The Adler Function Testing Renormalization Schemes

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Outline of Talk:

① **Status of the Non-Perturbative Adler Function**

② **Theoretical Prediction in Perturbative QCD**

③ **Application: Precision Determination of $\alpha(M_Z)$**

④ **Extension of the Perturbative Regime: APT**

⑤ **Conclusions and Outlook**

① Status of the Non-Perturbative Adler Function

Basic object: **photon vacuum polarization amplitude**

$$\Pi^{\gamma}_{\mu\nu}(q) \;=\; i\int d^4x e^{iqx} < 0|T J^{\gamma}_{\mu}(x)\, J^{\gamma}_{\nu}(0)\,|0 >= -\left(q^2\, g_{\mu\nu} - q_\mu q_\nu\right)\;\Pi'_\gamma\,(q^2)$$

Adler function:

$$D(-s) = -(12\pi^2)\, s\, \frac{d\Pi'_\gamma\,(s)}{ds} = \frac{3\pi}{\alpha} s \frac{d}{ds}\Delta\alpha_{\rm had}(s)$$

$\Delta\alpha_{\rm had}(s)$ **is the hadronic contribution to the shift of the fine structure constant at scale q^2, i.e.**

$$\alpha(s) = \frac{\alpha}{1-\Delta\alpha_{\rm had}(s)}$$

Adler function in terms of experimental data: (exploiting analyticity and unitarity)

$$D(Q^2) = Q^2\left(\int_{4m_\pi^2}^{E^2_{\rm cut}} \frac{R^{\rm data}(s)}{(s+Q^2)^2} ds \;+\; \int_{E^2_{\rm cut}}^{\infty} \frac{R^{\rm pQCD}(s)}{(s+Q^2)^2} ds\right)$$

$Q^2 = -q^2$ **is the squared Euclidean momentum transfer,**

s **is the center of mass energy squared for hadron production in e^+e^-–annihilation**

Adler 1974: excellent monitor for checking (perturbative) QCD

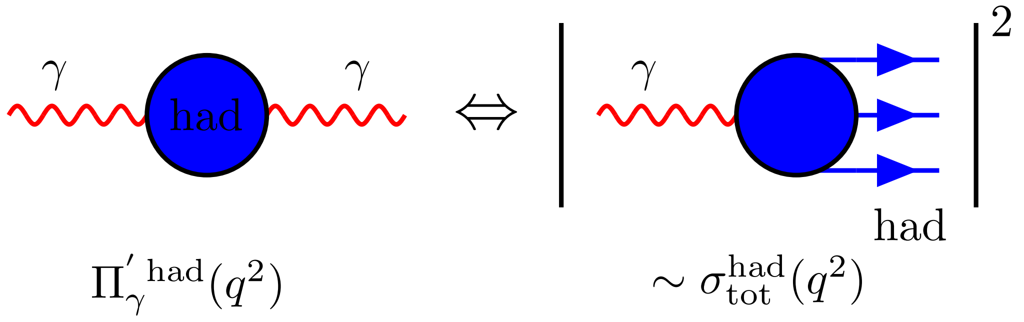
- smooth function away from resonances and thresholds
- UV and IR finite object at all scales
- RG–invariant, scheme independent

Its non-perturbative version can be evaluated in terms of known experimental data !

Major recent progress mainly radiative return measurements by BaBar, many (old and new) channels in problematic region 1.4 to 2.4 GeV

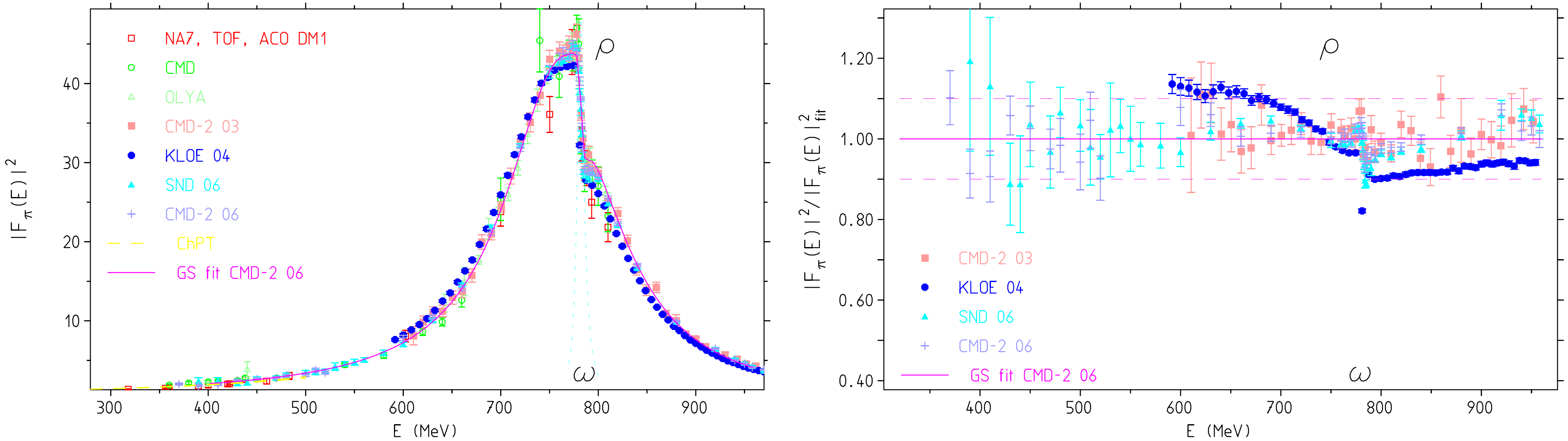
CMD-2/SND, KLOE, BaBar, Belle, CLEO and BES data and more

- standard evaluation of the non-perturbative hadronic contributions via DR in terms of measured cross-sections $\sigma(e^+e^- \to \mathrm{hadrons})$:

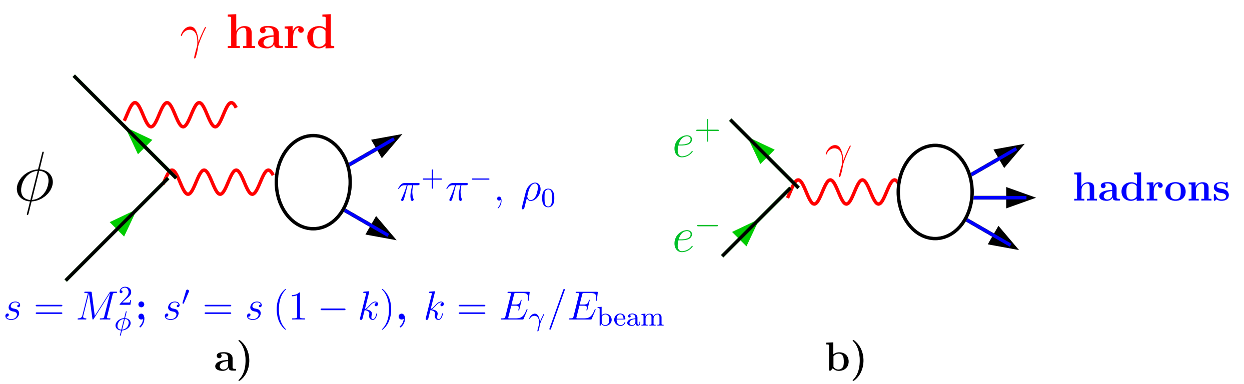


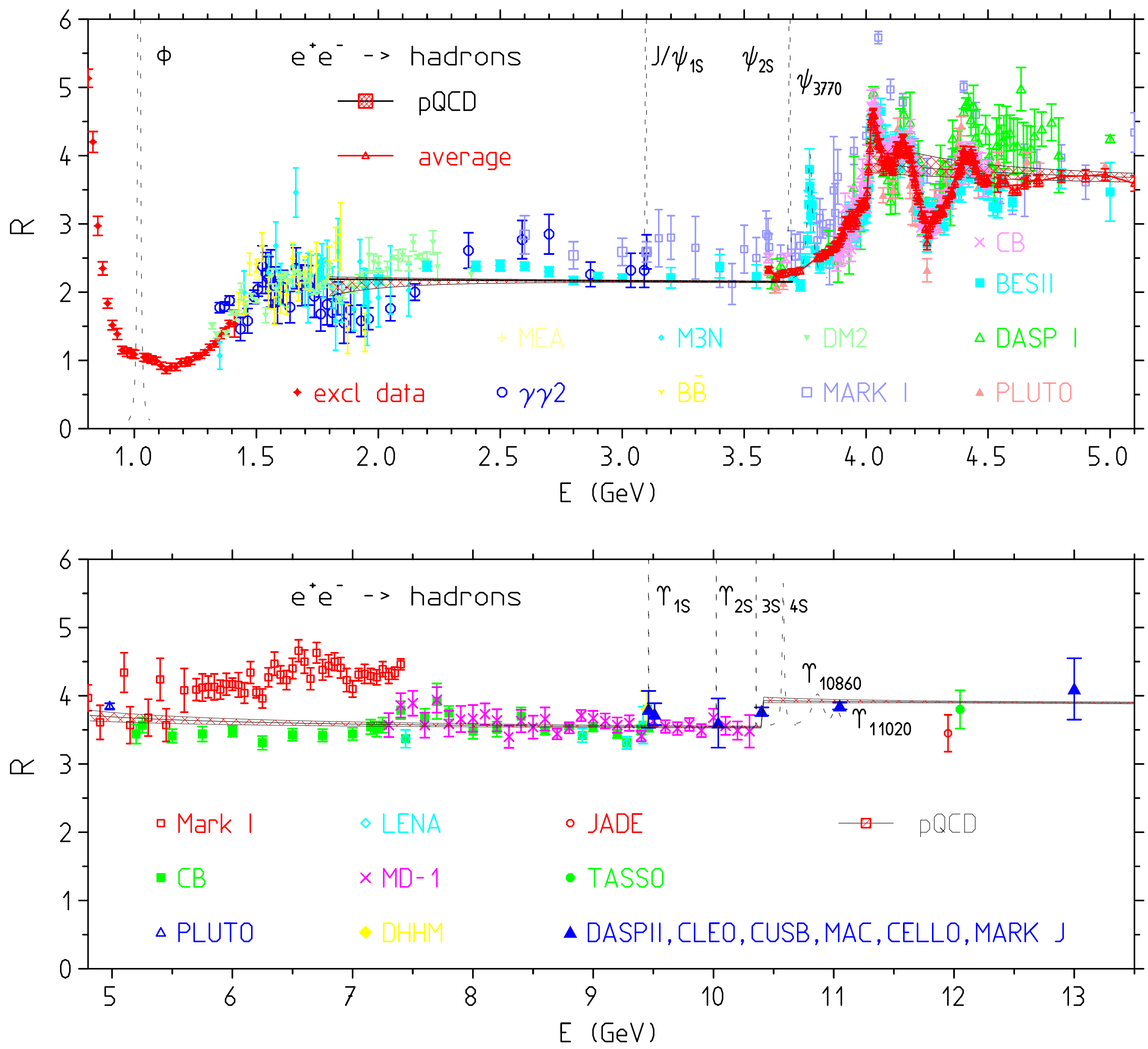
At low Q^2: $\propto \int ds/s^4 \cdots$ dominated by $\gamma^* \longrightarrow \pi^+\pi^-$ pion form factor $|F_\pi(s)|^2$

