Large-order asymptotes of Kraichnan model with a 'frozen' velocity field: renormalization constant.

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Introduction to the model

Turbulent advection of passive scalar in d - dimensional fluid.

Stochastic equation

$$(\partial_t + g\nabla_i V_i - \nu\Delta)\varphi(\mathbf{x}, t) = \xi(\mathbf{x}, t). \tag{1.1}$$

- $\mathbf{x} \in \mathbb{R}^d$ and t are space and time variables
- $\varphi(\mathbf{x}, t)$ is a passive scalar field
- $\mathbf{V}(\mathbf{x})$ is a random vector velocity field
- $\xi(\mathbf{x}, t)$ is a random force
- ν is a viscosity

The velocity correlator

Gaussian distribution of the velocity field

$$\langle \mathbf{V}_i(\mathbf{x})\mathbf{V}_j(\mathbf{x}')\rangle = D_{ij}(\mathbf{x}-\mathbf{x}') \tag{1.2}$$

Coordinate representation of the correlator

$$D_{ij}(\mathbf{z}) = a_1 \frac{\delta_{ij}}{\mathbf{z}^{2\beta}} + a_2 \frac{\mathbf{z}_i \mathbf{z}_j}{\mathbf{z}^{2\beta+2}}, \qquad \beta = 1 - \varepsilon/2, \tag{1.3}$$

ε is a regular expansion parameter

Renormalization constant

Minimal subtraction scheme (MS)

$$\nu \to \nu Z_\nu, \qquad u \equiv g^2$$

$$Z_\nu = 1 + \frac{\Box u + \Box u^2 + \ldots + \spadesuit u^N + \ldots}{\epsilon} + \frac{\Box u^2 + \Box u^3 + \ldots}{\epsilon^2} + \ldots$$

$$\ln(Z_\nu) = \frac{\Box u + \Box u^2 + \ldots + \spadesuit u^N + \ldots}{\epsilon} + \frac{\Box u^2 + \Box u^3 + \ldots}{\epsilon^2} + \ldots$$

Investigation of coefficients $\spadesuit$ at large N.

Martin-Siggia-Rose formalism

Response function in arbitrary **V** field

$$G_V^{(1,2)} = \int \mathcal{D}\varphi \mathcal{D}\varphi' \;\; \varphi(\mathbf{x}_1, t_1)\varphi'(\mathbf{x}_2, t_2)\exp(-S^{msr}) \tag{1.4}$$

$$S^{msr} = \frac{\varphi' D_\xi \varphi'}{2} + \varphi'(\partial_t + g\nabla_i \mathbf{V}_i - \nu Z_\nu \Delta)\varphi, \tag{1.5}$$

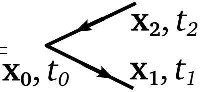
Response function (gaussian integration in **V**)

$$< \varphi\varphi' >= \int \mathcal{D}\mathbf{V} G_V^{(1,2)} e^{-\mathbf{V}_i D_{ij}^{-1} \mathbf{V}_j/2} \tag{1.6}$$

The usual way to extract Z_ν

$$\frac{\partial}{\partial \nu} < \varphi\varphi' >= Z_\nu < \varphi\varphi'(\varphi\Delta\varphi') >= Z_\nu G = \text{finite}$$

$$G(\mathbf{x}, T) \equiv < \varphi(\mathbf{x}_1, t_1)\varphi'(\mathbf{x}_2, t_2)\Delta\varphi'(\mathbf{x}_0, t_0)\varphi(\mathbf{x}_0, t_0) >=$$



$$x \equiv \mathbf{x}_2 - \mathbf{x}_1, \quad T \equiv t_2 - t_1$$

$$\underset{\varepsilon \to 0}{\text{res}} \ln Z_\nu = - \underset{\varepsilon \to 0}{\text{res}} \ln G. \tag{2.1}$$

Lagrange variables

The two fluid particles. $\mathbf{c}'_1$,$\mathbf{c}'_2$ are momenta, $\mathbf{c}_1$,$\mathbf{c}_2$ are coordinates.

