NEW RESULTS OF LARGE-ORDER INVESTIGATION IN DYNAMIC MODELS

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Standard models of critical dynamics with Gibbsian static limit may be described by the generic Langevin equation

$$\frac{\partial \varphi_a}{\partial t} + (\alpha_{ab} + \beta_{ab}) \frac{\delta S}{\delta \varphi_b} = \xi_a \,, \tag{2.1}$$

S is the static "action" (effective Hamiltonian). ξ is a Gaussian random field, $< \xi >= 0$

$$\langle \xi_a(t, \mathbf{x}) \xi_b(t', \mathbf{x}') \rangle = 2\alpha_{ab} \delta(t - t') \delta(\mathbf{x} - \mathbf{x}') \tag{2.2}$$

where:

$$\alpha^\top = \alpha \,, \qquad \beta^\top = -\beta \,, \qquad \sum_\alpha \frac{\delta \beta_{ab}}{\delta \varphi_a} = 0 \,. \tag{2.3}$$

Let us list the static actions S and the parameters α and β for models A-H:

Model A:

$$S = \frac{1}{2}\nabla\phi\nabla\phi + \frac{g}{4!}\phi^4 . \qquad \alpha = D/2,$$

where the correlator D is a constant.

Model B:

$$S = \frac{1}{2}\nabla\phi\nabla\phi + \frac{g}{4!}\phi^4 . \qquad \alpha = -\lambda\nabla^2 ,$$

where λ is a constant.

Model C:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2} + \frac{1}{2}v_2 m\phi^2 . \qquad \alpha = \begin{pmatrix} \Gamma & 0 \\ 0 & -\lambda\nabla^2 \end{pmatrix},$$

where m is an additional scalar field; Γ, λ are constants; v_2 is an additional coupling constant.

Model D:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2} + \frac{1}{2}\, v_2 m\phi^2 . \qquad \alpha = \begin{pmatrix} -\lambda\nabla^2 & 0 \\ 0 & -\lambda_1\nabla^2 \end{pmatrix},$$

where λ and λ_1 are constants.

Model F:

$$S = |\nabla\psi|^2 + \frac{g}{6}|\psi|^4 + \frac{m^2}{2} + v_2 m|\psi|^2 .$$

$$\alpha = \begin{pmatrix} 0 & \lambda_\psi & 0 \\ \lambda_\psi & 0 & 0 \\ 0 & 0 & -\lambda_m\nabla^2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & iv_3 & iv_4\psi \\ -iv_3 & 0 & -iv_4\psi^* \\ -iv_4\psi^* & iv_4\psi & 0 \end{pmatrix},$$

where ψ is a complex-valued field; λ_ψ, λ_m are constants; v_i $(i = 2, 3, 4)$ are additional coupling constants.

Model E is F model with $v_2 = v_3 = 0$.
Model G: There are two real vector fields forming the field φ by prescription: $\varphi_a = \phi_a$ and $\varphi_{3+a} = m_a$, where $a = 1, 2, 3$. Then

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{m^2}{2}, \qquad \alpha = \begin{pmatrix} \lambda_\phi & 0 \\ 0 & -\lambda_m \nabla^2 \end{pmatrix},$$
$$\beta_{ab} = 0, \quad \beta_{a\,3+b} = v_2 \epsilon_{abc}\phi_c, \quad \beta_{3+a\,3+b} = v_2 \epsilon_{abc} m_c,$$

$a, b, c = 1, 2, 3$.
Model H:

$$S = \frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}\phi^4 + \frac{c}{2}\mathbf{v}_\perp^2,$$
$$\alpha = \begin{pmatrix} -\lambda_\phi \nabla^2 & 0 \\ 0 & -\lambda_v \nabla^2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & v_2 \overrightarrow{\nabla}\phi \\ -v_2 \overleftarrow{\nabla}\phi & 0 \end{pmatrix},$$

where the transverse projection operator for the vector field **v** is implied; λ_ψ, λ_m, v_2 are constants.
In the MSR approach the dynamic action for the model (2.1)– (2.3) may be written symbolically [4] as

$$\overline{S} = -\varphi'_a \alpha_{ab} \varphi'_b + \varphi'_a \left[\frac{\partial \varphi_a}{\partial t} + (\alpha_{ab} + \beta_{ab}) \frac{\delta S}{\delta \varphi_b} \right] . \qquad (2.4)$$

the boundary conditions $\varphi(t = -\infty) = 0$ and arbitrary $\varphi(T)$, $\varphi'(t = -\infty)$, $\varphi'(T)$. T is a maximal time in the consideration.
The Nth order contribution to perturbation expansion in g of the

k-point Green function

$$G_k^{[N]} = \frac{1}{2\pi i}\oint \frac{du}{u}\,\frac{\iint \mathcal{D}\varphi\mathcal{D}\varphi'\,\Phi_{a_1}(t_1,\mathbf{x_1})\ldots\Phi_{a_k}(t_k,\mathbf{x_k})\,e^{-\overline{S}\,-\,N\ln u}}{\iint \mathcal{D}\varphi\mathcal{D}\varphi'\,e^{-\overline{S}_0}}\,, \tag{3.1}$$