mainly CMD2/SND Novosibirsk, KLOE Frascati



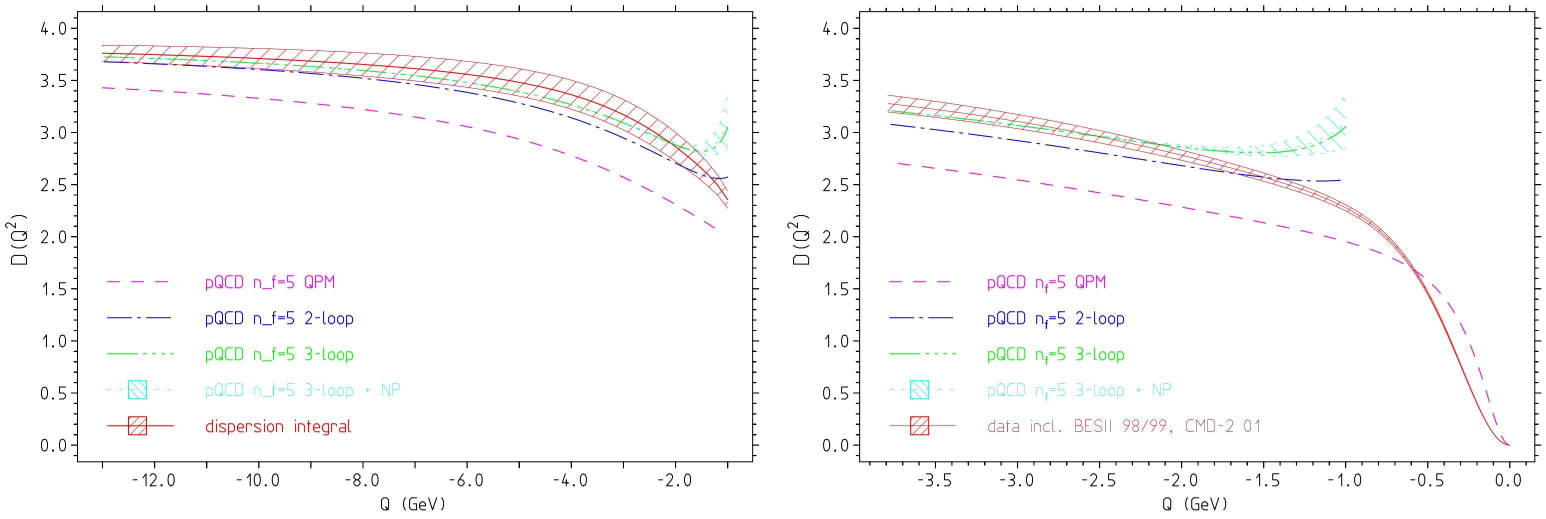
a) Radiative return determination of hadronic cross–section at fixed energy machines: KLOE ϕ–factory DAΦNEat Frascati , BaBar B–factory at SLAC; b) usual energy scan





e+e- -> hadrons
pQCD
average
ϕ
J/ψ_{1S}
ψ_{2S}
ψ_{3770}
R
E (GeV)
CB
BESII
MEA
M3N
DM2
DASP I
excl data
$\gamma\gamma2$
$B\bar{B}$
MARK I
PLUTO
e+e- -> hadrons
Υ_{1S}
Υ_{2S} 3S 4S
Υ_{10860}
Υ_{11020}
Mark I
LENA
JADE
pQCD
CB
MD-1
TASSO
PLUTO
DHHM
DASPII,CLEO,CUSB,MAC,CELLO,MARK J

- take $R(s)$ data up to $\sqrt{s} = E_{cut} = 5.2$ GeV and for the Υ resonance–region between 9.46 and 13 GeV and apply perturbative QCD from 5.2 to 9.46 GeV and for the high energy tail above 13 GeV.



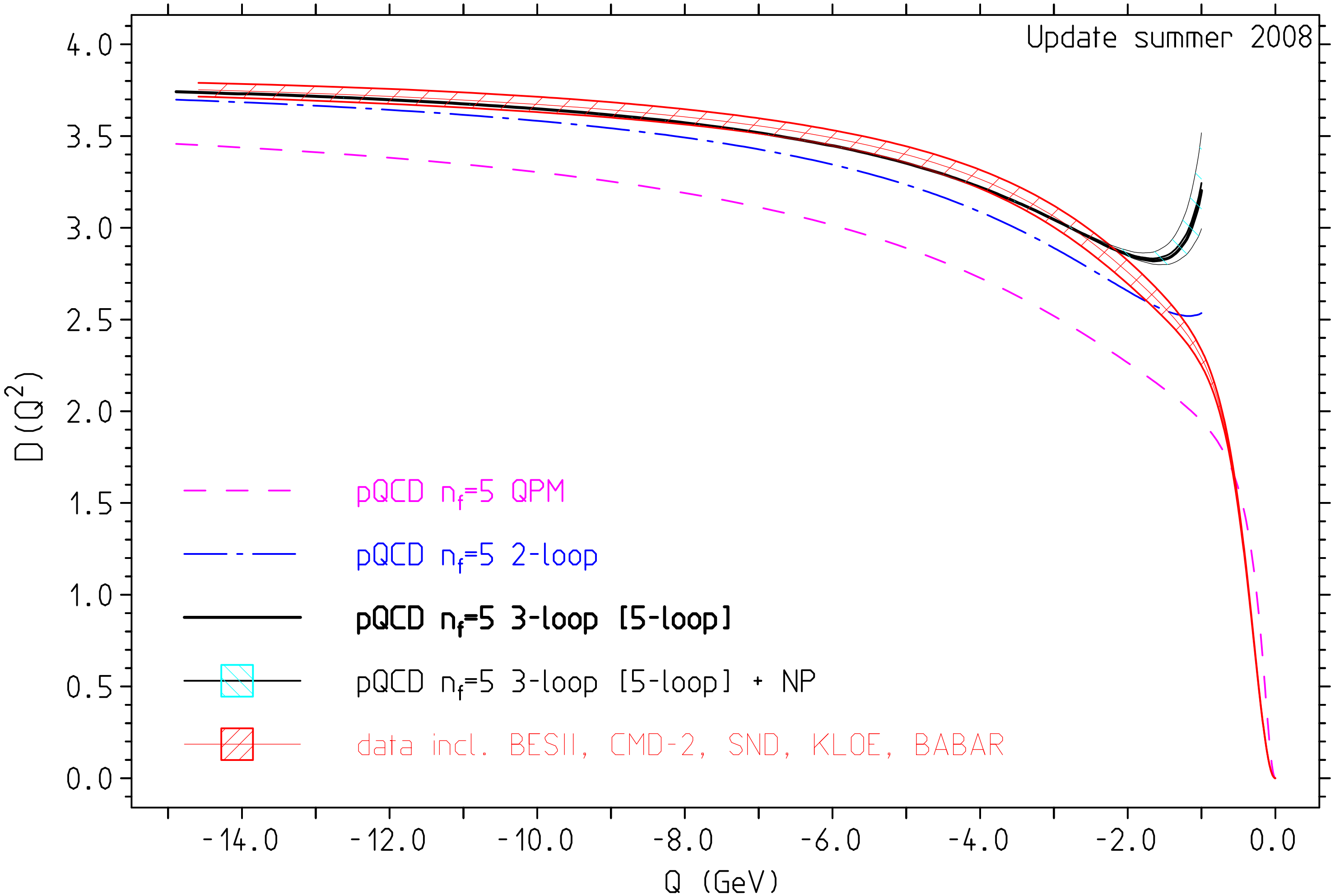
(Eidelman, FJ, Kataev, Veretin 1998)

- very much improved data were available (based on Eidelman & FJ 1995 data compilation)
- first calculation including massive 3–loop result
- first real pQCD test of $D(Q^2)$ [full massive QCD mandatory, FOPT with RG improvement]

10 years ago

"Experimental" Adler–function versus theory (pQCD + NP)

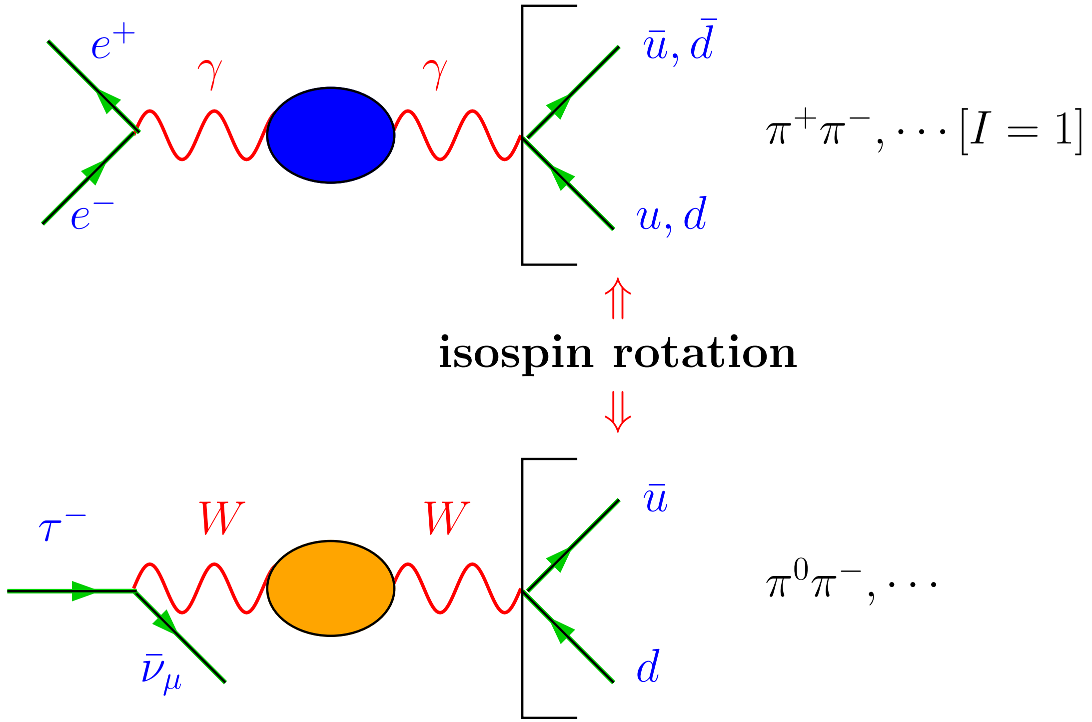
Error includes statistical + systematic here (in contrast to most R-plots showing statistical errors only)!



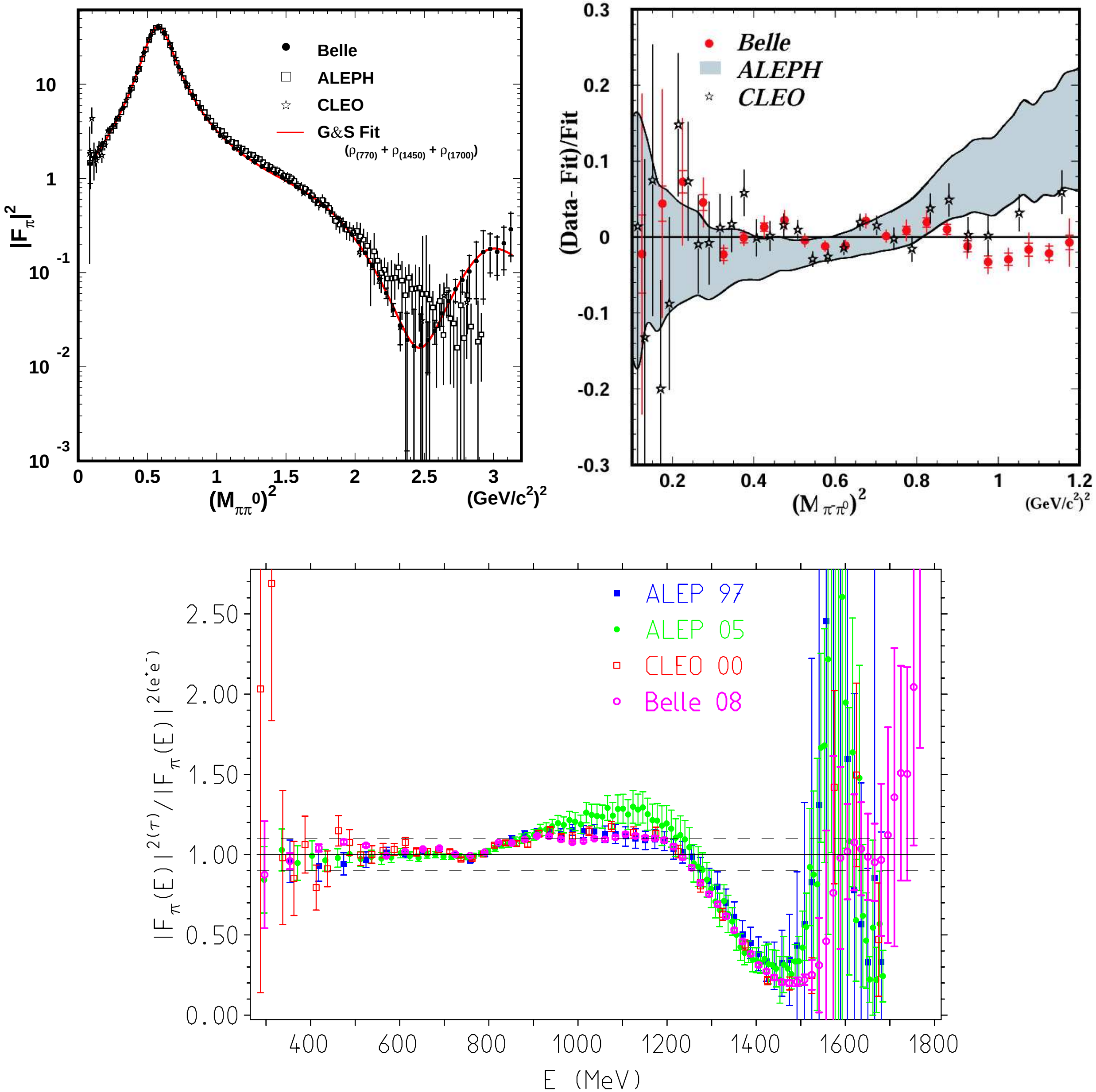
(Eidelman, FJ, Kataev, Veretin 98, FJ 08 update (incl. BaBar)) theory based on results by Chetyrkin, Kühn et al.

