On resummation of PT series in time-like momentum region

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Outline

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- Summary

1. Motivation: RenormGroup in SM

Concept of renormalization – renormalizable models of QFT as self consistent tool for particle interactions (...,Bogoliubov,Shirkov,...)

QED: Parameters of the Lagrangian as formal quantities to be fixed to observables. Finite RG transformations (Shirkov's preprint at this conf). Still "physical" (on-shell) scheme for α and m_e. RG is not crucial for quantitative analysis since no very different scales and $\alpha = 1/137$ is small

Current theory - SM requires heavy use of RG for precision analysis as scales are different and coupling constants (especially α_s) are quite large.

Indeed, EW physics is at $M_{Z,W} \sim 90~\mathrm{GeV}$ while $m_t = 175~\mathrm{GeV}$, $m_b = 5~\mathrm{GeV}$ and light hadrons are at $1~\mathrm{GeV}$. Also DIS allows for wide range of scales by varying Q. Most experiments include hadrons that implies importance of QCD at soft scales with $\alpha_s(1~\mathrm{GeV}) \approx 0.45$.

Actual topics of current PT analysis in SM:

- Higher order PT corrections: NNLO is a (almost) standard. Problems of (explicit) convergence and control/estimate over the residual

- Rigorous definition of expansion parameters (renormalization scheme), $\overline{\mathrm{MS}}$ is a standard for technical reasons as dim reg is main calculational framework. More physical ones are in use like $\alpha_V(r)$ from the Coulomb part of the heavy quark potential for dynamics in NonRelativ QCD for dynamics near the threshold

- Resummation of some infinite subsets of the series in α_s. The most direct - the very RG - powers of logs: $(\alpha_s \ln(Q/\mu))^n$, $\alpha_s(\alpha_s \ln(Q/\mu))^n$... Some others (still related to RG): β_0 dominance (naive nonabelization) $(\beta_0\alpha_s)^n$, effects of analytic continuation between Euclidean and Minkowskian regions $(\pi\beta_0\alpha_s)^{2n}$ (developed into APT - Shirkov,Solovtsov,Bakulev...)

Non-polynomial terms in α_s:
resummation may provide terms of the form $\exp(-1/\alpha_s)$ that interfere with nonPT expansions: higher twists in light-cone type expansions, or condensates at short distances. That makes extracted numerical values of nonPT parameters dependent on how PT series are treated.

Hadronic τ-lepton decays - a lab for high order PT analysis:

- Theoretical description is simple and related to e^+e^--annihilation
- Record number of PT terms is available - NNNNO
- Convergence is slow and interpretation of the series is needed
- nonPT terms are available in OPE

PT series for τ system is of practical importance (not a question of model QFT like, say, ϕ^4) as τ data are related to hadronic contributions to $\alpha_{EM}(M_Z)$ and muon $g-2$ which used for Higgs mass determination in SM and new physics searches.

2a. Phenomenology of hadronic τ-lepton decays

Differential decay rate of τ into hadrons H with total energy s

$$\frac{d\sigma(\tau \to \nu H(s))}{ds} \sim \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) \rho(s)$$

is determined by the spectral density $\rho(s)$ defined through the correlator of weak currents $j_\mu^W(x) = \bar{u}\gamma_\mu(1-\gamma_5)d$

$$i \int \langle T j_\mu^W(x) j_\nu^{W+}(0) \rangle e^{iqx} dx = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^{\rm had}(q^2)$$

with $\rho(s) = \frac{1}{\pi} {\rm Im}\ \Pi^{\rm had}(s+i0)$, $s = q^2$ and $\Pi^{\rm had}(q^2) = \int \frac{\rho(s)ds}{s-q^2}$

Total τ lepton rate is given by the integral over s

$$R_{\tau S=0} = \frac{\Gamma(\tau \to H_{S=0}\nu)}{\Gamma(\tau \to l\bar{\nu}\nu)} \sim \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) \rho(s) ds$$

and is an observable experimentally measured with high precision.
For charged weak hadronic current

$$j_\mu^w(x) = V_{ud}\bar{u}\gamma_\mu(1-\gamma_5)d + V_{us}\bar{u}\gamma_\mu(1-\gamma_5)s$$

one can extract information on: V_{ud}, V_{us}, $m_s(M_\tau)$, and $\alpha_s(M_\tau)$ from the moments

$$M_{kl} = \frac{(k+l+1)!}{k!l!} \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{\rho(s)ds}{M_\tau^2}$$