$\Phi = \{\varphi,\,\varphi'\}$. The stationarity equations of the method of steepest descent

$$\frac{\delta\overline{S}}{\delta\varphi_a} = -\frac{\partial\varphi'_a}{\partial t} + \varphi'_b\,(\alpha_{bc}+\beta_{bc})\,\frac{\delta^2 S}{\delta\varphi_c\delta\varphi_a} + \varphi'_b\,\frac{\delta\beta_{bc}}{\delta\varphi_a}\,\frac{\delta S}{\delta\varphi_c} = 0, \tag{3.2}$$

$$\frac{\delta\overline{S}}{\delta\varphi'_a} = -2\alpha_{ab}\varphi'_b + \frac{\partial\varphi_a}{\partial t} + (\alpha_{ab}+\beta_{ab})\,\frac{\delta S}{\delta\varphi_b} = 0\,, \tag{3.3}$$

with the boundary condition

$$\varphi'(T, \mathbf{x}) = 0\,. \tag{3.4}$$

The dynamic instanton φ_D for the basic field is the nontrivial solution of the equation

$$-\frac{\partial \varphi_a}{\partial t} + (\alpha_{ab} - \beta_{ab})\,\frac{\delta S}{\delta \varphi_b} = 0. \tag{3.5}$$

$\varphi_D(T, \mathbf{x}) = \varphi_{st}(\mathbf{x})$, where φ_{st} is the static instanton solution for the static action S. An iterative solution of the instanton equation (3.5) exists. Then

$$\overline{S}\,(\varphi_D, \varphi'_D) = S\,(\varphi_{st})\ .$$

Thus the dynamic model at leading order in N are determined by the static instanton solution which leads to

$$F^{[N]} = C\,N!a_M^N N^b\,, \tag{3.6}$$

The developed turbulence is described by the equation

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + (\mathbf{v}\partial)\mathbf{v} = \xi,$$

for $d = 2$: $< \xi\xi >= \mathrm{const}\Delta\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$. Then the static limit exists, it is Gibbsian $\exp(-v^2/2)$. There is no nonzero static instanton.

There is a model with derivative in the interaction.

The advection of a passive scalar field $\varphi(\mathbf{x}, t)$ is described by a stochastic equation:

$$\partial_t \varphi - \nu \Delta \varphi + g\partial_i(\mathbf{v}_i \varphi) = \xi(\mathbf{x}, t), \tag{5.1}$$

where ν is a molecular diffusivity coefficient, ξ is an Gaussian scalar noise, $\mathbf{v}(\mathbf{x}, t)$ is a velocity field. The field $\mathbf{v}(\mathbf{x}, t)$ obeys a Gaussian distribution

$$\mathbf{D}_v^{ij}(\mathbf{x}-\mathbf{x}') \equiv \langle \mathbf{v}_i(\mathbf{x},t)\mathbf{v}_j(\mathbf{x}',t')\rangle = D_0\delta(t-t')\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{P_{ij}^{\perp}(\mathbf{k})+\alpha P_{ij}^{\parallel}(\mathbf{k})}{(\mathbf{k}^2+m^2)^{\beta(\epsilon)}}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \tag{5.2}$$

or for the model with the “frozen” velocity

$$\mathbf{D}_v^{ij}(\mathbf{x}-\mathbf{x}') \equiv \langle \mathbf{v}_i(\mathbf{x},t)\mathbf{v}_j(\mathbf{x}',t')\rangle = D_0\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{P_{ij}^{\perp}(\mathbf{k})+\alpha P_{ij}^{\parallel}(\mathbf{k})}{(\mathbf{k}^2+m^2)^{\beta(\epsilon)}}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \tag{5.3}$$

