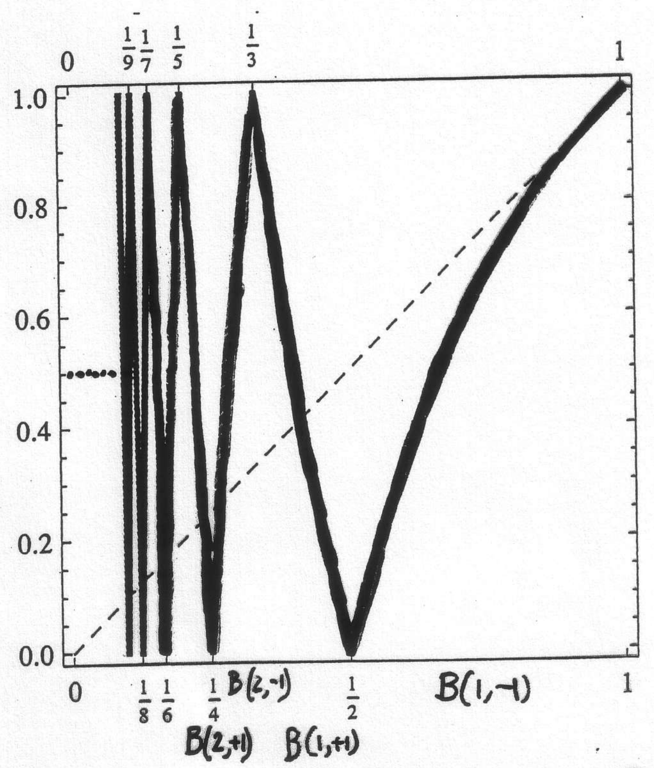
Notice: the A.R.F. is uniform in N.

For $|a|$ small, $\{S_a(n)\}_{n=0}^{N-1}$ contains approximately $|a| \cdot N$ spirals at level 0.

By the A.R.F., $\{S_a(n)\}_{n=0}^{N-1}$ can be approximated by $\{S_{a_1}(n)\}_{n=0}^{N_1-1}$ (up to scaling by $|a|^{-\frac{1}{2}}$ and rotating by $-\frac{\pi}{4}$). The geometric structure at level 0 for $\{S_{a_1}(n)\}_{n=0}^{N_1-1}$ corresponds to the structure at level 1 for $\{S_a(n)\}_{n=0}^{N-1}$.

We introduce a new map $T:(0,1] \circlearrowleft$,

$$T(\alpha) = \xi \cdot \left(\frac{1}{\alpha} - 2k\right) \quad \text{for } \alpha \in B(k,\xi), \quad k \in \mathbb{N}, \quad \xi \in \{-1,+1\}.$$



[Fixed points: $k-\sqrt{k^2-1}$, $-k+\sqrt{k^2-1}$]

The Approximate Renormalization Formula becomes

$$\left| S_\alpha(N) - e^{-\frac{\pi}{4}i}\alpha^{-\frac{1}{2}} S_{\alpha_1}^{(\eta)}(N_1)\right| \le C_1\alpha^{-\frac{1}{2}} + C_2,$$

$\alpha_1 = T(\alpha)$, $\eta = \eta(\alpha)$, $N_1 = \lfloor \alpha N \rfloor$.

Iterated A.R.F.

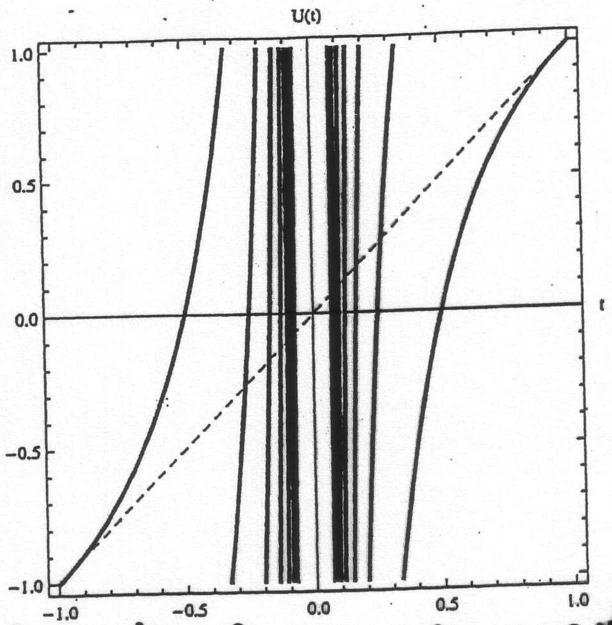
$\alpha = \alpha_0 \xmapsto{T} \alpha_1 \xmapsto{T} \alpha_2 \xmapsto{T} \ldots \xmapsto{T} \alpha_r$, $\quad \alpha_l = T^l(\alpha_0)$

