Improved ϵ-expansion in theory of turbulence:

inclusion of an infrared irrelevant operator as a way of summation of nearest singularities.

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results of work by L.Ts. Adzhemyan & M.Hnatič

ε-expansion in theory of turbulence

$$Q(\varepsilon, d) = \sum_{k=0}^{\infty} Q_k(d)\varepsilon^k$$

Coefficients $Q_k(d)$: singularities at $(d-2) \equiv 2\Delta = 0$

Laurent series:

$$Q_k(d) = \sum_{l=0}^{\infty} q_{kl}\, \Delta^{l-k}$$

L.Ts. Adzhemyan, N.V. Antonov, M.V. Kompaniets, and A.N. Vasil'ev, Int. J. Mod. Phys. B **17**, 2137 (2003)

Kolmogorov constant, skewness-factor:

two-loop contribution in comparison with one-loop approximation is lar-ge the nearest singularity strongy manifests itself at realistic value $d = 3 \Rightarrow$ improvement of ε-expansion by means of summation of singular contributions in all orders of this expansion

A version to solve this problem has been proposed in

L.Ts. Adzhemyan, J. Honkonen, M.V. Kompaniets, and A.N. Vasil'ev, Phys. Rev. E **71**, 036305 (2005)

– additional couter-terms $\Rightarrow$ elimmination of divergencies in Δ

– theory with two formally small parameters ε Δ, which satisfy relation $\zeta \equiv \varepsilon/\Delta = const$

J. Honkonen and M.Yu. Nalimov, Z. Phys. B **99**, 297 (1996)

$$Q(\varepsilon, \zeta) = \sum_{k=0}^{\infty} b_k(\zeta) \varepsilon^k \,, \qquad \zeta \equiv \varepsilon/\Delta$$

$$Q(\varepsilon, d) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \zeta^k q_{kl} \, \Delta^l$$

$$b_k(\zeta) = \sum_{l=0}^{\infty} q_{kl} \, \zeta^{l-k}$$

n-loop calculation in first sheme approximates sought quatity by expression

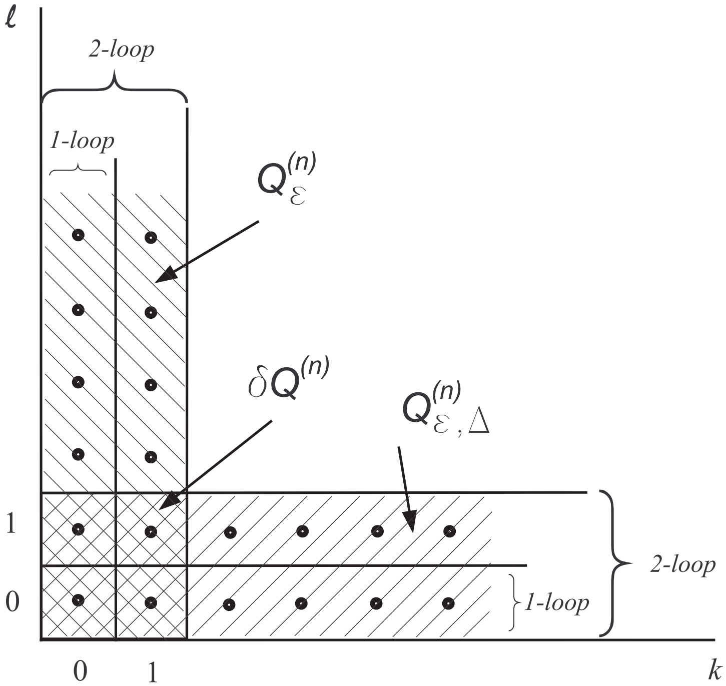
$$Q^{(n)}_{\varepsilon,\,d} \equiv \sum_{k=0}^{n-1} Q_k(d)\varepsilon^k$$

inclusion of all terms of double sum in first vertical bands (depicted on figure)

Calculations in the second scheme correspond to the approximation

$$Q^{(n)}_{\varepsilon,\,\zeta} \equiv \sum_{k=0}^{n-1} b_k(\zeta)\varepsilon^k$$

inclusion of all terms of double sum in first horizontal bands (depicted on figure)



ℓ
2-loop
1-loop
$Q_{\varepsilon}^{(n)}$
$\delta Q^{(n)}$
$Q_{\varepsilon,\Delta}^{(n)}$
1
0
2-loop
1-loop
0
1
k

Algorithm proposed by L. Adzemyan et al:

$$Q^{(n)}_{eff} = Q^{(n)}_{\varepsilon,d} + Q^{(n)}_{\varepsilon,\zeta} - \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \zeta^k q_{kl}\,\Delta^l$$

includes all terms in n vertical and horizontal bands simultaneously ("n-region"). Such calculations led to the marked improwement of agreement of Kolmogorov constant and skewness factor with their experimental values!

