Scattering amplitudes/Wilson loop duality in $\mathcal{N} = 4$ super Yang-Mills theory and beyond

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Based on work in collaboration with

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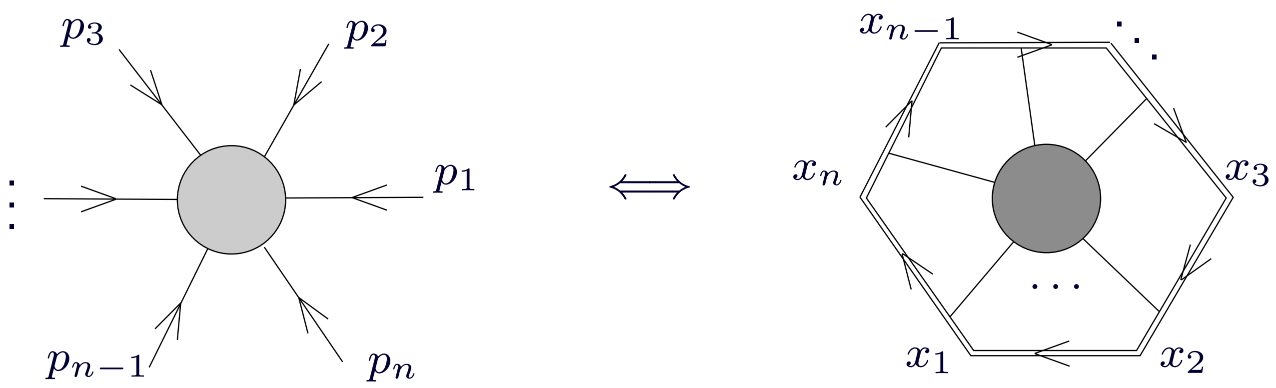
Outline

- On-shell gluon scattering amplitudes
- Iterative structure at weak/strong coupling in $\mathcal{N}=4$ SYM
- Dual conformal invariance – hidden symmetry of planar amplitudes
- Maximally helicity violating (MHV) scattering amplitude/Wilson loop duality in $\mathcal{N}=4$ SYM



$\langle 0|S|1^-2^-3^+\dots n^+\rangle$ $\qquad$ $\langle 0|\,\mathrm{tr}\,\mathrm{P}\exp\left(i\oint_C dx\cdot A(x)\right)|0\rangle$

Why $\mathcal{N}=4$ super Yang-Mills theory is interesting?

✔ Four-dimensional gauge theory with extended spectrum of physical states/symmetries

2 gluons with helicity $\pm 1\,,$ *6 scalars with helicity* $0\,,$ *8 gaugino with helicity* $\pm\frac{1}{2}$

all in the adjoint of the $SU(N_c)$ gauge group

✔ All classical symmetries survive at quantum level:

- ✗ Beta-function vanishes to all loops $\Longrightarrow$ the theory is (super)conformal!
- ✗ The theory contains only two free parameters: 't Hooft coupling constant $\lambda = g^2_{\rm YM} N_c$ and the number of colors N_c

✔ Why $\mathcal{N}=4$ SYM theory is fascinating?

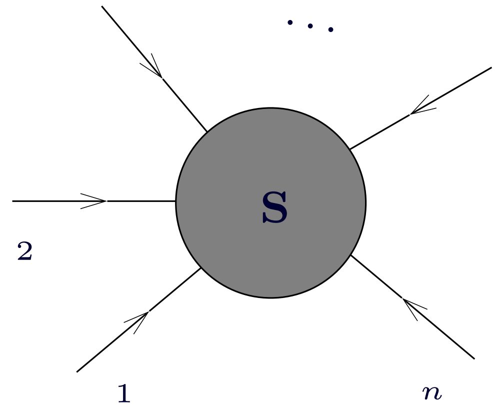
- ✗ *At weak coupling*, the number of contributing Feynman integrals is *MUCH* bigger compared to QCD ... but the final answer is *MUCH* simpler (examples to follow)
- ✗ *At strong coupling*, the conjectured AdS/CFT correspondence [Maldacena],[Gubser,Klebanov,Polyakov],[Witten]

Strongly coupled planar $\mathcal{N}=4$ *SYM* $\Longleftrightarrow$ *Weakly coupled string theory on* $\mathrm{AdS}_5 \times \mathrm{S}^5$

- ✗ Final goal (dream):

$\mathcal{N}=4$ SYM theory is a unique example of the four-dimensional gauge theory that can be/ should be/ would be solved exactly for arbitrary value of the coupling constant!!!