In massless limit PT expression for the correlator $\Pi(Q^2)$ reads

$$D(Q^2) = -Q^2\Pi'(Q^2) = 1 + a_s + k_1 a_s^2 + k_2 a_s^3 + k_3 a_s^4 + O(a_s^5)$$

with $a_s = \alpha_s(Q^2)/\pi$. The spectral density $\rho(s)$

$$\rho(s) = 1 + a_s(s) + k_1 a_s(s)^2 + \left(k_2 - \frac{\pi^2}{3}\beta_0^2\right) a_s(s)^3 + \ldots$$

where π^2 results from analytic continuation.
This $\rho(s)$ gives moments M_{kl} as polynomials in $\alpha_s(M_\tau^2)$ - Finite Order PT analysis. In $\overline{\mathrm{MS}}$-scheme the QCD correction to total width δ_P is

$$\delta_P^{\mathrm{th}} = a_s + 5.2023a_s^2 + 26.366a_s^3 + (78.003 + 49.08)a_s^4 + O(a_s^5)$$

Main problem - slow convergence of the series.
Can one do better than FOPT?

2b-1. Resummation on the cut: $q^2 > 0$

Going beyond FOPT by account of RG logs in $\rho(s)$ doesn't work. For a simple approximation $\rho(t) = \alpha_s(t)$ and

$$\alpha_s(t) = \frac{\alpha_s(s)}{1 - \beta_0 \alpha_s(s) \ln(s/t)} = \alpha_s(s) \sum_{n=0}^{\infty} (\beta_0 \alpha_s(s) \ln(s/t))^n$$

one finds $F(s) = \frac{1}{s} \int_0^s \alpha_s(t) dt = \alpha_s(s) \sum_{n=0}^{\infty} (\beta_0 \alpha_s(s))^n n!$
with factorial growth that is not Borel summable.
The reason is ghost pole in the expression for $\alpha_s(t)$ or divergence of the integrand outside the convergence circle $|\beta_0 a(s) \ln(s/t)| < 1$.

Thus higher order terms in $\alpha_s(t)$ are important for $\rho(t)$.

Possible solution: modification of running of effective coupling at low s by changing β-function

a) Example with explicit expression for the integral

Consider a model β-function

$$\beta(a) = \frac{-a^2}{1+2a}$$

In this case the integral

$$F(s) = \frac{1}{s}\int_0^s a(t)dt$$

can be found explicitly. RG equation for the effective charge $a(s)$ is

$$\ln(s/\Lambda^2) = \frac{1}{a(s)} - 2\ln a(s)$$

and $F(s)$ reads

$$F(s) = \frac{1}{s}\int_0^s a(t)dt = a(s) + a(s)^2 - a(s)^2 \exp(-\frac{1}{a(s)}).$$

The last term gives the "condensate" contribution to $F(s)$

$$F^{cond}(s) = a(s)^2 \exp(-\frac{1}{a(s)}) \sim \frac{\Lambda^2}{s} \qquad \text{at} \qquad s \to \infty$$

This term cannot be detected in series expansion at small $a(s)$.

Thus, a change of evolution in the IR region can help to resum factorials.

Another way to kill factorials is to sum up π^2 terms (APT, Shirkov,Solovtsov,...). In fact this can also be considered as a modification of the evolution.