Similar purpose can be achieved by another tools: using renormalization anologous to the second approach (Honkonen, Nalimov), but not in the framework of minimal substraction scheme as it was done by Adzemyan et al.

the scheme with normalization point.

n-loop calculation in such a scheme ($without\ expansion\ in\ \varepsilon$) allow us to guarantee true reproduction of terms from n-region and leads to the good agreement with experiment already in one-loop approximation.

From the point of view of renormalization of d-dimensional theory this approach correspond to the inclusion in action infrared-irrelevant operator at $d > 2$, which becomes to be relevant at $d \to 2$.

Stochastic model of developed turbulence

$$\nabla_t \varphi_i = \nu_0 \partial^2 \varphi_i - \partial_i \mathcal{P} + f_i, \qquad \nabla_t \equiv \partial_t + (\varphi\partial)$$

$$\begin{aligned} \left\langle f_i(t,\mathbf{x}) f_j(t',\mathbf{x}')\right\rangle &\equiv D_{ij}(t,\mathbf{x};t',\mathbf{x}') \\ &= \frac{\delta(t-t')}{(2\pi)^d} \int d\mathbf{k}\, P_{ij}(\mathbf{k})\, d_f(k)\, \exp\left[\mathrm{i}\mathbf{k}\,(\mathbf{x}-\mathbf{x}')\right] \end{aligned}$$

$$d_f(k) = D_0 k^{4-d-2\varepsilon}$$

physical value $\varepsilon = 2$, because as $\varepsilon \to 2,\ D_0 \sim (2-\varepsilon) \Rightarrow d_F(k) \sim \delta(\mathbf{k})$,

$$d_f(k) = D_0\, k^{4-d-2\varepsilon}\, h(m/k) \qquad h(0) = 1$$

$$h(m/k) = \Theta(k-m)$$

$$S(\Phi) = \varphi' D \varphi'/2 + \varphi'[-\partial_t\varphi + \nu_0\partial^2\varphi - (\varphi\partial)\varphi]$$

for $d > 2$ superficial UF-divergence is present only in one-irreducible function $\langle\varphi'\varphi\rangle$, and can be eliminated by counter-term $\varphi'\partial^2\varphi$

In the special case $d = 2$ a new UV divergence appears in the 1-irreducible function $\langle\varphi'\varphi'\rangle_{1-\mathrm{ir}}$

Introduction of couter-term $\varphi'\partial^2\varphi$ to the action is reproduced by multiplicative renormalization of parameters ν_0 g_0 :

$$\nu_0 = \nu Z_\nu, \qquad D_0 = g_0\nu_0^3 = g\mu^{2\varepsilon}\nu^3 \qquad g_0 = g\mu^{2\varepsilon}Z_g \qquad Z_g = Z_\nu^{-3}$$

Critical dimensions of field Δ_φ and frequency Δ_ω are truncated:

$$\Delta_\varphi = 1 - 2\varepsilon/3 \qquad \Delta_\omega = 2 - 2\varepsilon/3$$

These formulae are exact and they have not higher correctins in ε They are a result of relation between renormalization constants Z_g and Z_ν, which at the same time results from absence of renormalization of non-local contribution of correlator of random forcing in action

realistic value $\varepsilon = 2 \Rightarrow$ exponents acquire kolmogorov values

$$\Delta_\varphi = -1/3 \qquad \Delta_\omega = 2/3$$

Additional renormalization with inclusion of divergencies at $d \to 2$

additional divergencies in 1-irreducible fuction $\langle\varphi'\varphi'\rangle_{1-\mathrm{ir}}$ Divergencies manifest themselves in the form of singularities in $\Delta = (d-2)/2$

Conter-term $\varphi'\partial^2\varphi'$ is local and quite different from nonlocal contribution $\varphi' D\varphi'/2$

$$d_f(k) = D_{10}k^{2-2\Delta-2\varepsilon} + D_{20}k^2 \quad D_{i0} = g_{i0}\nu_0^3 \;\; i = 1,2$$

$$S(\Phi) = \varphi'(D_{10}k^{2-2\Delta-2\varepsilon} + D_{20}k^2)\varphi'/2 + \varphi'[-\partial_t\varphi + \nu_0\partial^2\varphi - (\varphi\partial)\varphi]$$