Isovector (I=1) Adler function: directly accessible in hadronic τ–decays $\boxed{\tau^\pm \to \nu_\tau \pi^\pm \pi^0, \cdots}$

Remarks on τ vs. e^+e^- data:



The new precise hadronic τ–decay spectral data for the $\pi\pi$ channel from Belle bring back into focus the τ vs. e^+e^- data discrepancy. Known isospin violation effects have been accounted for.


Belle
ALEPH
CLEO
G&S Fit
$(\rho_{(770)} + \rho_{(1450)} + \rho_{(1700)})$
$|F_\pi|^2$
$(M_{\pi\pi^0})^2$
$(GeV/c^2)^2$
(Data- Fit)/Fit
Belle
ALEPH
CLEO
$(M_{\pi\pi^0})^2$
$(GeV/c^2)^2$
ALEP 97
ALEP 05
CLEO 00
Belle 08
$|F_\pi(E)|^{2(\tau)}/|F_\pi(E)|^{2(e^+e^-)}$
E (MeV)

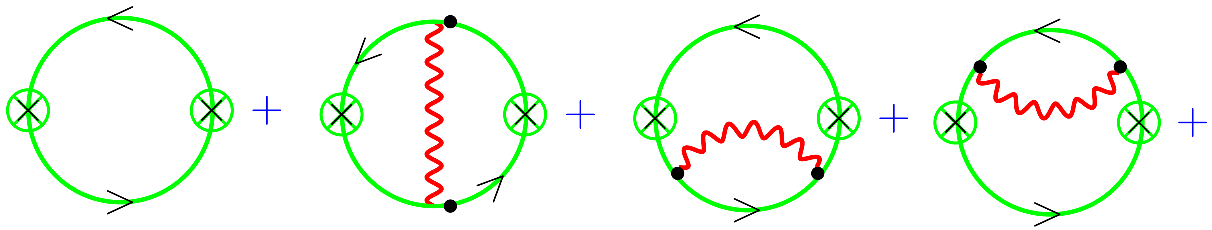
- good agreement below 800 MeV
- between 800 and 1200 MeV τ data appear enhanced by 10 to 20%
- pronounced dip at about 1500 MeV.

Apart from possible experimental problems (e.g. clean separation of different channels), reasons for the observed deviations could be unaccounted isospin breaking, inappropriate treatment of interferences (coherent vs incoherent) or radiative corrections (photon radiation from hadrons) of the hadrons involved.

② Theoretical Prediction in Perturbative QCD

Adler function from pQCD and OPE:

❶ **pQCD calculations of vacuum polarization amplitudes**



$+ \cdots$

up to 5–loops massless — **Groshny, Kataev, Larin 91, Baikov, Chetyrkin, Kühn 08**

up to 3–loops massive — **Chetyrkin, Kühn et al. 97**

up to 2–loops massive BF–MOM RG — **F. J., Tarasov 98, [Shirkov 92]**

❷ **implemented to Adler function** — **Eidelman, F. J., Kataev, Veretin 98**

pQCD $\leftrightarrow R(s)$	**pQCD** $\leftrightarrow D(Q^2)$
true R very difficult to obtain in theory	**smooth simple function in Euclidean region**

Contributions:

$$D(Q^2) = D^{(0)}(Q^2) + D^{(1)}(Q^2) + D^{(2)}(Q^2) + \cdots + D^{\mathrm{NP}}(Q^2)$$

One–loop:

$$D^{(0)}(Q^2) = \sum_f Q_f^2 N_{cf} H^{(0)}$$

with $H \equiv (12\pi^2)\,\dot{\Pi}'_V$, $\dot{\Pi}'_V \equiv -s\, d\Pi'_V/ds$, **in terms of the vector current amplitude** Π'_V**.**

Explicitly:

$$H^{(0)} = 1 + \frac{3y}{2} - \frac{3y^2}{4}\frac{1}{\sqrt{1-y}}\ln\xi \quad y = 4m_f^2/s \ , \quad \xi = \frac{\sqrt{1-y}-1}{\sqrt{1-y}+1}$$

where ξ **is taking values** $0 \leq \xi \leq 1$ **for** $s \leq 0$.

Asymptotically:

$$H^{(0)} \to \begin{cases} \frac{1}{5}\frac{Q^2}{m_f^2} - \frac{3}{70}\left(\frac{Q^2}{m_f^2}\right)^2 + \frac{1}{105}\left(\frac{Q^2}{m_f^2}\right)^3 + \cdots & Q^2 \ll m_f^2 \\ 1 - 6\frac{m_f^2}{Q^2} - 12\left(\frac{m_f^2}{Q^2}\right)^2 \ln\frac{m_f^2}{Q^2} + 24\left(\frac{m_f^2}{Q^2}\right)^3\left(\ln\frac{m_f^2}{Q^2}+1\right) + \cdots & Q^2 \gg m_f^2 \end{cases} \tag{1}$$

and this behavior determines the quark parton model (QPM) (leading order QCD) property of the Adler function: heavy quarks ($m_f^2 \gg Q^2$**) decouple like** Q^2/m_f^2 **while light modes (**$m_f^2 \ll Q^2$**) contribute** $Q_f^2 N_{cf}$ **to** $D^{(0)}$**.**

Two–loop: known analytically (Broadhurst 85 and others)

$$D^{(1)}(Q^2) = \frac{\alpha_s(Q^2)}{\pi} \sum_f Q_f^2 N_{cf}\, H^{(1)}$$

where $H^{(1)} = (12\pi^2)\, \dot{\Pi}_V'^{(2)}(-Q^2, m_f^2)$.

Three–loop: known as low and large momentum expansion (Chetyrkin, Harlander, Kühn, Steinhauser 96/97)

$$D^{(2)}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 \sum_f Q_f^2 N_{cf}\, H^{(2)}$$

where $H^{(2)} = (12\pi^2)\, \dot{\Pi}_V'^{(3)}(-Q^2, m_f^2)$.

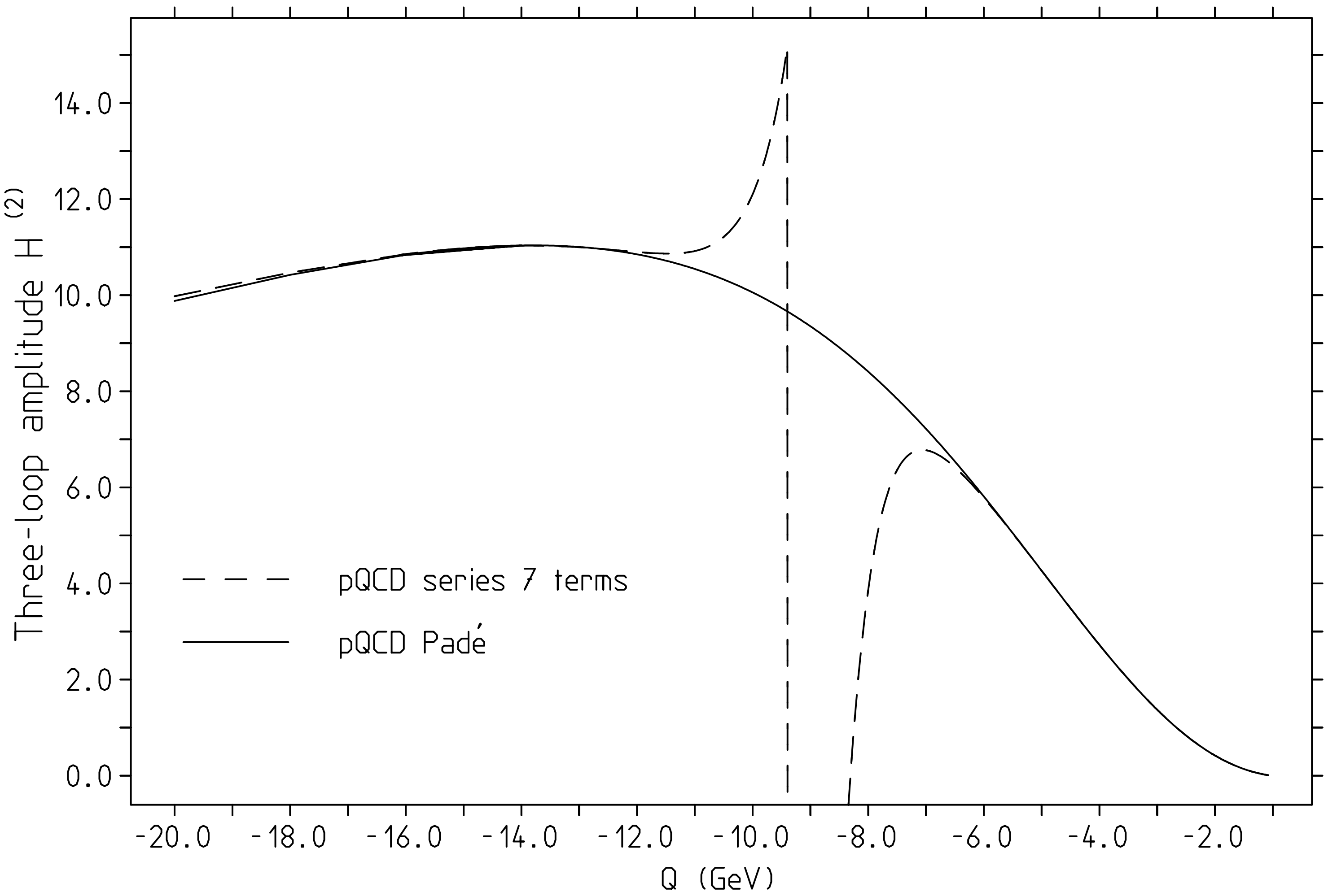
Both series expansions diverge at the boundary of the circle of convergence $Q^2 = 4m^2$ $\Rightarrow$ problem in the region where mass effects are of the order of unity in the Euclidean region. Apply a conformal mapping (Schwinger 48)

$$y^{-1} = \frac{-Q^2}{4m^2} \to \omega = \frac{1-\sqrt{1-1/y}}{1+\sqrt{1-1/y}}$$

from the complex negative q^2 half–plane to the interior of the unit circle $|\omega| < 1$ together with Padé resummation (Fleischer and Tarasov 94). The Padé approximant provides a good estimation to much higher values of $1/y$ up to about $1/y \sim 4$. This is displayed in $(\Rightarrow)$ the Figures. Padé improvement allows us to obtain reliable results also in the relevant Euclidean "threshold region", around $y = 1$.