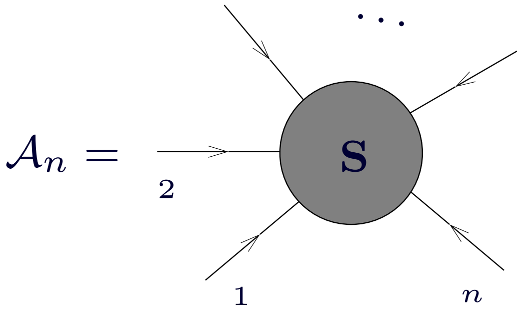
Why scattering amplitudes?

$\mathcal{A}_n \;=$ 

- ✔ On-shell matrix elements of $S-$matrix:
  - ✗ Probe (hidden) symmetries of gauge theory
  - ✗ Are independent on gauge choice
- ✔ Simpler than QCD amplitudes but they share many of the same properties
- ✔ In planar $\mathcal{N} = 4$ SYM theory they seem to have a remarkable structure
- ✔ All-order conjectures and a proposal for strong coupling via AdS/CFT (non-trivial tests)
- ✔ New symmetries (integrability) and a new duality - hints at much more to be understood

On-shell gluon scattering amplitudes in $\mathcal{N}=4$ SYM

- ✔ Gluon scattering amplitudes in $\mathcal{N}=4$ SYM



  - ✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($(p_i^\mu)^2 = 0$), helicity ($h = \pm 1$), color ($a$)
  - ✗ Suffer from IR divergences $\mapsto$ require IR regularization
  - ✗ Close cousin to QCD gluon amplitudes

- ✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1} T^{a_2} \dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1, p_2, \dots, p_n) + \text{[Bose symmetry]}$$

  - ✗ Color-ordered amplitudes are classified according to their helicity content $h_i = \pm 1$
  - ✗ Supersymmetry relations:

$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A^{--+\dots+}\,, \qquad A^{(\mathrm{next-MHV})} = A^{---+\dots+}\,, \quad \dots$$

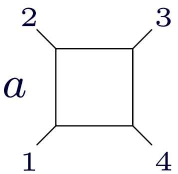
  - ✗ The $n = 4$ and $n = 5$ planar gluon amplitudes are all MHV

$$\{A_4^{++--}, \quad A_4^{+-+-}, \quad \dots\}, \quad \{A_5^{+++--}, \quad A_5^{+-+--}, \quad \dots\}$$

  - ✗ *Weak/strong coupling corrections to all MHV amplitudes are described by a single function of 't Hooft coupling and kinematical invariants!* [Parke,Taylor]

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$M_4 \equiv \mathcal{A}_4/\mathcal{A}_4^{(\text{tree})} = 1 + a$$



$$+ O(a^2)\,, \qquad a = \frac{g_{\rm YM}^2 N_c}{8\pi^2}$$

[Green,Schwarz,Brink'82]

All-loop planar amplitudes can be split into (universal) IR divergent and (nontrivial) finite part

$$M_4(s,t) = \text{Div}(s,t,\epsilon_{\rm IR})\,\text{Fin}(s/t)$$

✔ IR divergences appear to all loops as poles in $\epsilon_{\rm IR}$ (in dim.reg. with $D = 4 - 2\epsilon_{\rm IR}$)

✔ IR divergences exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],[GK,Radyushkin]...

$$\text{Div}(s,t,\epsilon_{\rm IR}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\rm cusp}^{(l)}}{(l\epsilon_{\rm IR})^2} + \frac{G^{(l)}}{l\epsilon_{\rm IR}}\right)\left[(-s/\mu^2)^{l\epsilon_{\rm IR}} + (-t/\mu^2)^{l\epsilon_{\rm IR}}\right]\right\}$$

✔ *IR divergences* are in one-to-one correspondence with *UV divergences* of cusped Wilson loops

[Ivanov,GK,Radyushkin'86]

$\Gamma_{\rm cusp}(a) = \sum_l a^l \Gamma_{\rm cusp}^{(l)} =$ *cusp anomalous dimension of Wilson loops*

$G(a) = \sum_l a^l G_{\rm cusp}^{(l)} =$ *collinear anomalous dimension*

✔ Scattering amplitudes satisfy *IR renormalization group equations*

$$\mu\frac{d}{d\mu}\ln M_4 = 2\Gamma_{\rm cusp}(a)\left[\ln(-s/\mu^2) + \ln(-t/\mu^2)\right] + G(a) + O(1/\epsilon)$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling II