$$\nu_0 = \nu Z_\nu \qquad D_{10} = g_{10}\nu_0^3 = g_1\mu^{2\varepsilon}\nu^3 \qquad D_{20} = g_{20}\nu_0^3 = g_2\mu^{-2\Delta}\nu^3 Z_{D_2}$$

$$g_{10} = g_1\mu^{2\varepsilon}Z_{g_1} \qquad g_{20} = g_2\mu^{-2\Delta}Z_{g_2} \qquad Z_{g_1}Z_\nu^3 = 1 \qquad Z_{g_2}Z_\nu^3 = Z_{D_2}$$

independent renormalization constants – viscosity ν_0 and amplitude D_{20}

the aplitude D_{10} of nonlocal correlator of random forcing is not renormalized!!! Considering ε and Δ as formally small parameters $\varepsilon \sim \Delta \to 0$, we can find renormalization constants Z_ν Z_{D_2} from condition of UV - finitness (absence of poles in ε at $\Delta/\varepsilon = \zeta = const$) 1-irreducible functions $\langle\varphi'\varphi\rangle_{1-\mathrm{ir}}\Big|_{\omega=0}$ $\langle\varphi'\varphi'\rangle_{1-\mathrm{ir}}\Big|_{\omega=0}$,
renormalized theory is free off poles in ε and Δ

Application of RG approach to this theory leads to the finding of physical characteristics in the form of double expansions in ε Δ

renormalization sheme with normalization point

$$\Gamma_{\varphi'\varphi} = \frac{\left\langle \varphi'_i \varphi_i \right\rangle_{1-\mathrm{ir}}\Big|_{\omega=0}}{\nu p^2 (1-d)}$$

$$\Gamma_{\varphi'\varphi'} = \frac{\left\langle \varphi'_i \varphi'_i \right\rangle_{1-\mathrm{ir}}\Big|_{\omega=0}}{\nu^3 p^2 (d-1)} - g_1 (\mu/p)^{2\Delta+2\varepsilon}$$

renormalization constants from conditions:

$$\Gamma_{\varphi'\varphi}\Big|_{p=0,\,\mu=m} = 1\,, \qquad \Gamma_{\varphi'\varphi'}\Big|_{p=0,\,\mu=m} = g_2$$

Because physical quantities – critical exponents and universal scaling functions – are independent of renormalization scheme, aplication of RG to the renormalized theory according to last conditions leads to same double ε, Δ-expansions, as it is MS scheme.

Our assumption consists in the use of this normalization scheme but without usage of ε, Δ-expansion.

Such a scheme properly reproduces leading terms of expansion in regime $\varepsilon \to 0\,, \Delta = const$, and simultaneously in the regime $\varepsilon \sim \Delta \to 0$.

At the same time in the first case additional term $\varphi' \partial^2 \varphi'$ plays the role of infrared irrelevant supplement to the action and constant Z_{D_2} describes renormalization of this additional operator.

One-loop calculation of renormalization constants

$$Z_\nu = 1 + \frac{d-1}{4(d+2)}\Big(- \frac{u_1}{2\varepsilon} + \frac{u_2}{2\Delta}\Big)$$

$$Z_{D_2} = 1 + \frac{d^2-2}{4d(d+2)}\Big(- \frac{u_1^2}{2(2\varepsilon+\Delta)u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta}\Big)$$

$$u_i \equiv \bar{S}_d g_i \qquad \bar{S}_d \equiv S_d/(2\pi)^d \qquad S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$$

$$Z_{u_1} = 1 + \frac{3(d-1)}{4(d+2)}\Big(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\Big)$$

$$Z_{u_2} = 1 + \frac{d^2-2}{4d(d+2)}\Big(- \frac{u_1^2}{2(2\varepsilon+\Delta)\,u_2} - \frac{u_1}{\varepsilon} + \frac{u_2}{2\Delta}\Big) + \frac{3(d-1)}{4(d+2)}\Big(\frac{u_1}{2\varepsilon} - \frac{u_2}{2\Delta}\Big)$$

RG-functions

$$\gamma_i = (\beta_1 \partial_{u_1} + \beta_2\, \partial_{u_2})\, ln Z_{u_i} \qquad i = 1,\, 2$$

$$\beta_1 = -u_1(2\varepsilon + \gamma_1) \qquad \beta_2 = -u_2(-2\Delta + \gamma_2)$$

fixed point u_* $\beta_i(u_*) = 0$:

$$u_1^* + u_2^* = \frac{8\varepsilon(d+2)}{3(d-1)}$$

$$u_2^* = \frac{\varepsilon^2}{\varepsilon + \Delta} \frac{8(d^2-2)(d+2)}{9d(d-1)^2}$$