Method may be tested with exact two loop; works perfectly!

Adler–function: pQCD three-loop



- **Four–loop in/and high energy limit:**

$$D(Q^2) \simeq 3 \sum_f Q_f^2 \, \left(1 + a + d_2 a^2 + d_3 a^3 + d_4 a^4\right)$$

with $a = \alpha_s(Q^2)/\pi$,

$$\begin{aligned} d_2 &= 1.9857 - 0.1153\, n_f \,, \\ d_3 &= 18.2428 - 4.2159\, n_f + 0.0862\, n_f^2 - 1.2395\, (\sum Q_f)^2/(3 \sum Q_f^2) \,, \\ d_4 &= -0.010\, n_f^3 \, + \, 1.88\, n_f^2 \, - \, 34.4\, n_f \, + \, 135.8 \,. \end{aligned}$$

The corresponding formula for $R(s)$ only differs at the 4–loop and 5–loop level due to the effect from the analytic continuation from the Euclidean to the Minkowski region which yields $r_3^R = d_3 - \pi^2 \beta_0^2 \frac{d_1}{3}$ with $\beta_0 = (11 - 2/3\, n_f)/4$, $d_1 = 1$ and $r_4^R = d_4 - \pi^2 \beta_0^2 \left(d_2 + \frac{5\beta_1}{6\beta_0}\, d_1\right)$ with $\beta_1 = (102 - 38/3\, n_f)/16$.

Numerically the 4–loop term proportional to d_3 amounts to -0.0036% at 100 GeV and increases to about 0.32% at 2.5 GeV.

- **Non perturbative effects**

Parametrize NP effects at sufficiently large energies and away from resonances as prescribed by the OPE. Non–vanishing gluon and light quark condensates (Shifman, Vainshtein, Zakharov 79) imply the leading power corrections

$$
\begin{aligned}
D^{\mathrm{NP}}(Q^2) \quad &= \quad \sum_{q=u,d,s} Q_q^2 N_{cq}\,(8\pi^2) \cdot \Bigg[\tfrac{1}{12}\left(1-\tfrac{11}{18}a\right) \frac{<\frac{\alpha_s}{\pi}GG>}{Q^4} \\
&+ \quad 2\left(1+\tfrac{1}{3}\,a+\left(\tfrac{47}{8}-\tfrac{3}{4}l_{q\mu}\right)\,a^2\right)\frac{<m_q\bar{q}q>}{Q^4} \\
&+\left(\tfrac{4}{27}\,a+\left(\tfrac{4}{3}\zeta_3-\tfrac{88}{243}-\tfrac{1}{3}l_{q\mu}\right)\,a^2\right)\sum_{q'=u,d,s}\frac{<m_{q'}\bar{q}'q'>}{Q^4}+\cdots\Bigg]
\end{aligned}
$$

where $a \equiv \alpha_s(\mu^2)/\pi$ and $l_{q\mu} \equiv \ln(Q^2/\mu^2)$.

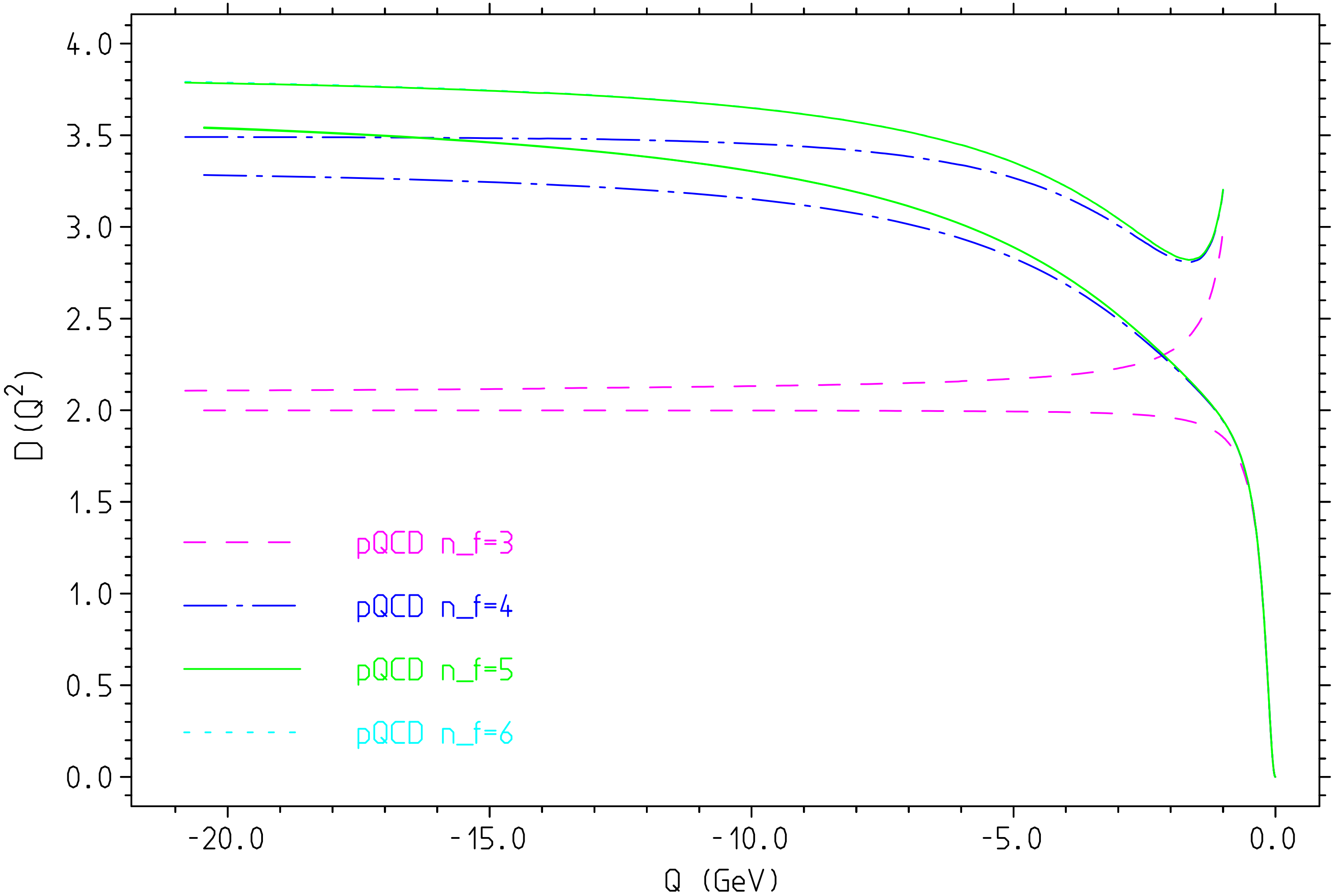
$<\frac{\alpha_s}{\pi}GG>$ and $<m_q\bar{q}q>$ are the scale-invariantly defined condensates. Typically

$$<\frac{\alpha_s}{\pi}GG> \sim \quad (0.389\ \mathbf{GeV})^4,$$

$$<m_q\bar{q}q> \sim -(0.098\ \mathbf{GeV})^4 \quad q=u,d\,,\ <m_q\bar{q}q> \sim -(0.218\ \mathbf{GeV})^4 \quad q=s$$

Parametrizes the high energy tail of NP effects associated with the existence of non–vanishing condensates. Does not include other NP effects like bound states, resonances, instantons etc. The dilemma with is that it works only for Q^2 large enough and it has been successfully applied in heavy quark physics. It fails do describe NP physics at lower Q^2, once it starts to be numerically relevant pQCD starts to fail because of the growth of the strong coupling.

Adler–function: pQCD in $\overline{\mathrm{MS}}$



Which renormalization scheme?

minimal subtraction scheme (MS or $\overline{\mathrm{MS}}$) **simple but unphysical, no decoupling, matching of N_f-effective approximants, Euclidean matching scales?**

momentum substraction scheme (MOM), **violating ST-identities, but satisfy decoupling (smooth in the Euclidean)**

Way out:
background field method **(BFM) (DeWitt). The latter takes advantage of the freedom to chose a gauge fixing function in a particular way, namely, such that the canonical Slavnov-Taylor identities remain valid also after momentum subtractions. The gauge invariant physical quantities are not affected by the gauge fixing, however, the** "background field gauge" **selects a particular representative of the gauge variant off–shell amplitudes. The resaturation of the Slavnov-Taylor identities in the BFM is achieved solely by changing the vertices with external gluons appropriately.**

For a mass–dependent renormalization schemes the RG equations

$$\mu \frac{d}{d\mu} g_s(\mu) = \beta[g_s(\mu)] \quad , \quad \mu \frac{d}{d\mu} m_i(\mu) = -\gamma_m[g_s(\mu)]\, m_i(\mu)$$

in general can be solved only by numerical integration. However, an approximate solution for the mass dependent effective QCD coupling was proposed by Shirkov 1992. Indeed, at the two-loop level the

expression

$$\overline{\alpha_s}(Q^2) = \frac{\alpha_s}{1+\alpha_s/(4\pi)U_1+\alpha_s/(4\pi)(U_2/U_1)\ln(1+\alpha_s/(4\pi)U_1)}\ ,$$

with $U_{1,2}$ given below, correctly sums up all leading as well as "next-to-leading" terms $\alpha_s U_2 (\alpha_s U_1)^n$ though it is not an exact solution of the two-loop differential RG equation. We will compare this with the result of the numerical integration of the differential RG equation.

Exact 2–loop calculation: in BF-MOM only need gluon propagator.

The renormalized gluon self–energy amplitude $\Pi(Q^2)$ has the form:

$$\Pi(Q^2) = \left(\frac{\alpha_s}{4\pi}\right) U_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 U_2$$

where

$$U_1 = \frac{11}{3} C_A \ln\frac{Q^2}{\mu^2} + T_F \sum_{i=1}^{n_f} \left(\Pi_1\left(\frac{Q^2}{m_i^2}\right) - \Pi_1\left(\frac{\mu^2}{m_i^2}\right)\right)$$

$$U_2 = \frac{34}{3} C_A^2 \ln\frac{Q^2}{\mu^2} + T_F \sum_{i=1}^{n_f} \left(\Pi_2\left(\frac{Q^2}{m_i^2}\right) - \Pi_2\left(\frac{\mu^2}{m_i^2}\right)\right)$$

C_A, C_F **and** T_F **Casimir invariants of the gauge group,** n_f **number of flavors.**
The results of our calculations for $\Pi_{1,2}$ **read**

$$\begin{aligned}
\Pi_1\left(\frac{Q^2}{m^2}\right) &= \frac{4}{3z}\left[1-(1+2z)(1-z)G(z)\right]\,, \\
\Pi_2\left(\frac{Q^2}{m^2}\right) &= \frac{(1+2z)}{3z^2}\left[(C_A+4C_F)\,\sigma(z)-(C_A-2C_F)(1-2z)\,I(z)\right] \\
&+ \frac{2}{9z}\left\{39+3\tilde{I}_3^{(4)}(z)-\left[4z^2+134z+57-12(2-5z)zG(z)\right](1-z)G(z)\right. \\
&+ \left.2[z^2+18z+9-3(3+8z)(1-z)G(z)]\ln(-4z)\right\}C_A \\
&+ \frac{2}{3z}\left\{13-[6(3+2z)+(7+8z-48z^2)G(z)](1-z)G(z)\right\}C_F
\end{aligned}$$

Notation:

Kinematical variables:

$$Q^2=-q^2\,,\qquad z=\frac{q^2}{4m^2}\,,\qquad y=\frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}\,,$$