The action for composite operator calculation

$$S = -i\mathbf{q}(\mathbf{x}_2 - \mathbf{x}_1) - \nu Z_\nu(\mathbf{c}_1'^2 + \mathbf{c}_2'^2) + i\mathbf{c}'_1\partial\mathbf{c}_1 + i\mathbf{c}'_2\partial\mathbf{c}_2 + S_u. \quad (2.2)$$

A nonlinear part of the action is collected in the term

$$u \equiv g^2, \qquad S_u = -\frac{u}{2}\Big(\mathbf{c}'_1(\tau_1)D(\mathbf{c}_1(\tau_1) - \mathbf{c}_1(\tau'_1))\mathbf{c}'_1(\tau'_1) +$$

$$+\mathbf{c}'_2(\tau_2)D(\mathbf{c}_2(\tau_2) - \mathbf{c}_2(\tau'_2))\mathbf{c}'_2(\tau'_2) + 2\mathbf{c}'_1 D(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}'_2\Big)$$

The correct order of limit operations

There are two large parameters

$$\frac{1}{\varepsilon} \to \infty \qquad N \to \infty \qquad N\varepsilon \to 0$$

Renormalization constants

$$Z_\nu = 1 + \frac{\Box u + \Box u^2 + \ldots + \Box u^N + \ldots}{\epsilon} + \frac{\Box u^2 + \Box u^3 + \ldots}{\epsilon^2} + \ldots$$

How to handle

$$\exp(S) = \exp(S_{reg} + S_{sing}) = \exp(S_{reg}) \sum_{p=0}^{\infty} \frac{1}{p!} (S_{sing})^p, \qquad \oint \frac{du}{u^{N+1}} \ldots$$

Instanton analyses

$G(\mathbf{x}, T)$ composite operator

- The stationarity equation can be written (7)
- The symmetry of the model can be considered
- A particular solution can be found

$$F = 0 \qquad \Leftrightarrow \qquad G(\mathbf{x}, T), \quad |\mathbf{x}| = x_{st}$$

- Momentum frequency representation $G(q, \omega)$. Additional integrals in T and x. $\int dTd\mathbf{x}e^{iqx}G(\mathbf{x}, T)$

$$q = q_0 = \frac{iD_0 u}{(1-\varepsilon)x_s t^{1-\varepsilon}\nu^2} \qquad \omega = 0$$

The action at the instanton

$$S_{st} = -\frac{uD_0 x_{st}^{\epsilon}}{\nu^2 \varepsilon} = -\frac{u}{\nu^2}\left(\frac{A(x_{st}^{\varepsilon}-1)}{\varepsilon} + \frac{A}{\varepsilon} + B(\varepsilon)x^{\varepsilon}\right) \qquad (3.1)$$

The velocity correlator formulae reminder

$$D_{ij}(\mathbf{z}) = a_1 \frac{\delta_{ij}}{\mathbf{z}^{2\beta}} + a_2 \frac{\mathbf{z}_i \mathbf{z}_j}{\mathbf{z}^{2\beta+2}}, \qquad D_0 = a_1(\varepsilon) + a_2(\varepsilon) = A + \varepsilon B(\varepsilon), \quad (3.2)$$

How to handle

$$\exp(S) = \exp(S_{reg} + S_{sing}) = \exp(S_{reg}) \sum_{p=0}^{\infty} \frac{1}{p!} (S_{sing})^p,$$

Simple poles in ε

$$G^{(N)} \sim \oint \frac{du}{u^{N+1}} \int dT \;\; \mathcal{Z}(T,u) \;\; e^{N S_{reg}(\varepsilon,T)} \sum_{p=0}^{\infty} \frac{1}{p!} \left(S_{sing}\right)^p$$

$$S_{reg} = -\frac{u}{\nu^2} B(\varepsilon) x^{\varepsilon} \sim -\frac{u}{\nu^2} B(\varepsilon) (6uTA/\nu)^{\varepsilon/2}$$