The stability of fixed point is determined by sign of real part of eigen-values $\omega_\pm$ of matrix $\partial_j\beta_i\big|_{u=u_*}$:

$$\omega_\pm = \Delta + \frac{2\varepsilon(2d^2-3d+2)}{3d(d-1)} \pm$$

$$\pm\sqrt{\Delta^2 - \frac{4(d^2-2)}{3d(d-1)}\epsilon\,\Delta - \frac{4(d^2-2)(2d^2-3d+2)}{9d^2(d-1)^2}\varepsilon^2}$$

regime $\varepsilon \sim \Delta \to 0$:

$$\omega_{\pm} = \Delta + \frac{4\varepsilon}{3} \pm \sqrt{\Delta^2 - \frac{4}{3}\varepsilon\,\Delta - \frac{8}{9}\varepsilon^2} + O(\varepsilon^2)$$

At $\varepsilon \to 0,\ \Delta = const$

$$\omega_{-} = 2\varepsilon + \frac{2(d^2-2)}{3d(d-1)} \cdot \frac{\varepsilon^2}{\Delta} + O(\varepsilon^3)$$

$$\omega_{+} = 2\Delta + \frac{2(d^2-3d+4)}{3d(d-1)}\varepsilon - \frac{2(d^2-3d+4)}{3d(d-1)} \cdot \frac{\varepsilon^2}{\Delta} + O(\varepsilon^3)$$

quantity ω_- is correction index ω in usual ε-expansion, ω_+ critical dimension of infrared-irrelevant composite operator $\varphi'\partial^2\varphi'$

Terms $\sim \varepsilon$ are reliable - for ω_- it is known one-loop expression for index ω , for ω_+ result was checked by direct calculations. Terms $\sim \varepsilon^2/\Delta$ give true singular part in Δ in coeficients at ε^2 – for ω_- it was confirmed by two-loop calculation by L.A. et.al. for ω_+ it was checked in the framework of usual ε-expansion by calculation of renormalization constant of composite operator $\varphi'\partial^2\varphi'$ in two-loop approximation.

Expression for ω in fact, properly involves main singular parts of the form $\varepsilon(\varepsilon/\Delta)^k$ and all leading terms ε-expansion, i.e. the first terms corresponding Laurent series for coefficients $Q_k(d)$

Next "loop calculation" in the our renormalization scheme guarantees that results become more precise step by step in the sence of increase of number of true terms of ε-expansion and numbers of singular in Δ contributions: n-loop calculation properly reproduces n coefficients $Q_k(d)$ and simultaneously in the rest $Q_k(d)$ the first n terms of Laurent series properly will be reproduced.

$$Q(\varepsilon, d) = \sum_{k=0}^{\infty} Q_k(d)\varepsilon^k$$

$$Q_k(d) = \sum_{l=0}^{\infty} a_{kl}\, \Delta^{l-k}$$

Calculation of Kolmogorov constant

Structure functions

$$S_n(r) \equiv \langle [\varphi_r(t, \mathbf{x} + \mathbf{r}) - \varphi_r(t, \mathbf{x})]^n \rangle, \qquad \varphi_r \equiv (\varphi_i \cdot r_i)/r, \quad r \equiv |\mathbf{r}|$$

According to Kolmogorov theory, structure function $S_2(r)$ has in inertial range form

$$S_2(r) = C_K \overline{\mathcal{E}}^{2/3} r^{2/3}$$

The amplitude of triple structure function $S_3(r)$ is determined in Kolmogorov theory exactly:

$$S_3(r) = -\frac{12}{d(d+2)} \overline{\mathcal{E}}\, r$$

From the point of view of RG the skewness-factor is more suitable

$$\mathcal{S} \equiv S_3 / S_2^{3/2}$$

$$\mathcal{S} = -\frac{d(d+2)}{12\, C_K^{3/2}}$$

Skewness-factor ia an analog of the universal ratios of amplitudes in theory of critical behaviour, it is dimensionless and is independent of amplitudes D_{i0}. Using some additional reasons it can be shown that an couterpart of skewness-factor is more suitable

$$\mathcal{DS}(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}}$$

Calculation of this quantity in one-loop approximation gives

$$\mathcal{DS}(\varepsilon) = \left[4(d-1)/9/u_{1*}^2\right]^{1/3} \cdot \frac{u_{1*}+u_{2*}}{2}$$

$\mathcal{DS} = 1.461$ Kolmogorov constant $C_K = 1.889$

fairish agreement with its experimental value $C_K \simeq 2.01$

usual $\epsilon - expansion$:

$C_K \simeq 1.47$

improved $\epsilon - expansion$ (L.A. et al):

$C_K \simeq 1.79$