Basic integrals:

$$G(z)=\frac{2y\ln y}{y^2-1}\,,$$

$$
\begin{aligned}
I(z) &= 6[\zeta_3 + 4\mathrm{Li}_3(-y) + 2\mathrm{Li}_3(y)] - 8[2\mathrm{Li}_2(-y) + \mathrm{Li}_2(y)] \ln y \\
&\qquad - 2[2\ln(1+y) + \ln(1-y)] \ln^2 y \;, \\
\tilde{I}_3^{(4)}(z) &= 6\zeta_3 - 6\mathrm{Li}_3(y) + 6\ln y \mathrm{Li}_2(y) + 2\ln(1-y)\ln^2 y \;, \\
\sigma(z) &= \frac{1-y^2}{y}\left\{2\mathrm{Li}_2(-y) + \mathrm{Li}_2(y) + [\ln(1-y) + 2\ln(1+y) - 3/4\ln y]\ln y\right\},
\end{aligned}
$$

$C_A = 0, C_F = T = 1, n_f = 1$ **and taking the limit** $\mu^2 \to 0$ **yields correct photon propagator in the on-shell scheme**

The BFMOM RG equation

RG β–function in BFMOM:

$$
\begin{aligned}
\mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_s}{4\pi}\right) &= \frac{g_s}{(4\pi)^2}\,\beta\left(\frac{\mu^2}{m_i^2},\alpha_s\right) = \lim_{\varepsilon\to 0}\ \alpha_s\,\mu\frac{\partial}{\partial\mu}\ln Z_A \\
&= -\beta_0\left(\frac{\alpha_s}{4\pi}\right)^2 - \beta_1\left(\frac{\alpha_s}{4\pi}\right)^3 - \cdots
\end{aligned}
$$

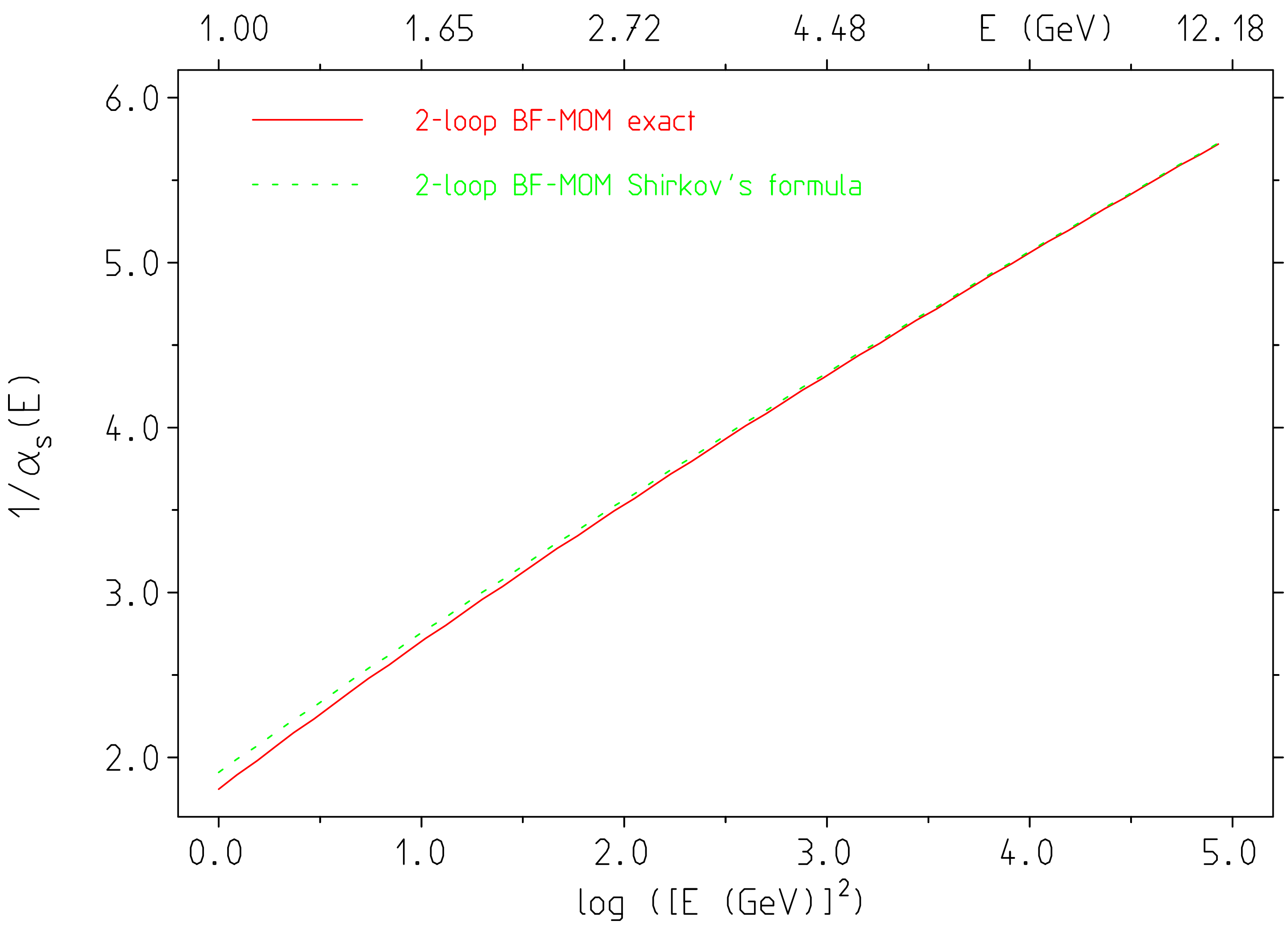
$$
\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F\sum_{i=1}^{n_f} b_0\left(\frac{\mu^2}{m_i^2}\right)\;,\quad \beta_1 = \frac{34}{3}C_A^2 - \ T_F\sum_{i=1}^{n_f} b_1\left(\frac{\mu^2}{m_i^2}\right),
$$

where

$$b_0\left(\frac{\mu^2}{m^2}\right)=1+\frac{3}{2x}(1-G(x))\;,$$

$$b_1\left(\frac{\mu^2}{m^2}\right)=-[16(1-x^2)C_F+(1+8x^2)C_A]\frac{\sigma(x)T_F}{6x^2(1-x)}+\frac{2}{3x^2}(C_A-2C_F)I(x)$$
$$-\frac{2}{3x}\tilde{I}_3^{(4)}C_AT_F-[(1+3x-10x^2+12x^3)C_A-3(3-3x-4x^2+8x^3)C_F]\frac{4}{3x}G^2(x)$$
$$+[(147-4x-100x^2+8x^3)C_A+168(1-x)C_F+6(9+4x)\ln(-4x)C_A]\frac{1}{9x}G(x)$$
$$-[(99+62x)C_A+12(11+3x)C_F+2(27+24x-2x^2)\ln(-4x)C_A]\frac{1}{9x}\;,$$

with $x=-\mu^2/(4m^2)$.



1.00
1.65
2.72
4.48
E (GeV)
12.18
6.0
5.0
4.0
3.0
2.0
2-loop BF-MOM exact
2-loop BF-MOM Shirkov's formula
1/αs(E)
0.0
1.0
2.0
3.0
4.0
5.0
log ([E (GeV)]2)

$\overline{\mathrm{MS}}$ **versus** MOM

Relation between renormalized coupling constants

$$h_{\mathrm{MOM}} \;=\; \frac{g^2_{\mathrm{MOM}}}{(4\pi)^2} = H(h,\mu^2) = h + k_1(\mu^2)h^2 + (k_2(\mu^2) + k_1^2(\mu^2))h^3 \qquad (1)$$

where

$$k_1(\mu^2) = \frac{205}{36}C_A + T_F\sum_{i=1}^{n_F} z_{1i}\,,\;\; k_2(\mu^2) = \left(\frac{2687}{72} - \frac{57}{8}\zeta_3\right) C_A^2 + T_F\sum_{i=1}^{n_F} z_{2i}$$

with

$$z_{1i} = -\Pi_1\left(r_i\right) - \frac{20}{9} - \frac{4}{3}\, l_i$$

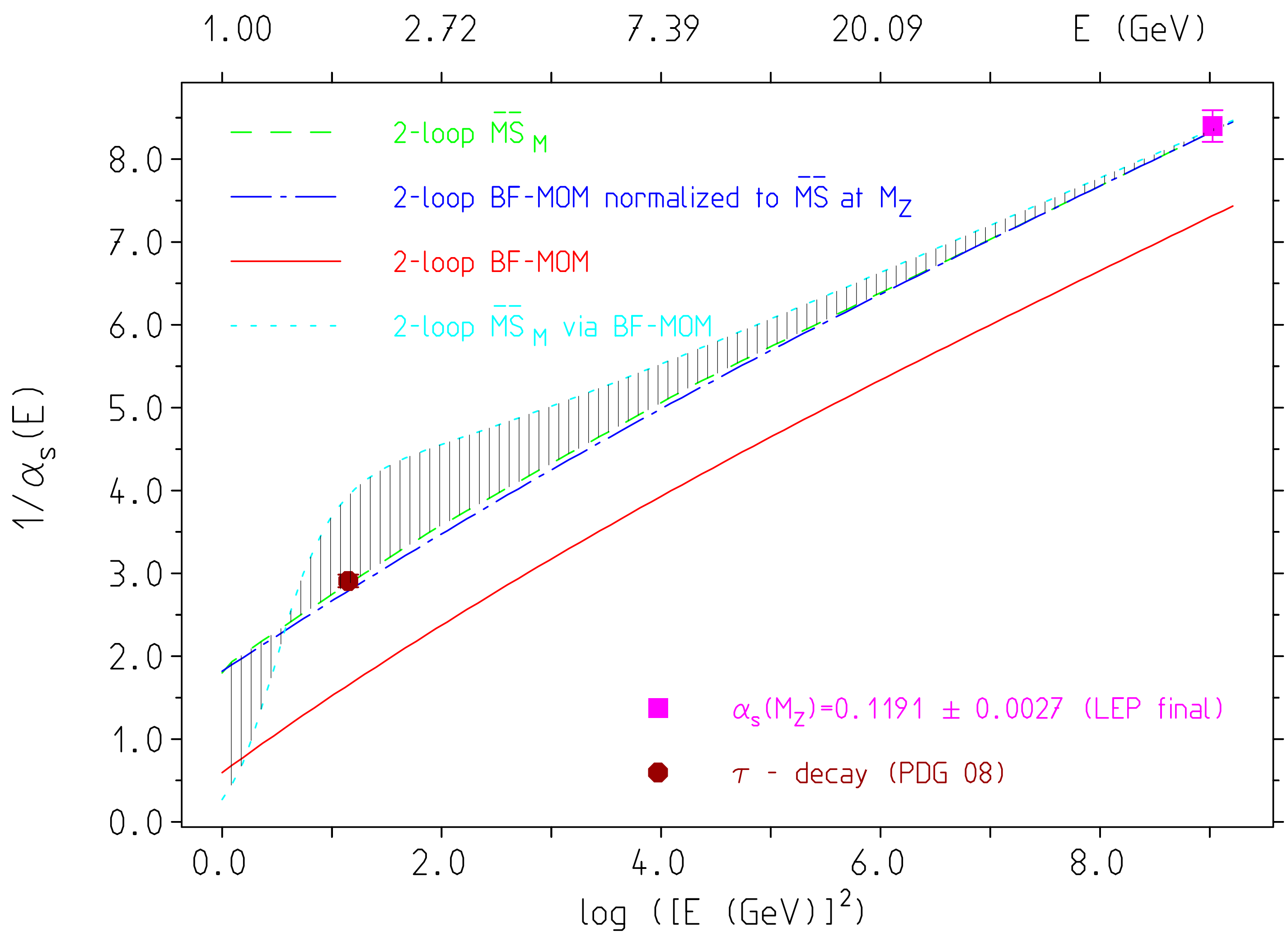
$$z_{2i} = -\Pi_2\left(r_i\right) - \left(\frac{52}{3} + \frac{20}{3}l_i\right) C_A - \left(\frac{55}{3} + 4l_i\right) C_F$$

where

$$l_i = \ln r_i, \qquad r_i = \frac{\mu^2}{m_i^2}$$

From the above equation we obtain

$$\beta_{0\,\mathrm{MOM}} = \beta_0 - \mu^2\frac{\partial}{\partial\mu^2}k_1(\mu^2)\,,\;\; \beta_{1\,\mathrm{MOM}} = \beta_1 - \mu^2\frac{\partial}{\partial\mu^2}k_2(\mu^2).$$



Comparison of α_s in the $\mathrm{BF-MOM}$ and the $\overline{\mathrm{MS}}$ schemes with $\alpha^{(5)}_{s\,\overline{\mathrm{MS}}}$ = 0.12 at scale M_Z=91.19 GeV and $\alpha_{s\ \mathrm{BF-MOM}}$ at this scale calculated using (??). The dotted line is the $\overline{\mathrm{MS}}$ coupling calculated from $\alpha_{s\ \mathrm{BF-MOM}}(E)$ by inverting (??).