$N = N_0 \longmapsto N_1 \longmapsto N_2 \longmapsto \ldots \longmapsto N_r$, $\quad N_{l+1} = \lfloor \alpha_l N_l \rfloor$

$N_0 \ge N_1 \ge N_2 \ge \ldots \ge N_r$, $\quad \eta_{l+1} = \eta(\alpha_l)$.

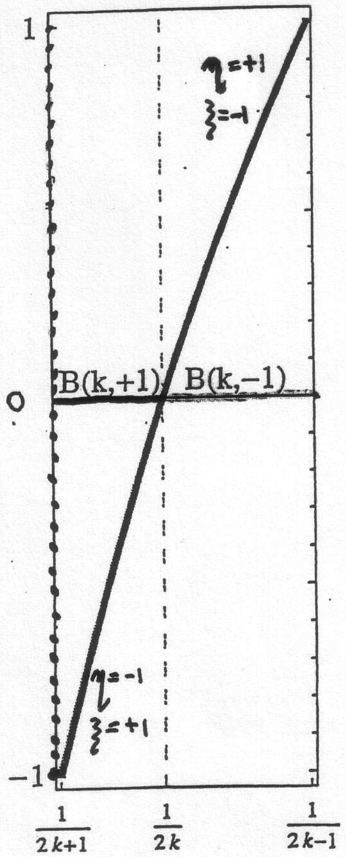
$$e^{r\frac{\pi}{4}i}(\alpha_0 \cdot \ldots \cdot \alpha_{r-1})^{\frac{1}{2}} S_{\alpha_0}(N_0) \sim S_{\alpha_r}^{(\eta_1 \eta_2 \cdots \eta_r)}(N_r).$$

We want to study the map $U:[-1,1]\setminus\{0\}\circlearrowleft$,
$U(t)=-\frac{1}{t} \pmod 2$.



Let's look at the k-th positive branch of U:

$$U_k:\left(\frac{1}{2k+1},\frac{1}{2k-1}\right]\to(-1,1],\quad t\mapsto -\frac{1}{t}+2k.$$



Define $B(k,-1)=\left(\frac{1}{2k},\frac{1}{2k-1}\right]$
and $B(k,+1)=\left(\frac{1}{2k+1},\frac{1}{2k}\right]$.

For $\alpha=|a|\in(0,1)$ define
$\eta(\alpha)=\operatorname{sgn} U(\alpha)$,
$\xi(\alpha)=-\eta(\alpha)$ and

$$S_\alpha^{(\eta)}(N)=\begin{cases} S_\alpha(N) & \text{if } \eta=+1\\ \overline{S_\alpha(N)} & \text{if } \eta=-1\end{cases}$$

2. CONTINUED FRACTIONS WITH EVEN PARTIAL QUOTIENTS

Recall: $(0,1] = \bigsqcup_{(k,\xi)\in\mathbb{N}\times\{\pm1\}} B(k,\xi)$,

$\alpha\in B(k,\xi) \Rightarrow T(\alpha) = \xi\left(\frac{1}{\alpha} - 2k\right).$

<u>Property of T:</u> $\boxed{\alpha\in B(k,\xi) \Rightarrow \alpha = \dfrac{1}{2k+\xi\cdot T(\alpha)}}$

$\Rightarrow$ T generates the following C.F. expansion of $\alpha\in(0,1]$

$$\alpha = \cfrac{1}{2k_1 + \cfrac{\xi_1}{2k_2 + \cfrac{\xi_2}{2k_3 + \cfrac{\xi_3}{\ddots}}}} = [[(k_1,\xi_1),(k_2,\xi_2),(k_3,\xi_3),\dots]]$$

(ECF-EXPANSION)

$(k_n,\xi_n)\in\mathbb{N}\times\{\pm1\} =: \Omega$

T acts as a <u>SHIFT</u> over the space $\Omega^{\mathbb{N}}$:

$\alpha = \alpha_0 = [[(k_1,\xi_1),(k_2,\xi_2),\dots]]$

$\Rightarrow \alpha_n = T^n(\alpha_0) = [[(k_{n+1},\xi_{n+1}),(k_{n+2},\xi_{n+2}),\dots]]$

<u>References</u>: Schweiger (1982, 84), Kraaikamp - Lopes (1996).