b) Resummation of π^2 terms as modification of the evolution

Resummation of analytic continuation effects gives

$$\rho(s) = \frac{1}{\beta_0} \arctan(\beta_0 \alpha(s)) \equiv \alpha_M(s).$$

The evolution equation for effective charge $\alpha_M(s)$ reads

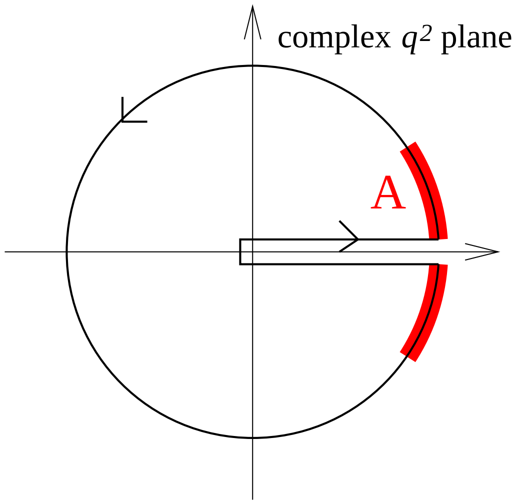
$$s \frac{d\alpha_M(s)}{ds} = -\frac{1}{\pi^2 \beta_0} \sin^2(\pi \beta_0 \alpha_M(s)).$$

$\alpha_M(s)$ is finite at small s and can be integrated over IR region.

There is still one more option to avoid factorials:

Integration in the complex q^2 plane

2b-2. Resummation on the contour



Integrating the function $\Pi(q^2)$ over a contour in the complex q^2 plane beyond the physical cut $s > 0$ one finds

$$\oint_C \Pi(z)dz = \int_{\text{cut}} \rho(s)ds$$

with $\rho(s) = \frac{1}{2\pi i}\left(\Pi(s+i0) - \Pi(s-i0)\right)$ for $s > 0$

Using the approximation $\Pi(z)|_{z\in C} \approx \Pi^{\mathrm{PT}}(z)|_{z\in C}$ which is justified far from the physical cut one obtains

$$\oint_C \Pi(z)dz = \int_{\mathrm{cut}} \rho(s)ds = \oint_C \Pi^{\mathrm{PT}}(z)dz$$

Thus, the integral over the hadronic spectrum can be evaluated in pQCD. The total decay rate of the τ-lepton and spectral moments M_{kl} are the quantities that can be computed this way.

Parameterizing the contour by $Q^2 = M_\tau^2 e^{i\varphi}$ one obtains for M_{kl}

$$M_{kl} = \frac{(-1)^l}{2\pi}\frac{(k+l+1)!}{k!l!}\int_{-\pi}^{\pi} \Pi(M_\tau^2 e^{i\varphi})(1+e^{i\varphi})^k e^{i(l+1)\varphi}d\varphi.$$

This program realized for extraction of $m_s(M_\tau^2)$ from τ decays data

2c. Numerical values for $m_s(M_\tau^2)$ from resummed PT analysis

For $j_\mu(x) = \bar{u}\gamma_\mu(1-\gamma_5)s$ with $m_s \neq 0$

$$i\int dx e^{iqx}\langle T j_\mu(x) j_\nu^\dagger(0)\rangle = q_\mu q_\nu \Pi_q(q^2) + g_{\mu\nu}\Pi_g(q^2)\,.$$

Retaining m_s^2/q^2 term at $q^2 \to \infty$ one finds

$$\Pi_q(q^2) = \Pi(q^2) + 3\frac{m_s^2}{q^2}\Pi_{mq}(q^2), \quad \Pi_g(q^2) = -q^2\Pi(q^2) + \frac{3}{2}m_s^2\Pi_{mg}(q^2)$$

where $\Pi(q^2)$ is as in ud (massless) case. $\Pi_{q,g}(Q^2)$ are computable in pQCD. Thus, data on τ decays are theoretically described by three functions which can be analyzed simultaneously that may help to reduce uncertainties due to renormalization scheme freedom.

Indeed one can factor out the renormalization scheme freedom to large extent by introducing an effective scheme with definitions of effective quantities a, m_q^2, m_g^2 through the relations

$$
\begin{aligned}
-Q^2\frac{d}{dQ^2}\Pi(Q^2) &= 1 + a(Q^2)\,, \\
-m_s^2(M_\tau^2)Q^2\frac{d}{dQ^2}\Pi_{mg}(Q^2) &= m_g^2(M_\tau^2)C_g(Q^2)\,, \\
m_s^2(M_\tau^2)\Pi_{mq}(Q^2) &= m_q^2(M_\tau^2)C_q(Q^2)\,.
\end{aligned}
$$

Here $C_{q,g}(Q^2)$ are coefficient functions of mass corrections. They are normalized by the requirement $C_{q,g}(M_\tau^2) = 1$.