The rescaling scheme

- **Problem: large scale independent constants in** $\mathrm{MOM} \leftrightarrow \overline{\mathrm{MS}}$
- **At high Q^2 the β–functions become identical**
- **Scale parameters μ in MOM and $\overline{\mathrm{MS}}$ need not have identical meaning**
- **Rescale μ_{MOM} relative to $\mu_{\overline{\mathrm{MS}}}$ such that couplings essentially agree at high energies**

$$\bar{h}(\mu^2) = h(\mu^2) + k_1 h^2(\mu^2) + \tilde{k}_2 h^3(\mu^2) + O(h^4)$$

with $\tilde{k}_2 = k_2 + k_1^2$. We may absorb the disturbing large term k_1 into a rescaling of μ by a factor x_0 such

$$\bar{h}((x_0\mu)^2) = h(\mu^2) + 0 + O(h^3) \ .$$

and the rescaling factor x_0 is determined by the equation

$$k_1 = U_1(x_0^2, \{m_i^2/\mu^2\}) \ .$$

In our mass dependent scheme we require this to be true only at very large scales $\mu^2 \gg m_f^2$ for all flavors f including the top quark.

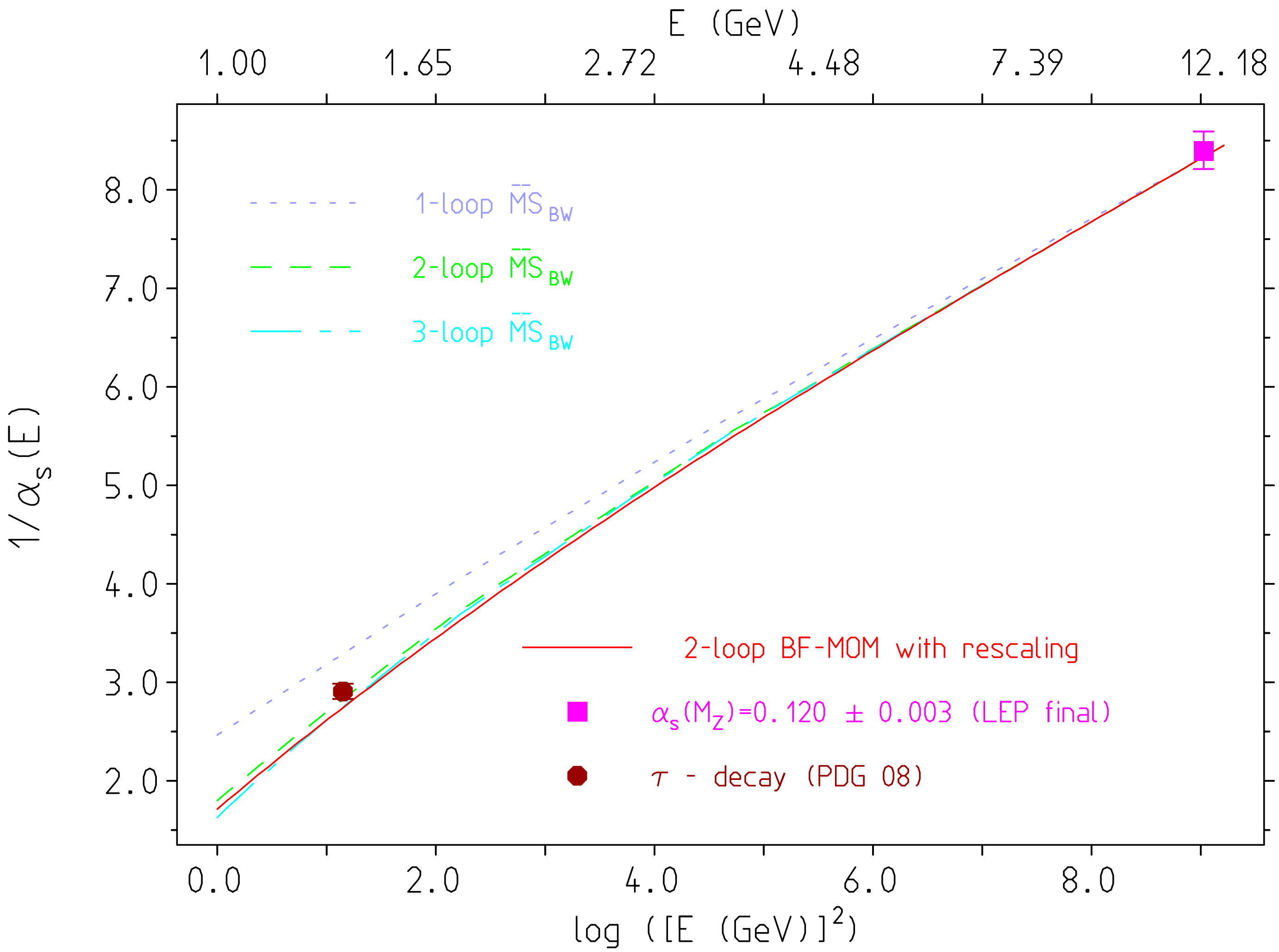
For the $\mathrm{BF-MOM}$ scheme the rescaling factor x_0 is determined by

$$\ln(x_0^2) = \frac{\frac{205}{36}C_A - \frac{20}{9}T_F n_F}{\frac{11}{3}C_A - \frac{4}{3}T_F n_F} = 125/84$$

for QCD with $n_F = 6$ flavors. Numerically: $x_0 \simeq 2.0144$

Rescaling changes the coefficients from $k_1 \simeq 10.42, \quad \tilde{k}_2 \simeq 126.35 \rightarrow$ $k_{1\,\mathrm{eff}} = 0, \quad \tilde{k}_{2\,\mathrm{eff}} \simeq -32.46$

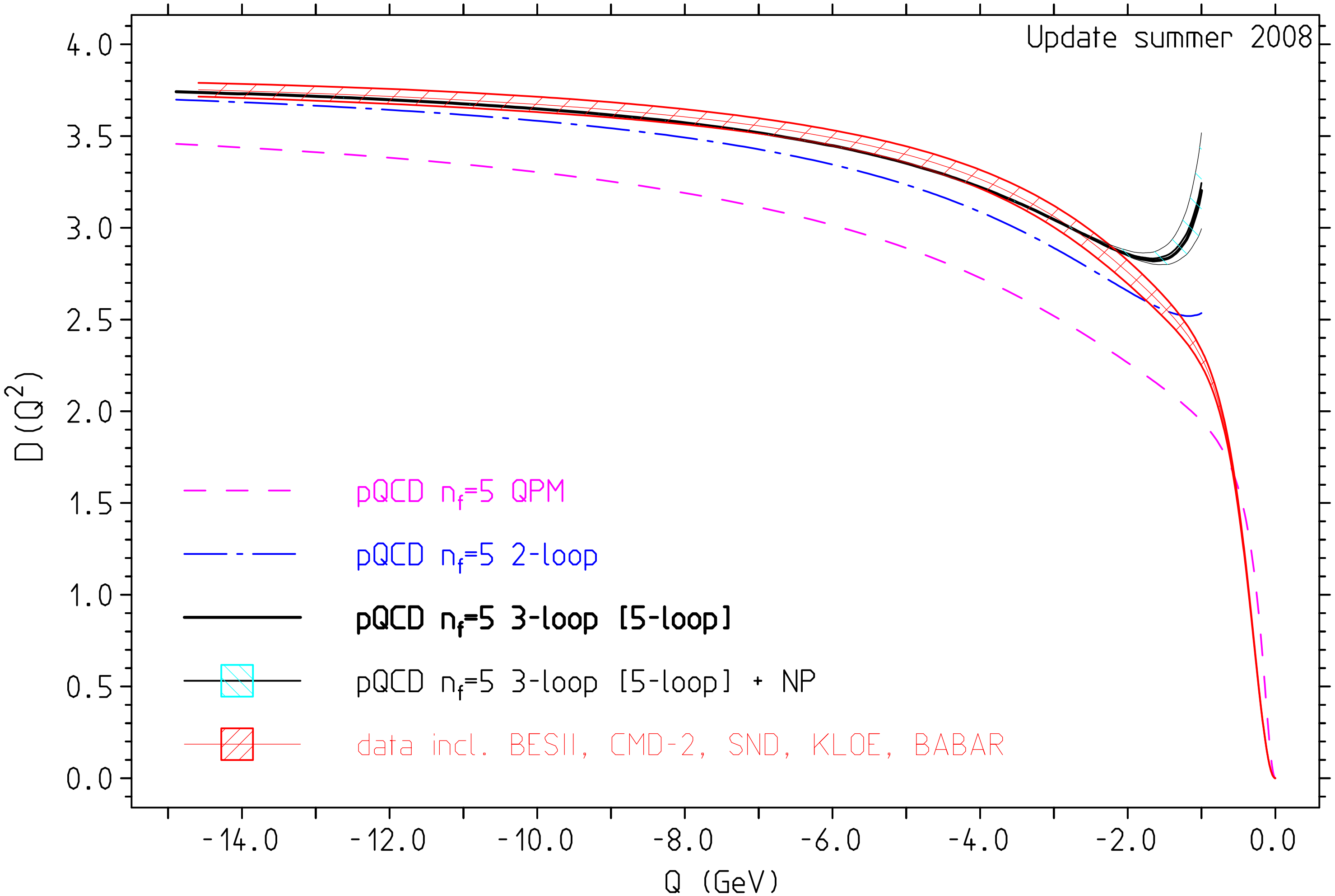
- substantial improvement also for the next to leading coefficient
- no large deviation between rescaling improved MOM versus $\overline{\mathrm{MS}}$ relationship
- however, size of mass effects as large as higher order effects
- estimate of theoretical uncertainties (scheme dependence) need inclusion of mass effects (set in very slowly in Euclidean region, no θ–function jumps)



Comparison of the α_s evolution in the space–like region normalized to a common value α_s = 0.12 at scale M_Z = 91.19 GeV. The dotted, dashed, dash–dot and the dash–dot–dot–dot curves show, respectively, the one–loop, two–loop, three–loop and the four–loop $\overline{\mathrm{MS}}$ evolution for BW–matching. The full line represents the exact $\mathrm{BF}-\mathrm{MOM}$ running coupling.

“Experimental” Adler–function versus theory (pQCD + NP)

Error includes statistical + systematic here (in contrast to most R-plots showing statistical errors only)!



(Eidelman, FJ, Kataev, Veretin 98, FJ 06 update (BES, CMD-2)) theory based on results by Chetyrkin, Kühn et al.

③ Application: Precision Determination of $\alpha(M_Z)$

✗ **Adler function as monitor for comparing theory and data**

- **space-like approach: pQCD works well for $\sqrt{Q^2=-q^2} > 2.5$ GeV (see plot)**

$\Rightarrow$ **pQCD works well to predict $D(Q^2)$ down to $s_0 = (2.5\,\text{GeV})^2$; use this to calculate**

$$\Delta\alpha_{\rm had}(-Q^2) \sim \frac{\alpha}{3\pi}\int dQ'^2\,\frac{D(Q'^2)}{Q'^2}$$

$$\boxed{\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) \;=\; & \left[\Delta\alpha^{(5)}_{\rm had}(-M_Z^2)-\Delta\alpha^{(5)}_{\rm had}(-s_0)\right]^{\rm pQCD} \\ & + \Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data}\end{aligned}}$$

and obtain, for $s_0 = (2.5\,\text{GeV})^2$: **(FJ 98-08)**

$$\Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data} \;=\; 0.007354 \pm 0.000107$$

$$\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) = 0.027477 \pm 0.000149 \qquad \Delta\alpha^{(5)}_{\rm had}(M_Z^2) = 0.027515 \pm 0.000149$$

$$\boxed{\alpha^{-1}(M_Z^2) = 128.957 \pm 0.020}$$

The virtues of this analysis are obvious:

- no problems with physical threshold and resonances
- pQCD is used only where we can check it to work (Euclidean, $Q^2 \gtrsim$ 2.5 GeV).
- no manipulation of data, no assumptions about global or local duality.
- non–perturbative "remainder" $\Delta\alpha^{(5)}_{\rm had}(-s_0)$ is mainly sensitive to low energy data !!!