The ECF-convergents are defined as

$$\frac{p_n}{q_n} = \cfrac{1}{2K_1 + \cfrac{\xi_1}{2K_2 + \dots + \cfrac{\xi_{n-2}}{2K_{n-1} + \cfrac{\xi_{n-1}}{2K_n}}}} \quad , \qquad GCD(p_n, q_n) = 1$$

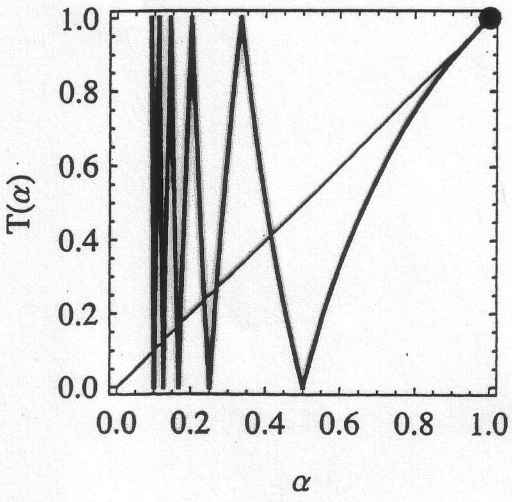
and they satisfy the REURRENCE RELATIONS:

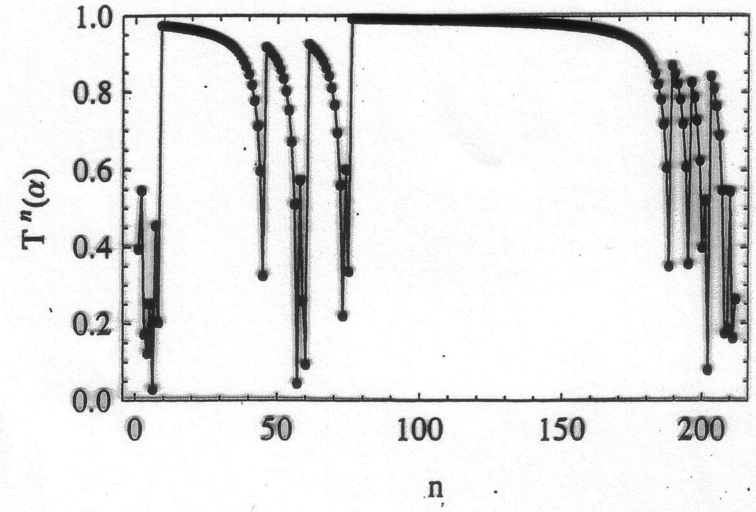
$$\begin{aligned} p_n &= 2K_n\, p_{n-1} + \xi_{n-1}\, p_{n-2} \\ q_n &= 2K_n\, q_{n-1} + \xi_{n-1}\, q_{n-2} \end{aligned} \qquad n \geq 1$$

with $q_{-1} = p_0 = 0$, $p_{-1} = q_0 = \xi_0 = 1$.

Despite its similarities with the GAUSS MAP, the map $\alpha \mapsto T(\alpha)$ is <u>INTERMITTENT</u>, i.e. it has an indifferent fixed point at $\alpha = 1$.

In its behavior, the map T is similar to the FAREY MAP.





Theorem (Schweiger)

The map T has a σ-finite invariant measure with infinite mass over $(0,1]$. Its density is $h(\alpha) = \dfrac{1}{\alpha+1} - \dfrac{1}{\alpha-1}$.