✔ *What about finite part of the amplitude* $\mathrm{Fin}(s/t)$*? Does it have a simple structure?*

$$\text{Fin}_{\text{QCD}}(s/t) = [\text{4 pages long mess}]\,, \qquad \text{Fin}_{\mathcal{N}=4}(s/t) = \text{BDS conjecture}$$

✔ Bern-Dixon-Smirnov (BDS) conjecture:

$$\text{Fin}(s/t) = a\left[\tfrac{1}{2}\ln^2(s/t) + 4\zeta_2\right] + O(a^2) \quad \stackrel{\text{all loops}}{\Longrightarrow} \quad \tfrac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2(s/t) + \text{const}$$

✗ Compared to QCD,

(i) the complicated functions of s/t are replaced by the elementary function $\ln^2(s/t)$;

(ii) no higher powers of logs appear in Fin(s/t) at higher loops;

(iii) the coefficient of $\ln^2(s/t)$ is determined by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$ just like the coefficient of the double IR pole.

✗ The conjecture has been verified up to three loops [Anastasiou,Bern,Dixon,Kosower'03],[Bern,Dixon,Smirnov'05]

✗ A similar conjecture exists for n-gluon MHV amplitudes [Bern,Dixon,Smirnov'05]

✗ It has been confirmed for $n=5$ at two loops [Cachazo,Spradlin,Volovich'04], [Bern,Czakon,Kosower,Roiban,Smirnov'06]

✗ Agrees with the strong coupling prediction from the AdS/CFT correspondence [Alday,Maldacena'06]

✔ Surprising features of the finite part of the MHV amplitudes in planar $\mathcal{N}=4$ SYM:

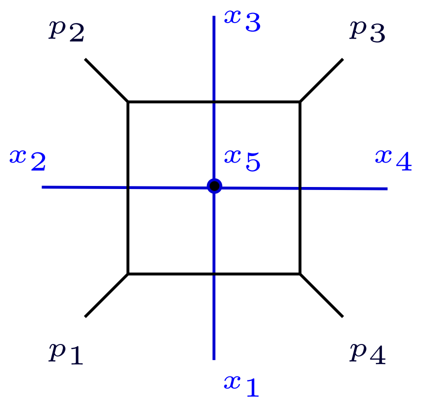
Why should finite corrections exponentiate? and be related to the cusp anomaly of Wilson loop?

Dual conformal symmetry

Examine one-loop 'scalar box' diagram

- ✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_5\; x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Check conformal invariance by inversion $x_i^\mu \to x_i^\mu / x_i^2$

[Broadhurst],[Drummond,Henn,Smirnov,Sokatchev]

- ✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!
- ✔ The symmetry *is not related* to conformal $SO(2,4)$ symmetry of $\mathcal{N}=4$ SYM
- ✔ All scalar integrals contributing to A_4 up to four loops possess the dual conformal invariance!
- ✔ The dual conformal symmetry allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,GK,Sokatchev],[Alday,Maldacena]
- ✔ Dual conformality is slightly broken by the infrared regulator
- ✔ For *planar* integrals only!

From gluon amplitudes to Wilson loops

Properties of gluon scattering amplitudes in $\mathcal{N}=4$ SYM:

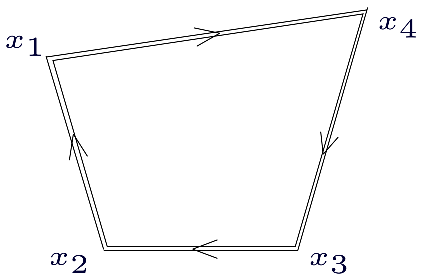
(1) IR divergences of M_4 are in one-to-one correspondence with UV div. of *cusped Wilson loops*

(2) Perturbative corrections to M_4 possess a hidden *dual conformal symmetry*

☞ *Is it possible to identify the object in $\mathcal{N}=4$ SYM for which both properties are manifest ?*

Yes! The expectation value of light-like Wilson loop in $\mathcal{N}=4$ SYM [Drummond-Henn-GK-Sokatchev]

$$W(C_4) = \frac{1}{N_c}\langle 0|\mathrm{Tr\,P}\exp\left(ig\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle\,, \qquad C_4 =$$



✔ Gauge invariant functional of the integration contour C_4 in Minkowski space-time

✔ The contour is made out of 4 light-like segments $C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ joining the cusp points x_i^μ