Status of QCD parameters: (Kühn, Steinhauser, Sturm)

$$\alpha_s(M_Z) = 0.1189(20)$$

$$m_c(m_c) = 1.286(13) \text{ GeV} \quad \Leftrightarrow \quad M_c^{3-\rm loop} = 1.666(17) \text{ GeV}$$

$$m_b(m_c) = 4.164(25) \text{ GeV} \quad \Leftrightarrow \quad M_b^{3-\rm loop} = 4.800(29) \text{ GeV}$$

Lattice QCD very close to be competitive [crucial cross check!]

Needed improvements (integral part of this strategy):

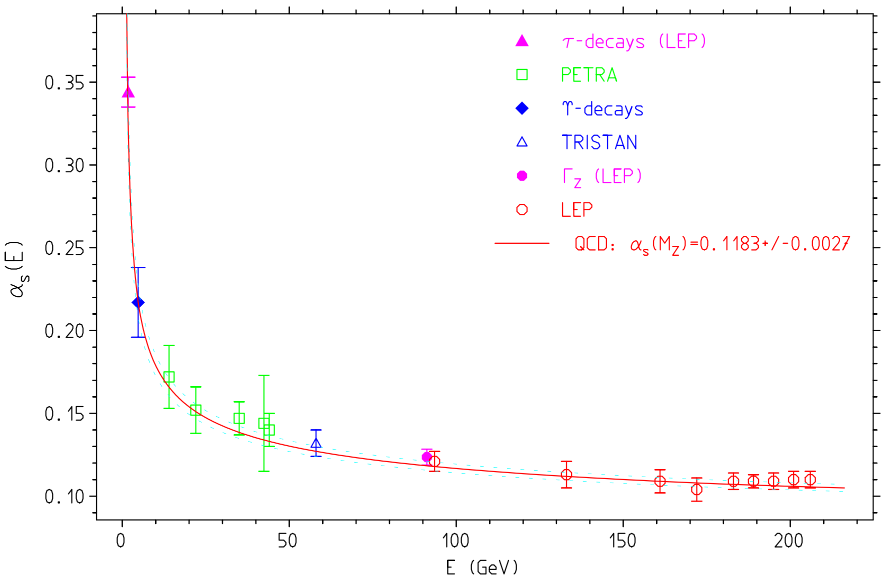
– 4–loop massive pQCD calculation of Adler function $\Leftrightarrow$ series expansion plus Padé improvement; essentially equivalent to 4–loop calculation of $R(s)$

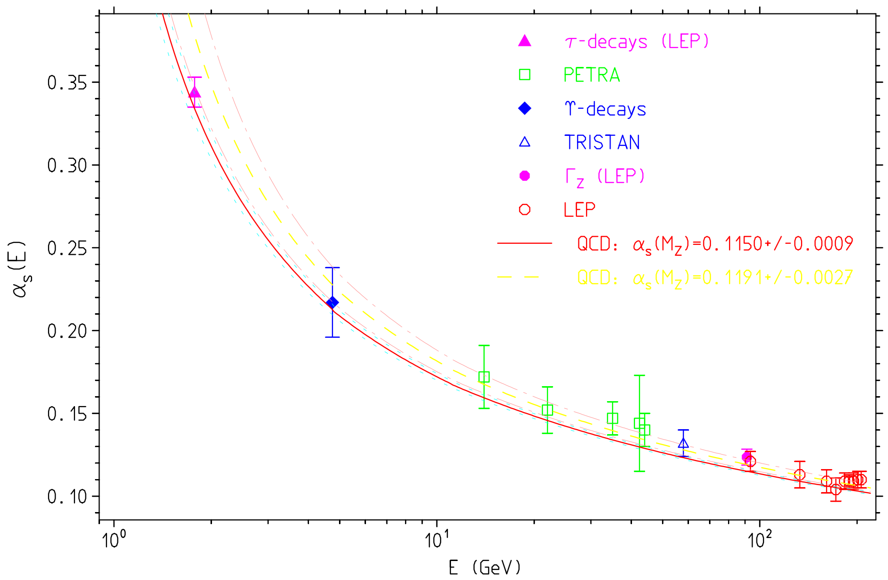
– m_c improvement; sum rule and lattice QCD evaluations

– α_s in low Q^2 region

Testable models:	–	"analytized" α_s	Shirkhov, Solovtsov, . . .
	–	instanton liquid model current vs. constituent quark masses	Dorokhov, . . .

④ Extension of the Perturbative Regime: APT





(Compilation S. Bethke)

- strong interactions at low scales; pQCD fails below about M_τ
- perturbative QCD is **not** full QCD; not Borel summable, no pions (spontaneous breaking of chiral symmetry), no condensates, no instantons etc.
- Non-perturbative Adler function provides a convincing constraint on strong coupling strength [Landau pole is artefact of perturbation theory]
- Minimal analytic extension by Shirkov and Solovtsov 1996: APT Minimal APT=Maximal ATP=ATP (Stefanis): renormalization group invariance plus causality

- Removes Landau Pole in Euclidean
- Resumms π^2–terms in Minkowski
- Improves convergence of pQCD, reduces scale dependence
- Be aware: does not promote pQCD to work down to very low scales

Power correction term cures singularity.

Analytization: $a^n \to \mathcal{A}_n^{(\ell)}$

Leading order:

$$\alpha_{s\,\mathrm{E}}^{(1)}(Q^2) = \mathcal{A}_1^{(1)} = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\sigma}{\sigma+Q^2}\,\rho_1(\sigma) = \frac{1}{\beta_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2-Q^2}\right]\,, \quad Q^2>0$$

$$\alpha_{s\,\mathrm{M}}^{(1)}(s) = \mathfrak{A}_1^{(1)} = \frac{1}{\pi}\int_s^\infty \frac{\mathrm{d}\sigma}{\sigma}\,\rho_1(\sigma) = \frac{1}{\beta_0\pi}\,\arccos\frac{\ln(s/\Lambda^2)}{\sqrt{\ln(s/\Lambda^2)^2+\pi^2}}\,, \quad s>0$$

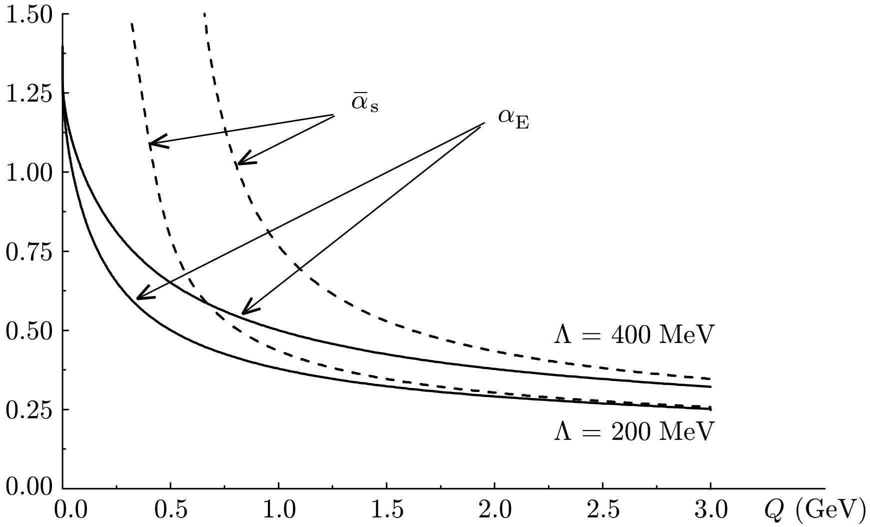
with $\rho_1(s) = \mathrm{Im}\;\alpha_s(-s-\mathrm{i}\epsilon)$ allows for consistent RG resummation under the dispersion integral.

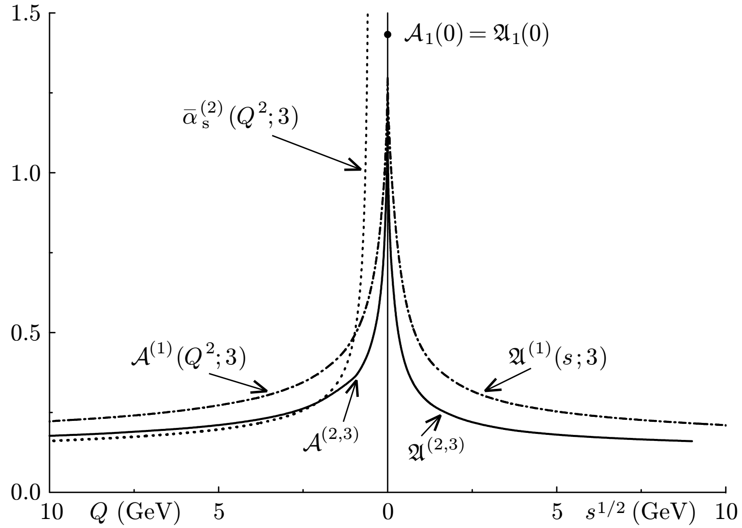
Recurrence relations:

$$\mathcal{A}_n^{(\ell)} = (-1)^n\,\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}L}\right)^n \mathcal{A}_1^{(\ell)}$$

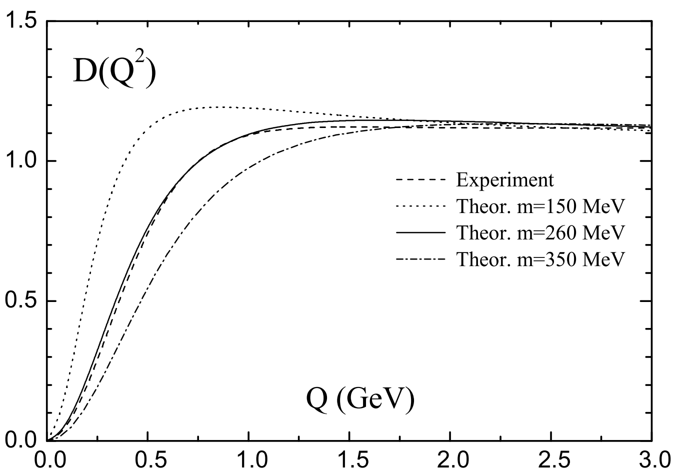
etc.