The Lagrangian variables

$$< \phi(\mathbf{x},s)\phi'(\mathbf{y},t) >= \Theta(s-t)M \int\limits_{\mathbf{c}(t)=\mathbf{y}}^{\mathbf{c}(s)=\mathbf{x}} D\mathbf{c}D\mathbf{c}' \exp \int_t^s d\tau(-\nu\mathbf{c}'^2 + i\mathbf{c}'\dot{\mathbf{c}} + ig\mathbf{c}'\mathbf{v}(\mathbf{c},\tau)), \quad (5.4)$$

The stationarity equations in these variables have a form

$$-i\dot{\mathbf{c}}' = u\nu \int d\tau' \mathbf{c}' \frac{\partial D(\mathbf{c}-\mathbf{c}(\tau'))}{\partial(\mathbf{c}-\mathbf{c}(\tau'))} \mathbf{c}(\tau')',$$

$$i\dot{\mathbf{c}} = u\nu \int d\tau' D(\mathbf{c}-\mathbf{c}(\tau'))\mathbf{c}(\tau')' + 2\nu\mathbf{c}'.$$

These equations have a solution

$$\mathbf{p}(\tau) = \frac{i\dot{\mathbf{q}}(\tau)}{2\nu - g\nu D_v(\mathbf{q}(\tau))}, \dot{\mathbf{q}}(\tau) = \frac{I_1(\mathbf{x})}{T}\sqrt{2\nu - g\nu D_v(\mathbf{q}(\tau))},$$

$$I_1(\mathbf{x}) = \int_0^x \frac{dz}{\sqrt{2\nu - g\nu D_v(z)}}$$

where $\mathbf{p} = \mathbf{c}_1' - \mathbf{c}_2'$, $\mathbf{q} = \mathbf{c}_1 - \mathbf{c}_2$, or

$$c'(\tau) = \frac{i\partial_\tau c(\tau)}{\nu}. \quad (5.5)$$

$$\int_0^c \frac{dy}{\int_0^{|x|} dz D(z-y)} = -\frac{u_{st}\tau}{\nu}, \qquad c \equiv c_{st}(\tau). \quad (5.6)$$

The hight-order asymptotes in the Kraichnan model show the convergent perturbation series.

Recently was calculated for dynamic index z in the model A

$$z = 2 + R\eta, \tag{6.1}$$

$$R = (h-1)\left[1 - 0.188483\epsilon + \left(-0.0999529 + \frac{b_1 n + b_0}{(n+8)^2}\right)\epsilon^2\right] + O(\epsilon^3),$$

$b_1 = 4.772294768,\ b_0!.49187274,\ h = 6\ln(4/3),$

$$\eta = \frac{(n+2)\epsilon^2}{2(n+8)^2}\left[1 + \frac{\epsilon}{4(n+8)^2}(-n^2+56n+272) + \frac{\epsilon^2}{16(n+8)^4}(-5n^4-230n^3+1124n^2+17920n+46144-384(n+8)(5n+22)\zeta(3))\right] + O(\epsilon^5), \quad (6.2)$$

Table: $z = z^{(k)}\epsilon^k$.

$z_\epsilon^{(k)}$	$n = 1$	$n = 2$	$n = 3$
$z_\epsilon^{(2)}$	0.0134462	0.0145218	0.0150019
$z_\epsilon^{(3)}$	0.0110364	0.0110588	0.0105317
$z_\epsilon^{(4)}$	-0.0055807	-0.0052671	-0.0049778

Table: Direct calculation of z in ϵ expansion.

d	N	$n=1$	$n=2$	$n=3$
2	2	2.0537846	2.0580874	2.0600076
	3	2.1420758	2.1465574	2.1442609
	4	2.0527838	2.0622836	2.0646160
3	2	2.0134462	2.0145218	2.0150019
	3	2.0244826	2.0255806	2.0255336
	4	2.0189018	2.0203135	2.0205558

Table: z, PBL P_2^2-approximation ($b = b_z + 1.5$).

d	N	$n = 1$	$n = 2$	$n = 3$
2	2	2.0537846	2.0580874	2.0600076
	3	2.0313245	2.0371809	2.0424021
	4	2.0195447	2.0232205	2.0260295
3	2	2.0134462	2.0145218	2.0150019
	3	2.0151405	2.0169805	2.0181449
	4	2.0111743	2.0127178	2.0136547

Table: z, PBL P_1^3-approximation ($b = b_z + 1.5$).

d	N	$n=1$	$n=2$	$n=3$
2	2	2.0537846	2.0580874	2.0600076
	3	2.1420758	2.1465574	2.1442609
	4	2.0988800	2.1045005	2.1043143
3	2	2.0134462	2.0145218	2.0150019
	3	2.0244826	2.0255806	2.0255336
	4	2.0208646	2.0220921	2.0222247