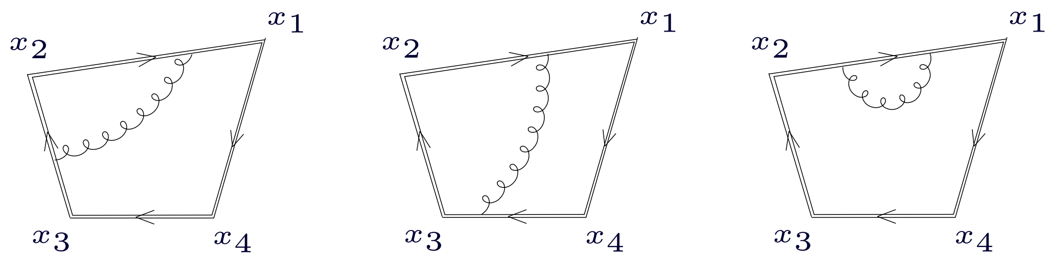
$$x_i^\mu - x_{i+1}^\mu = p_i^\mu = \text{on-shell gluon momenta}$$

✔ The contour C_4 has four light-like cusps $\mapsto$ $W(C_4)$ has UV divergencies

✔ Conformal symmetry of $\mathcal{N}=4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates x^μ

MHV scattering amplitudes/Wilson loop duality I

The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$)

$\ln W(C_4) =$ 

$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{{\epsilon_{\rm UV}}^2} \left[(-x_{13}^2\mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2\mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2}\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln M_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{{\epsilon_{\rm IR}}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2}\ln^2\left(\frac{s}{t}\right) + \text{const} \right\} + O(g^4)$$

✔ Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$:

$$x_{13}^2\,\mu^2 := s/\mu_{\rm IR}^2\,, \qquad x_{24}^2\,\mu^2 := t/\mu_{\rm IR}^2\,, \qquad x_{13}^2/x_{24}^2 := s/t$$

- ☞ UV divergencies of the light-like Wilson loop match IR divergences of the gluon amplitude
- ☞ the finite $\sim \ln^2(s/t)$ corrections coincide to one loop!

MHV scattering amplitudes/Wilson loop duality II

MHV amplitudes are dual to light-like Wilson loops [Drummond,Henn,GK,Sokatchev], [Brandhuber,Heslop,Travaglini]

$$\ln M_n^{(\mathrm{MHV})} = \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n-\text{(poly)gon}$$

- ✔ At strong coupling, the relation holds to leading order in $1/\sqrt{\lambda}$ [Alday,Maldacena]
- ✔ At weak coupling, the duality relation was verified for:
  - ✗ $n = 4$ (rectangle) to two loops [Drummond,Henn,GK,Sokatchev]
  - ✗ $n \geq 5$ to one loop [Brandhuber,Heslop,Travaglini]
  - ✗ $n = 5$ (pentagon) to two loops [Drummond,Henn,GK,Sokatchev]
- ✔ For arbitrary coupling, conformal symmetry of light-like Wilson loops in $\mathcal{N} = 4$ SYM + duality relation impose constraints on the finite part of the MHV amplitudes
- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the MHV amplitudes
  $\mathbb{D}$ = dilatations, $\mathbb{K}^\mu$ = special conformal transformations [Drummond,Henn,GK,Sokatchev]

$$\mathbb{D}\, F_n \equiv \sum_{i=1}^{n} (x_i \cdot \partial_{x_i}) F_n = 0$$

$$\mathbb{K}^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\mathrm{cusp}}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of MHV amplitudes

The consequences of the conformal Ward identity for the finite part of the Wilson loop/ MHV scattering amplitudes: [Drummond, Henn, GK, Sokatchev]

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
$\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\rm cusp}(a) \ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\rm cusp}(a) \sum_{i=1}^{5} \ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right) \ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

Exactly the functional forms of the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x^2_{13}x^2_{46}}{x^2_{14}x^2_{36}}, \qquad u_2 = \frac{x^2_{24}x^2_{15}}{x^2_{25}x^2_{14}}, \qquad u_3 = \frac{x^2_{35}x^2_{26}}{x^2_{36}x^2_{25}}$$