- Finite renormalization $uB(\varepsilon)/\nu^2 \to \bar{u}$

Simple poles in ε

$$G^{(N)} \sim \oint \frac{du}{u^{N+1}} \int \frac{dT}{T} \; \bar{\mathcal{Z}}(u) \qquad e^{NS_{reg}(\varepsilon,T)} \sum_{p=0}^{\infty} \frac{1}{p!} \left(S_{sing}\right)^p$$

$$S_{reg} = -\frac{u}{\nu^2} B(\varepsilon) x^{\varepsilon} \sim -\frac{u}{\nu^2} B(\varepsilon) (6uTA/\nu)^{\varepsilon/2}$$

- Finite renormalization $uB(\varepsilon)/\nu^2 \to \bar{u}$

Simple poles in ε extraction

The term corresponding to $p = 0$

$$\oint \frac{d\bar{u}}{\bar{u}^{N+1}} \int \frac{dT}{T} \bar{\mathfrak{Z}}(\bar{u}) \exp\left(-N\bar{u}(T\bar{u})^{\varepsilon/2}\right) =$$

$$\oint \frac{d\bar{u}}{\bar{u}^{N+1}} \bar{\mathfrak{Z}} \int \frac{d\chi}{\chi} \exp\left(-N\bar{u}\chi^{\varepsilon/2}\right) \sim \frac{N^N}{N!} \int\limits_0^1 \frac{d\chi}{\chi^{1-N\varepsilon/2}} \sim N^n \frac{1}{N\varepsilon}$$

Scale integration is not saddle-point one!

Replica trick

Logarithm representation

$$\ln \int dT f(T) = \lim_{r \to 0} \frac{\partial}{\partial r} \prod_{\alpha=0}^{r-1} \int dT_\alpha f(T_\alpha). \tag{3.3}$$

T becomes an r-dimensional vector in a replica space.

$$\oint \frac{du}{u^{N+1}} \int \Big(\prod_{\alpha=0}^{r-1} dT_\alpha\Big) \exp\left(Nu \sum_{\alpha=0}^{r-1} (T_\alpha u)^{\varepsilon/2}\right) \mathcal{Z}(\{T_\alpha\}_{\alpha=0}^{r-1}), \tag{3.4}$$

Saddle-point method and replica trick

Let's try to calculate all integrations in (1) by saddle-point method.

To be variated

$$Nu\sum_{\alpha=0}^{r-1}(T_\alpha u)^{\varepsilon/2} - N\ln u$$

With respect to u

$$u_{st}^{-1-\varepsilon} = (1+\varepsilon)\sum T_\alpha.$$

With respect to T_α

$$T_\alpha = 0$$

Replica trick results

- At least one of integrations has a non-saddle-point structure.
- Let's exclude the integration in T_0 from the saddle-point method consideration.
- The set of variables $\{T_\alpha\}_{\alpha=1}^{r-1} = 0$ at the stationary point.
- $u_{st}^{-1-\varepsilon} = (1+\varepsilon)T_0$.
- The same result!
- The factor r allows us to produce the operation $\lim\limits_{r\to 0} \partial/\partial r$ correctly.

Summary

- We have calculated the large order asymptotics for $\ln Z_{\nu}$ constants
- Though the number of diagrams has a factorial growth at large orders the series has a finite convergence radius.
- This was a difficult problem both from idealogical and technical point of view.

Thank you!

Some papers

- *Lipatov L. N., J. Exp. and Theor. Phys (1977).*
- *Juha Honkonen and Esa Karjalainen, J. Phys. A: Math. Gen. (1988), Phys Lett. A (1988)*
- *M.V.Komarova, M.Yu.Nalimov, Theor. and Math. Phys. (2001)*
- *A.Yu. Andreanov, M.Komarova, M.Nalimov, J. Phys. A (2006).*
- *To be publish (2008).*