In terms of the $\overline{\mathrm{MS}}$-scheme quantities $\alpha_s \equiv \alpha_s(M_\tau^2)$ and $m_s \equiv m_s(M_\tau^2)$ the effective parameters read (finite RG transformation)

$$a(M_\tau^2) = \frac{\alpha_s}{\pi} + k_1 \left(\frac{\alpha_s}{\pi}\right)^2 + k_2 \left(\frac{\alpha_s}{\pi}\right)^3 + k_3 \left(\frac{\alpha_s}{\pi}\right)^4 + \mathcal{O}(\alpha_s^5)$$

$$m_g^2(M_\tau^2) = m_s^2(M_\tau^2)(1 + \frac{5}{3}\frac{\alpha_s}{\pi} + k_{g1} \left(\frac{\alpha_s}{\pi}\right)^2 + k_{g2} \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4))$$

$$m_q^2(M_\tau^2) = m_s^2(M_\tau^2)(1 + \frac{7}{3}\frac{\alpha_s}{\pi} + k_{q1} \left(\frac{\alpha_s}{\pi}\right)^2 + k_{q2} \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4))$$

Coefficients k_j contain PT information on the correlation function

One can think of $a(M_\tau^2)$, $m_q^2(M_\tau^2)$ and $m_g^2(M_\tau^2)$ as proper physical parameters relevant for the description of the τ system

The RG equations for the set of quantities $\{a(Q), C_q(Q), C_g(Q)\}$ are

$$Q^2 \frac{d}{dQ^2} a(Q^2) = \beta(a), \quad Q^2 \frac{d}{dQ^2} C_{g,q}(Q^2) = 2\gamma_{g,q}(a) C_{g,q}(Q^2) \,.$$

The RG functions $\beta(a)$ and $\gamma_{g,q}(a)$ are given by the expressions

$$\frac{-4\beta(a)}{9a^2} = 1 + 1.78a + 5.24a^2 + 64.1a^3$$

$$\frac{-\gamma_g(a)}{a} = 1 + 4.03a + 17.45a^2 + 200.6a^3$$

$$\frac{-\gamma_q(a)}{a} = 1 + 4.78a + 32.99a^2 + 385.3a^3$$

Convergence is slow.