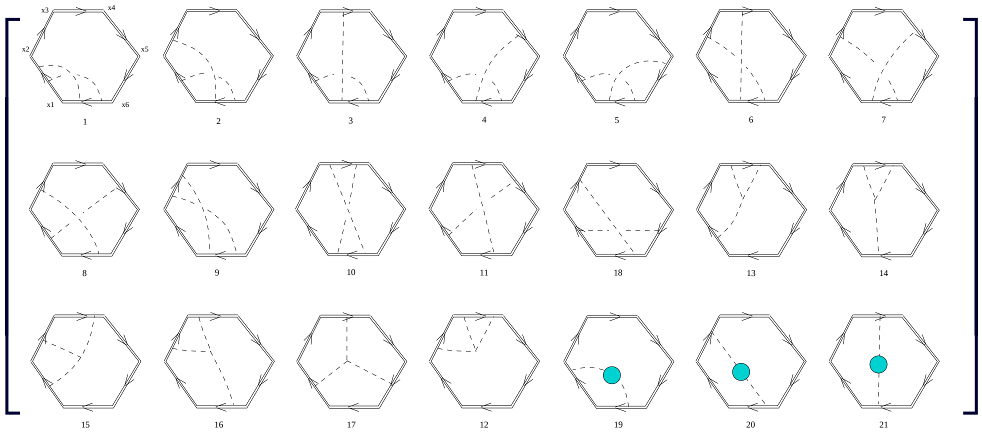
Hence the general solution of the Ward identity for $W(C_n)$ with $n \geq 6$ contains *an arbitrary function* of the conformal cross-ratios.

- ✔ The BDS ansatz is a solution of the conformal Ward identity for *arbitrary* n but does it actually work for $n \geq 6$ [Alday, Maldacena] [Bartels, Lipatov, Sabio Vera]? if not *what is a missing function of* $u_{1,2,3}$*?*

Discrepancy function

✔ We computed the two-loop hexagon Wilson loop $W(C_6)$ ... [Drummond, Henn, GK, Sokatchev'07]

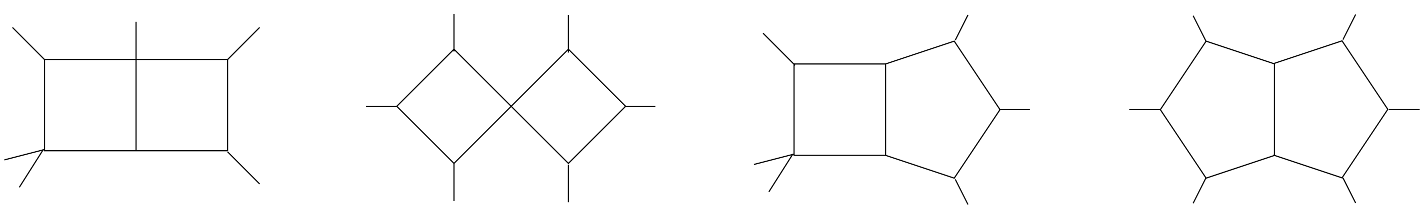
$$\ln W(C_6) = \left[\quad \right]$$



... and found a **discrepancy** $\ln W(C_6) \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

✔ Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich computed 6-gluon amplitude to 2 loops

$$\mathcal{M}_6^{(\mathrm{MHV})} =$$



$$+ \ldots$$

... and found a **discrepancy** $\ln \mathcal{M}_6^{(\mathrm{MHV})} \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

☞ The BDS ansatz **fails** for $n = 6$ starting from two loops.

☞ *What about Wilson loop duality?* $\ln \mathcal{M}_6^{(\mathrm{MHV})} \stackrel{?}{=} \ln W(C_6)$

6-gluon amplitude/hexagon Wilson loop duality

- ✔ Comparison between the DHKS discrepancy function $\Delta_{\rm WL}$ and the BDKRSVV results for the six-gluon amplitude $\Delta_{\rm MHV}$:

Kinematical point	(u_1, u_2, u_3)	$\Delta_{\rm WL} - \Delta_{\rm WL}^{(0)}$	$\Delta_{\rm MHV} - \Delta_{\rm MHV}^{(0)}$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	$< 10^{-5}$	-0.018 ± 0.023
$K^{(2)}$	$(0.547253, 0.203822, 0.88127)$	-2.75533	-2.753 ± 0.015
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74460	-4.7445 ± 0.0075
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.09138	4.12 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	9.72553	10.00 ± 0.50

evaluated for different kinematical configurations, e.g.

$$K^{(1)}: \quad x_{13}^2 = -0.7236200\,, \quad x_{24}^2 = -0.9213500\,, \quad x_{35}^2 = -0.2723200\,, \quad x_{46}^2 = -0.3582300\,, \quad x_{36}^2 = -0.4825841\,,$$
$$x_{15}^2 = -0.4235500\,, \quad x_{26}^2 = -0.3218573\,, \quad x_{14}^2 = -2.1486192\,, \quad x_{25}^2 = -0.7264904\,.$$

- ✔ Two nontrivial functions coincide with an accuracy $< 10^{-4}$!