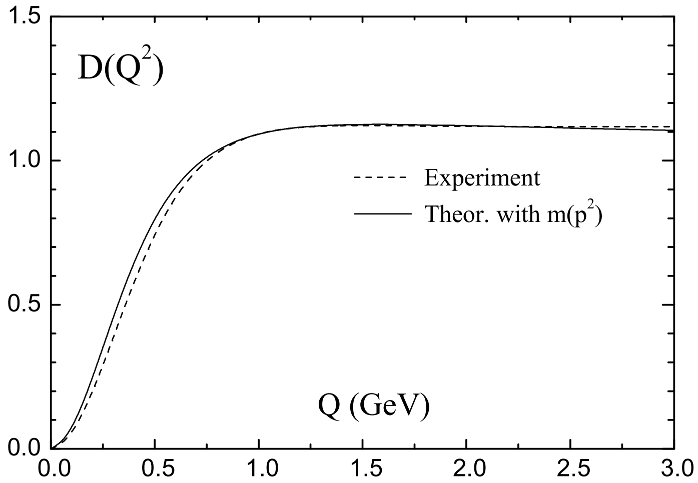
see presentations by Solovtsova, Stefanis, Valenzuella. Nesterenko, Bakulev





(Shirkov, Solovtsov)





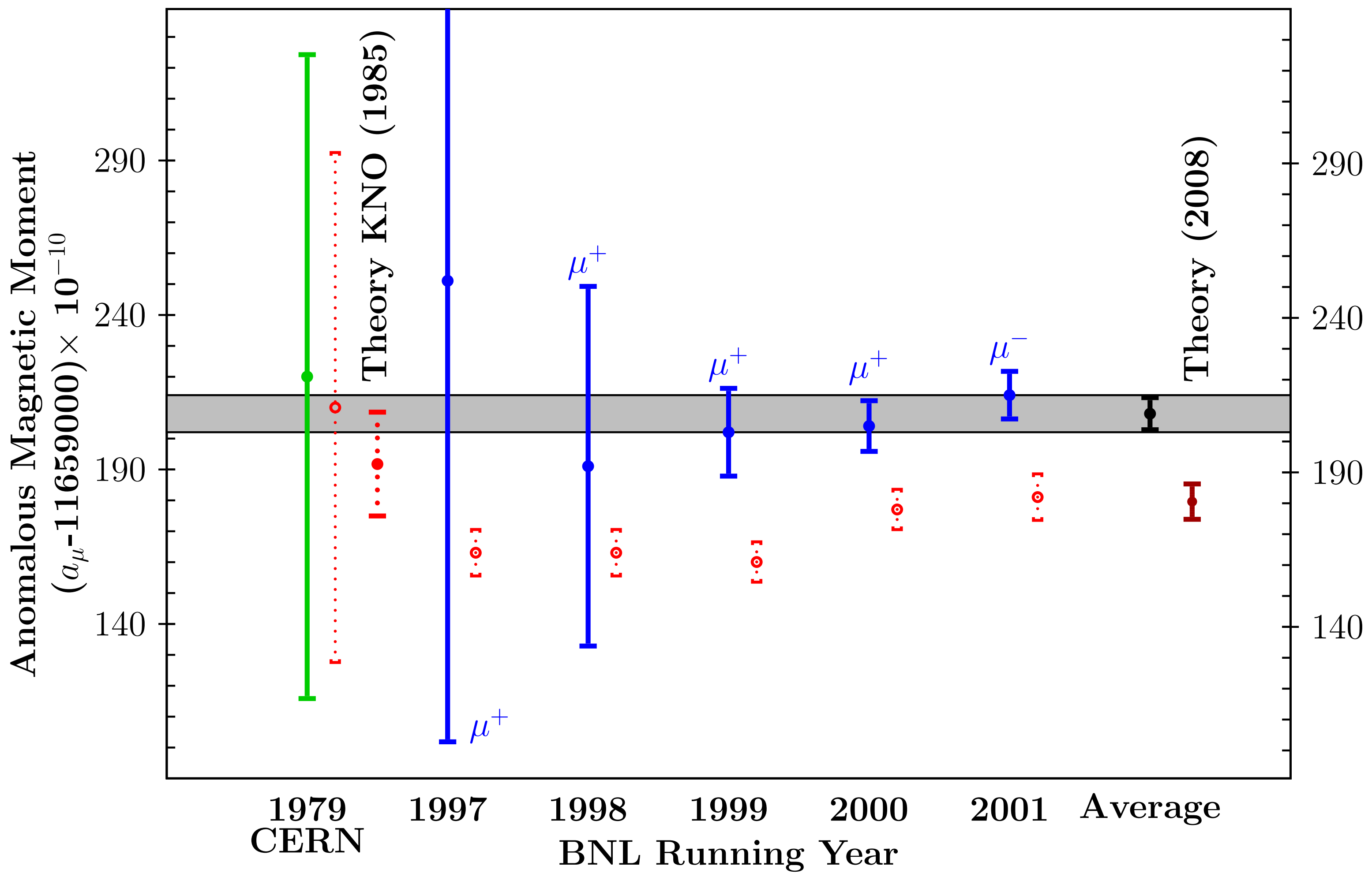
(Milton, Solovtsov, Solovtsova)

❑ Precision determination of hadronic contribution to Muon $g-2$

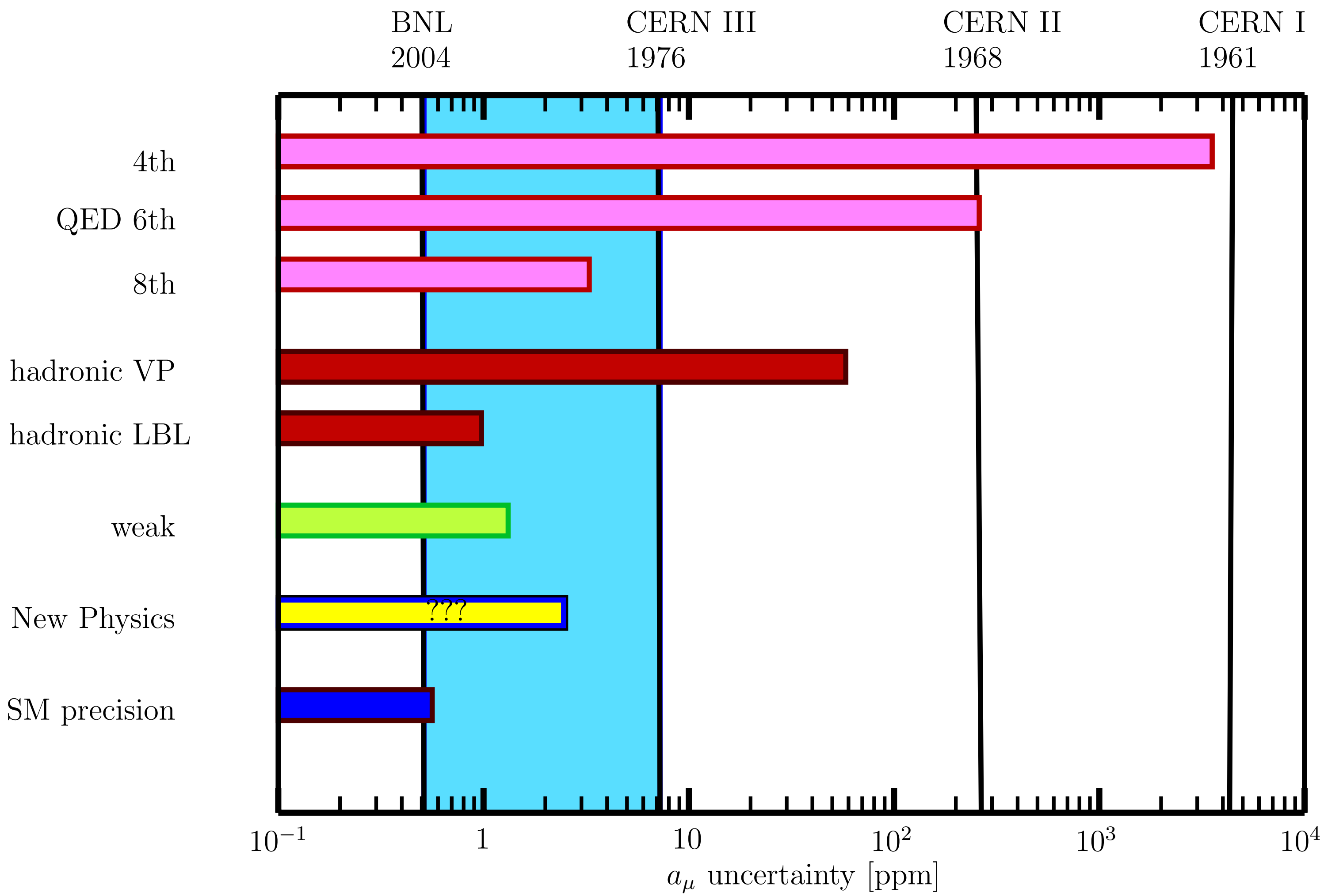
Using known representation

$$a_\mu^{\rm had} = \frac{\alpha}{\pi} m_\mu^2 \int\limits_0^1 dx\, x\,(2-x)\,\left(D(Q^2(x))/Q^2(x)\right)$$

we may apply a cut at $s_0 = 2.5$ GeV^2 and we safely can use pQCD to calculate about 50% of the contribution. The rest is evaluated in terms of the experimental Adler function.



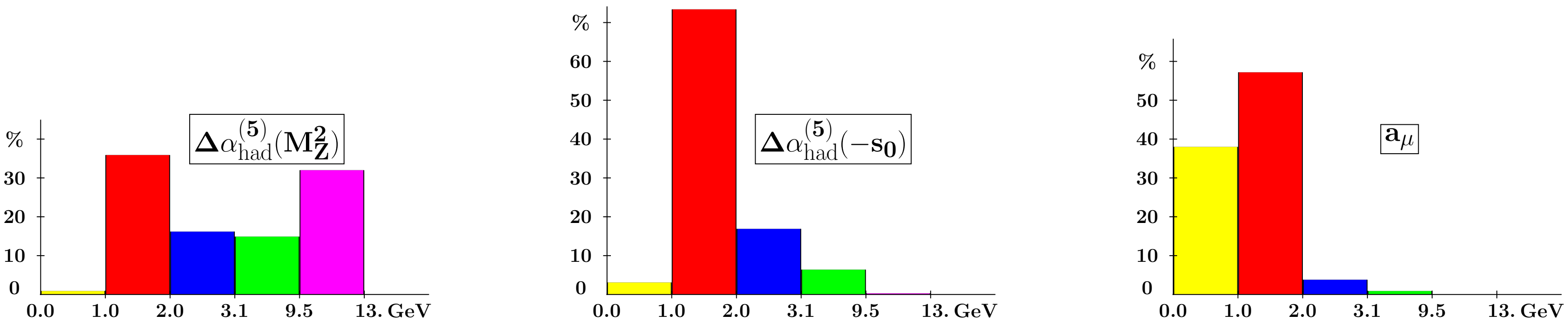
Anomalous Magnetic Moment
$(a_\mu\text{-}11659000)\times 10^{-10}$
290
240
190
140
Theory KNO (1985)
Theory (2008)
μ^+
μ^+
μ^+
μ^+
μ^+
μ^-
1979
CERN
1997
1998
1999
2000
2001
Average
BNL Running Year



Sensitivity of $g-2$ experiments to various contributions. The increase in precision with the BNL $g-2$ experiment is shown as a gray (blue) vertical band. New Physics is illustrated by the deviation $(a_\mu^{\rm exp}-a_\mu^{\rm the})/a_\mu^{\rm exp}$.

⑤ Conclusions and Outlook

The role of the Adler function approach to determine related precision observables may be illustrated by the error profiles between $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$, $\Delta\alpha^{(5)}_{\rm had}(-s_0)$ and a_μ:



- **Adler function will play key role in improving the effective fine structure constant in future. Presently $\alpha(M_Z^2)$ is a factor of 10 less well known than the next worse SM parameter M_Z.**

- **Key role in reaching this goal play low energy $\gamma^* \to$ hadrons experiments like VEPP-2000/Novosibirsk and proposed DAFNE2/Frascati and others (BES3, CLEOc, Belle).**

- **Radiative correction calculations needed for high precision determination of $R(s)^{\rm exp}$ absolutely mandatory.**

- **the Adler function is an excellent object to test pQCD as well as improvements of it towards low energy, and a test for all kinds of hadronic models. Analyticized perturbation theory is able to extend the range where pQCD works. Down to which scale?**

- **Note however: the Adler function (vacuum polarization function) is a single energy scale quantity [in the high energy limit just a constant] testing only limited aspects of hadronic models [at low energy just the ρ vector meson gives 75% of the contribution, as e.g. in the hadronic part of the muon $g-2$].**

- **Nevertheless: it remains a highly nontrivial task to get it from first principle calculations [lattice needed].**

The Adler function remains a big challenge

for perturbative and non-perturbative QCD

Dear Dmitry I wish you all the best and and many more happy years to come!

Dmitry V. Shirkov Fest 80