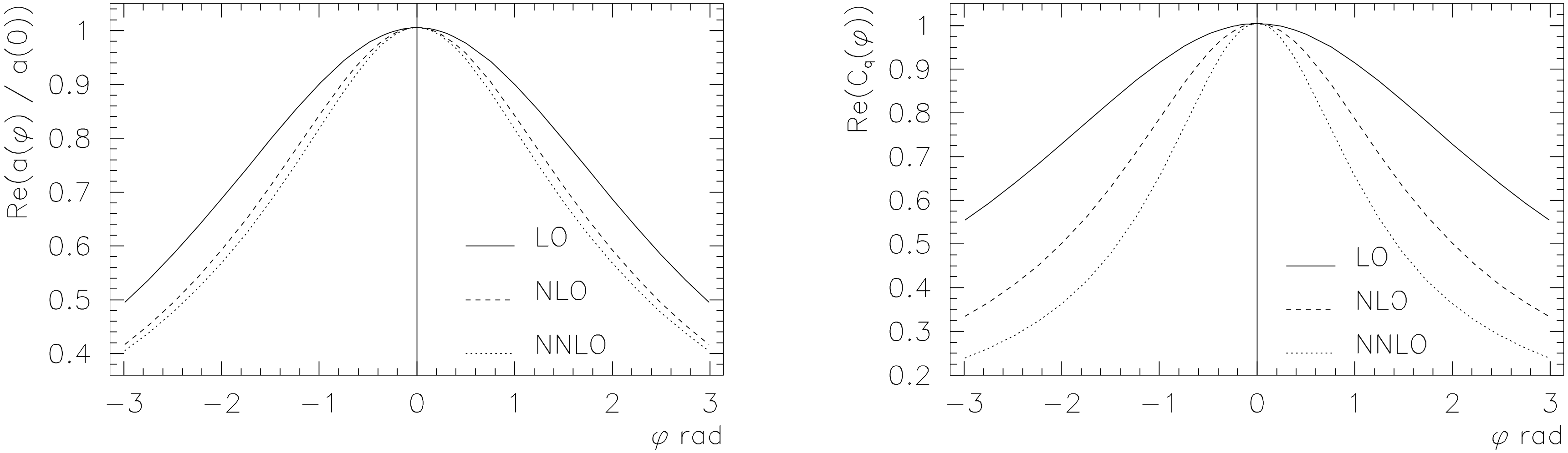


Figure 1: $a(\phi)$ and $C_q(\phi)$ on a contour calculated at LO, NLO and NNLO (left: $a(\phi)$; right: $C_q(\phi)$; real parts only)

One sees that $a(\phi)$ converges well in the entire range $-\pi < \phi < \pi$
$C_g(\phi)$ (not shown) related to the contributions of spin one particles is rather similar to $a(\phi)$
$C_q(\phi)$ is much worse. It seems γ_q has shown up an asymptotic growth in the NNLO which limits the precision of PT results.

Numerical value for m_s is extracted from the difference between Cabibbo-favored and Cabibbo-suppressed rates

$$\delta R_\tau^{kl} = \frac{R_{\tau S=0}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau S=1}^{kl}}{|V_{us}|^2}$$

The result of theory prediction reads

$$\delta R_\tau^{kl} = 3S_{EW}\left(6\frac{m_s^2}{M_\tau^2}(\omega_q A_{kl} + \omega_g B_{kl}) - 4\pi^2 \frac{m_s}{M_\tau}\frac{\langle \bar{s}s\rangle}{M_\tau^3} T_{kl}\right)$$

with $S_{EW} = 1.0194$. Here $m_{q,g}^2 = \omega_{q,g} m_s^2$ with $\omega_q = 1.73 \pm 0.04$, $\omega_g = 1.42 \pm 0.03$. Numerical values for A_{kl} and B_{kl} are found by numerical integration along the contour while T_{kl} can be found analytically

Experimental moments and extracted m_s mass

(k,l)	$(\delta R_\tau^{kl})^{\rm exp}$	$m_s(M_\tau^2)$ MeV
$(0,0)$	0.394 ± 0.137	$130 \pm \delta_{00}^{\rm th}(=6)$
$(1,0)$	0.383 ± 0.078	$111 \pm \delta_{10}^{\rm th}(=?)$
$(2,0)$	0.373 ± 0.054	$95 \pm \delta_{20}^{\rm th}(=?)$

Theoretical prediction for the moment $(0,0)$ is the most reliable from PT point of view as $\delta_{20}^{\rm th}(=?) > \delta_{10}^{\rm th}(=?) > \delta_{00}^{\rm th} = 6$ MeV. The higher moments with $(1 - s/M_\tau^2)^k$ have an uncontrollable admixture of higher condensates that makes them strongly nonperturbative and unreliable for applications based on PT.

Finally, extraction of the numerical value for $m_s(M_\tau^2)$ reads

$$m_s(M_\tau^2) = 130 \pm 27_{\rm exp} \pm 3_{\langle \bar{s}s \rangle} \pm 6_{\rm th} \ {\rm MeV}.$$

4. Summary

High order PT analysis allows for accurate description of precision data available in τ decays. However series converge slowly since expansion parameter is large $\alpha_s(M_\tau^2) \approx 0.35$ that requires modification of FOPT:

- resummation of different kinds
- manipulation with schemes for close observables avoiding artificial intermediaries as $\overline{\mathrm{MS}}$ quantities

One heavily uses RG that is powerful and flexible tool of analysis. Relating different scales and different schemes relevant for different physical systems (like $\alpha_s(M_\tau^2)|_\tau \approx 0.35$ vs $\alpha_s(M_Z^2)|_{Z\to b\bar{b}} \approx 0.12$)
Renorm Group plays a key part in checking whether particles interact according to SM rules