✌ *The Wilson loop/MHV amplitude duality holds at $n = 6$ to two loops!!*

✌ *We expect that the duality relation should also hold for arbitrary n to all loops!!!*

Conclusions and recent developments

- ✔ MHV amplitudes in $\mathcal{N} = 4$ theory
  - ✗ possess the dual conformal symmetry both at weak and at strong coupling
  - ✗ Dual to light-like Wilson loops

  ... but what about NMHV, NNMHV, *etc.* amplitudes?

- ✔ This symmetry is a part of much bigger dual superconformal symmetry of all planar superamplitudes in $\mathcal{N} = 4$ SYM [Drummond,Henn,GK,Sokatchev]
  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Imposes non-trivial constraints on the loop corrections

- ✔ Dual superconformal symmetry is now exlained better through the AdS/CFT correspondence by a combined bosonic and fermionic T duality symmetry [Berkovits,Maldacena]

- ✔ What is the generalisation of the Wilson loop/amplitude duality beyond MHV?

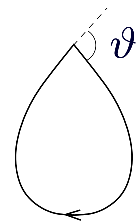
Back-up slides

What is the cusp anomalous dimension

- ✔ Cusp anomaly is a very ‘unfortunate’ feature of Wilson loops evaluated over an *Euclidean* closed contour with a cusp – generates the anomalous dimension [Polyakov’80]

$$\langle \mathrm{tr}\, \mathsf{P} \exp \left(i \oint_C dx \cdot A(x) \right) \rangle \sim (\Lambda_{\mathrm{UV}})^{\Gamma_{\mathrm{cusp}}(g,\vartheta)} \,,$$

$C =$



- ✔ A very ‘fortunate’ property of Wilson loop – the cusp anomaly controls the *infrared* asymptotics of scattering amplitudes in gauge theories [GK, Radyushkin’86]
  - ✗ The integration contour C is defined by the particle momenta
  - ✗ The cusp angle ϑ is related to the scattering angles in *Minkowski* space-time, $|\vartheta| \gg 1$

$$\Gamma_{\mathrm{cusp}}(g,\vartheta) = \vartheta\, \Gamma_{\mathrm{cusp}}(g) + O(\vartheta^0) \,,$$

- ✔ *The cusp anomalous dimension* $\Gamma_{\mathrm{cusp}}(g)$ is an ubiquitous observable in gauge theories: [GK’89]
  - ✗ Logarithmic scaling of anomalous dimensions of high-spin Wilson operators;
  - ✗ IR singularities of on-shell gluon scattering amplitudes;
  - ✗ Gluon Regge trajectory;
  - ✗ Sudakov asymptotics of elastic form factors;
  - ✗ ...

Four-gluon amplitude/Wilson loop duality in QCD

Finite part of four-gluon amplitude in QCD at two loops

$$\mathrm{Fin}_{\mathrm{QCD}}{}^{(2)}(s,t,u) = A(x,y,z) + O(1/N_c^2, n_f/N_c)$$

[Glover,Oleari,Tejeda-Yeomans'01]

with notations $x = -\frac{t}{s}$, $y = -\frac{u}{s}$, $z = -\frac{u}{t}$, $X = \log x$, $Y = \log y$, $S = \log z$