Table: z PBL $P^4_{1,-1/a}$-approximation ($b = b_z$).

d	N	$n = 1$	$n = 2$	$n = 3$
2	2	2.0537846	2.0580874	2.0600076
	3	2.1420758	2.1465574	2.1442609
	4	2.1274836	2.1322986	2.1303831
3	2	2.0134462	2.0145218	2.0150019
	3	2.0244826	2.0255806	2.0255336
	4	2.0229318	2.0240719	2.0240707

Table: z CM ($b = b_z + 1.5$).

d	N	$n = 1$	$n = 2$	$n = 3$
2	2	2.0064932	2.0073128	2.0078256
	3	2.0198686	2.0220831	2.0233094
	4	2.0369831	2.0406409	2.0423686
3	2	2.0032593	2.0036430	2.0038735
	3	2.0081436	2.0089683	2.0093953
	4	2.0127529	2.0138750	2.0143549

Table: $z^{(k)}_{asymp}$ (6.7).

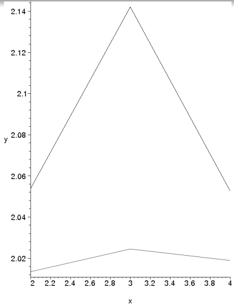
$z^{(k)}_{asymp}$	$n = 1$	$n = 2$	$n = 3$
$z^{(2)}_{asymp}$	0,0000034	0,0000011	0,0000003
$z^{(3)}_{asymp}$	-0,0000141	-0,0000051	-0,0000017
$z^{(4)}_{asymp}$	0,0000516	0,0000195	0,0000067

Table: Results for z (d=3).

d=3	$n = 1$	$n = 2$	$n = 3$
P_2^2	$2.011 \pm 0, 012$	2.013 ± 0.012	2.014 ± 0.011
P_1^3	$2.021 \pm 0, 006$	2.022 ± 0.005	2.022 ± 0.005
$P_{1,-1/a}^4$	2.023 ± 0.006	2.024 ± 0.005	2.024 ± 0.005
CM	$2.0133 \ {}^{+0.011}_{-0.0}$	2.014 ± 0.011	2.014 ± 0.011

Table: Results for z (d=2).

d=2	$n=1$	$n=2$	$n=3$
P_2^2	$2.020 \pm 0,045$	2.023 ± 0.053	2.026 ± 0.055
P_1^3	$2.100 \pm 0,089$	2.105 ± 0.084	2.104 ± 0.080
$P_{1,-1/a}^4$	2.130 ± 0.089	2.134 ± 0.084	2.132 ± 0.080
CM	2.037 ± 0.033	2.041 ± 0.040	2.042 ± 0.041



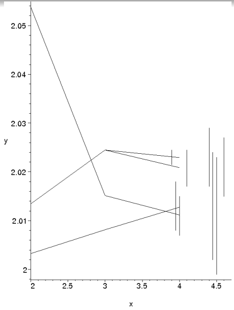
Borel-transformation

$$A(\epsilon) = \sum_{k\geq 0} A_k \epsilon^k, \tag{6.3}$$

$$B(\epsilon) = \sum_{k\geq 0} B_k \epsilon^k, \quad B_k = \frac{A_k}{\Gamma(k+b+1)}, \tag{6.4}$$

$$A(\epsilon) = \int_0^{+\infty} t^b e^{-t} B(\epsilon t) dt. \tag{6.5}$$

$$u(\epsilon) = \frac{\sqrt{1 + a\epsilon} - 1}{\sqrt{1 + a\epsilon} + 1} \quad \Leftrightarrow \quad \epsilon(u) = \frac{4u}{a(u-1)^2}. \tag{6.6}$$



Why the results are inconsistent?

$$z^{(k)}_{asymp} \approx (h-1)\eta^{(k)}_{asymp}. \tag{6.7}$$

$d = 3$, $z = 2.1 \pm 0.1$ [16]. $z = 1.97 \pm 0.08$ [17],
$z = 2.04 \pm 0.03$ [18], $z = 2.04 \pm 0.01$ [19] $z = 2.032 \pm 0.004$ [20].
$d = 2$, $z = 2.14 \pm 0.02$ [21], $z = 2.13 \pm 0.03$ [22],
$z = 2.076 \pm 0.005$ [23], $z = 2.24 \pm 0.04$ [24], $z = 2.24 \pm 0.07$ [25],
$z = 2.16 \pm 0.04$ [26] $z = 2.1667 \pm 0.0005$ [27]. $d = 2$ $z = 2.125$ [28].

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