Properties of $\dfrac{p_n}{q_n}$ $\qquad \alpha = [[(k_1,\xi_1),(k_2,\xi_2),\dots]]$

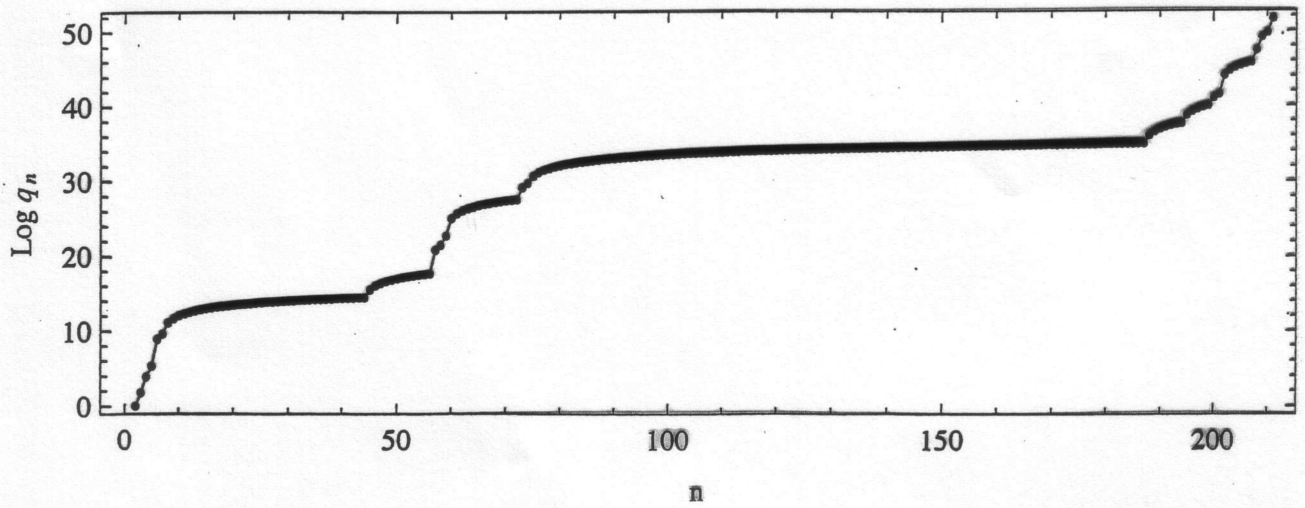
i) $q_n \geq n+1$

ii) $p_{n+1} q_n - p_n q_{n+1} = (-1)^n \cdot \prod_{j=0}^{n} \xi_j$

iii) $$\alpha - \frac{p_n}{q_n} = \frac{\alpha_n (-1)^n \prod_{j=0}^{n} \xi_j}{q_n^2 \left(1 + \xi_n \alpha_n \dfrac{q_{n-1}}{q_n}\right)}$$

iv) $$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n}$$

v) $$(\alpha_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_{n-1})^{-1} = q_n \left(1 + \xi_n \alpha_n \frac{q_{n-1}}{q_n}\right)$$



RENEWAL-TYPE LIMIT THEOREM FOR $\{q_n\}$

Fix $L>0$ and define the renewal time

$n_L = n_L(\alpha) = \min\{n \in \mathbb{N} : q_n > L\}$.

Denote by $\omega_n = (K_n, \xi_n) \in \Omega$ the entries of the ECF-expansion of α.

Theorem (C)

Fix $N_1, N_2 \in \mathbb{N}$. The ratio $\frac{q_{n_L}}{L}$ and the entries ω_{n_L+j}, $-N_1 < j \le N_2$, have a joint limiting probability distribution as $L \to \infty$ w.r.t. the Lebesgue measure on $(0,1]$.

In other words:

$\exists$ probability measure $P = P_{N_1, N_2}$ on $(1,\infty) \times \Omega^{N_1+N_2}$ s.t.

$\forall\, a, b > 1$, $\forall\, \underline{d} = (d_j)_{j=-N_1+1}^{N_2} \in \Omega^{N_1+N_2}$

$$\mathrm{Leb}\left(\left\{\alpha : a < \frac{q_{n_L}}{L} < b,\ (\omega_{n_L+j})_{j=-N_1+1}^{N_2} = \underline{d}\right\}\right) \xrightarrow[L\to\infty]{} P\left((a,b) \times \underline{d}\right).$$

For the sequence of denominators generated by the GAUSS map, the corresponding theorem was proven by Sinai and Ulcigrai (2007). Their proof uses the mixing property of a suitably defined special flow over the natural extension of the Gauss map.