$$\begin{aligned} A \;=\; & \Big\{ \Big(48\,\mathrm{Li}_4(x) - 48\,\mathrm{Li}_4(y) - 128\,\mathrm{Li}_4(z) + 40\,\mathrm{Li}_3(x)\,X - 64\,\mathrm{Li}_3(x)\,Y - \frac{98}{3}\,\mathrm{Li}_3(x) + 64\,\mathrm{Li}_3(y)\,X - 40\,\mathrm{Li}_3(y)\,Y + 18\,\mathrm{Li}_3(y) \\
& + \frac{98}{3}\,\mathrm{Li}_2(x)\,X - \frac{16}{3}\,\mathrm{Li}_2(x)\,\pi^2 - 18\,\mathrm{Li}_2(y)\,Y - \frac{37}{6}\,X^4 + 28\,X^3\,Y - \frac{23}{3}\,X^3 - 16\,X^2\,Y^2 + \frac{49}{3}\,X^2\,Y - \frac{35}{3}\,X^2\,\pi^2 - \frac{38}{3}\,X^2 \\
& - \frac{22}{3}\,S\,X^2 - \frac{20}{3}\,X\,Y^3 - 9\,X\,Y^2 + 8\,X\,Y\,\pi^2 + 10\,X\,Y - \frac{31}{12}\,X\,\pi^2 - 22\,\zeta_3\,X + \frac{22}{3}\,S\,X + \frac{37}{27}\,X + \frac{11}{6}\,Y^4 - \frac{41}{9}\,Y^3 - \frac{11}{3}\,Y^2\,\pi \\
& - \frac{22}{3}\,S\,Y^2 + \frac{266}{9}\,Y^2 - \frac{35}{12}\,Y\,\pi^2 + \frac{418}{9}\,S\,Y + \frac{257}{9}\,Y + 18\,\zeta_3\,Y - \frac{31}{30}\,\pi^4 - \frac{11}{9}\,S\,\pi^2 + \frac{31}{9}\,\pi^2 + \frac{242}{9}\,S^2 + \frac{418}{9}\,\zeta_3 + \frac{2156}{27}\,S \\
& - \frac{11093}{81} - 8\,S\,\zeta_3 \Big)\,\frac{t^2}{s^2} + \Big(-256\,\mathrm{Li}_4(x) - 96\,\mathrm{Li}_4(y) + 96\,\mathrm{Li}_4(z) + 80\,\mathrm{Li}_3(x)\,X + 48\,\mathrm{Li}_3(x)\,Y - \frac{64}{3}\,\mathrm{Li}_3(x) - 48\,\mathrm{Li}_3(y)\,X \\
& + 96\,\mathrm{Li}_3(y)\,Y - \frac{304}{3}\,\mathrm{Li}_3(y) + \frac{64}{3}\,\mathrm{Li}_2(x)\,X - \frac{32}{3}\,\mathrm{Li}_2(x)\,\pi^2 + \frac{304}{3}\,\mathrm{Li}_2(y)\,Y + \frac{26}{3}\,X^4 - \frac{64}{3}\,X^3\,Y - \frac{64}{3}\,X^3 + 20\,X^2\,Y^2 \\
& + \frac{136}{3}\,X^2\,Y + 24\,X^2\,\pi^2 + 76\,X^2 - \frac{88}{3}\,S\,X^2 + \frac{8}{3}\,X\,Y^3 + \frac{104}{3}\,X\,Y^2 - \frac{16}{3}\,X\,Y\,\pi^2 + \frac{176}{3}\,S\,X\,Y - \frac{136}{3}\,X\,Y - \frac{50}{3}\,X\,\pi^2 \\
& - 48\,\zeta_3\,X + \frac{2350}{27}\,X + \frac{440}{3}\,S\,X + 4\,Y^4 - \frac{176}{9}\,Y^3 + \frac{4}{3}\,Y^2\,\pi^2 - \frac{176}{3}\,S\,Y^2 - \frac{494}{9}\,Y\,\pi^2 + \frac{5392}{27}\,Y - 64\,\zeta_3\,Y + \frac{496}{45}\,\pi^4 \\
& - \frac{308}{9}\,S\,\pi^2 + \frac{200}{9}\,\pi^2 + \frac{968}{9}\,S^2 + \frac{8624}{27}\,S - \frac{44372}{81} + \frac{1864}{9}\,\zeta_3 - 32\,S\,\zeta_3 \Big)\,\frac{t}{u} + \Big(\frac{88}{3}\,\mathrm{Li}_3(x) - \frac{88}{3}\,\mathrm{Li}_2(x)\,X + 2\,X^4 - 8\,X^3\,Y \\
& - \frac{220}{9}\,X^3 + 12\,X^2\,Y^2 + \frac{88}{3}\,X^2\,Y + \frac{8}{3}\,X^2\,\pi^2 - \frac{88}{3}\,S\,X^2 + \frac{304}{9}\,X^2 - 8\,X\,Y^3 - \frac{16}{3}\,X\,Y\,\pi^2 + \frac{176}{3}\,S\,X\,Y - \frac{77}{3}\,X\,\pi^2 \\
& + \frac{1616}{27}\,X + \frac{968}{9}\,S\,X - 8\,\zeta_3\,X + 4\,Y^4 - \frac{176}{9}\,Y^3 - \frac{20}{3}\,Y^2\,\pi^2 - \frac{176}{3}\,S\,Y^2 - \frac{638}{9}\,Y\,\pi^2 - 16\,\zeta_3\,Y + \frac{5392}{27}\,Y - \frac{4}{15}\,\pi^4 - \frac{308}{9}\,S \\
& - 20\,\pi^2 - 32\,S\,\zeta_3 + \frac{1408}{9}\,\zeta_3 + \frac{968}{9}\,S^2 - \frac{44372}{81} + \frac{8624}{27}\,S \Big)\,\frac{t^2}{u^2} + \Big(\frac{44}{3}\,\mathrm{Li}_3(x) - \frac{44}{3}\,\mathrm{Li}_2(x)\,X - X^4 + \frac{110}{9}\,X^3 - \frac{22}{3}\,X^2\,Y \\
& + \frac{14}{3}\,X^2\,\pi^2 + \frac{44}{3}\,S\,X^2 - \frac{152}{9}\,X^2 - 10\,X\,Y + \frac{11}{2}\,X\,\pi^2 + 4\,\zeta_3\,X - \frac{484}{9}\,S\,X - \frac{808}{27}\,X + \frac{7}{30}\,\pi^4 - \frac{31}{9}\,\pi^2 \\
& + \frac{11}{9}\,S\,\pi^2 - \frac{418}{9}\,\zeta_3 - \frac{242}{9}\,S^2 - \frac{2156}{27}\,S + 8\,S\,\zeta_3 + \frac{11093}{81} \Big)\,\frac{u\,t}{s^2} + \Big(-176\,\mathrm{Li}_4(x) + 88\,\mathrm{Li}_3(x)\,X - 168\,\mathrm{Li}_3(x)\,Y - \ldots \end{aligned}$$

Four-gluon amplitude/Wilson loop duality in QCD II

- Planar four-gluon QCD scattering amplitude in the Regge limit $s \gg -t$ [Schnitzer'76],[Fadin,Kuraev,Lipatov'76]

$$\mathcal{M}_4^{(\mathrm{QCD})}(s,t) \sim (s/(-t))^{\omega_R(-t)} + \dots$$

The Regge trajectory $\omega_R(-t)$ is known to two loops [Fadin,Fiore,Kotsky'96]

- The all-loop gluon Regge trajectory in QCD [GK'96]

$$\omega_R^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{IR}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_R(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{IR}}] \,,$$

- Rectangular Wilson loop in QCD in the Regge limit $|x_{13}^2| \gg |x_{24}^2|$

$$W^{(\mathrm{QCD})}(C_4) \sim \left(x_{13}^2/(-x_{24}^2)\right)^{\omega_{\mathrm{W}}(-x_{24}^2)} + \dots$$

- The all-loop Wilson loop 'trajectory' in QCD

$$\omega_{\mathrm{W}}^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{UV}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_{\mathrm{W}}(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{UV}}] \,,$$

- *The duality relation holds in QCD in the Regge limit only!* [GK'96]

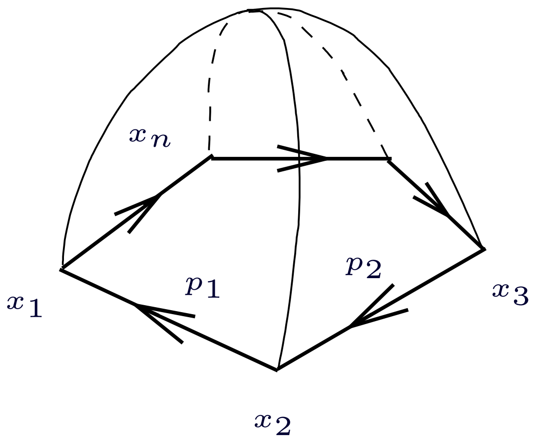
$$\ln \mathcal{M}_4^{(\mathrm{QCD})}(s,t) = \ln W^{(\mathrm{QCD})}(C_4) + O(t/s)$$

while in $\mathcal{N}=4$ SYM it is exact for arbitrary t/s

Four-gluon amplitude from AdS/CFT

Alday-Maldacena proposal:

✔ On-shell scattering amplitude is described by a classical string world-sheet in AdS$_5$



✗ On-shell gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary

✗ The closed contour has n cusps with the *dual coordinates* x_i^μ (the same as at weak coupling!)

$$x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$$

The dual conformal symmetry also exists at strong coupling!

✔ Is in agreement with the Bern-Dixon-Smirnov (BDS) ansatz for $n = 4$ amplitudes

✔ Admits generalization to arbitrary $n-$gluon amplitudes but it is difficult to construct explicit solutions for ‘minimal surface’ in AdS

✔ Agreement with the BDS ansatz is also observed for $n = 5$ gluon amplitudes [Komargodski] but disagreement is found for $n \to \infty$ $\mapsto$ *the BDS ansatz needs to be modified* [Alday,Maldacena]

The same questions to answer as at weak coupling:

☞ *Why should finite corrections exponentiate?*

☞ *Why should they be related to the cusp anomaly of Wilson loop?*