Applications of RG methods
to the analysis of gauge invariant TMD PDF's

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The completely gauge-invariant transverse-momentum dependent parton distribution functions (PDF's) are analyzed from the point of view of their renormalization properties. The UV anomalous dimension is calculated in the one-loop order in the light-cone gauge and the consistent treatment of the additional singularities, which produce undesirable contributions in the anomalous dimensions, is discussed. The generalized renormalization of TMD PDF based on the renormalization procedure for the Wilson exponentials with obstructions is proposed. The reduction of the re-defined TMD PDF to the standard integrated PDF's is considered. The probabilistic interpretation of the parton distribution functions is discussed in terms of their anomalous dimensions.

- **Integrated PDF's:** definition; gauge invariance; RG properties

- **Unintegrated (TMD) PDF's**: shortcomings (complete gauge invariance, undesirable divergences)

- **Towards a solution:** gauge invariance; factorization; reduction to the integrated case

- On the **"completely correct"** TMD PDF′s: anomalous dimensions sum rule as a starting point; calculation of AD in the light-cone gauge; generalized renormalization and cancelation of extra divergences

- Conclusions and outlook

Integrated PDF's: definition; gauge invariance; RG properties

$$q_{i/h}(x) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-, \mathbf{0}_\perp)\gamma^+\psi_i(0^-, \mathbf{0}_\perp)|h(P)\rangle \sim$$

$$\sim \langle h(P)|a_i^\dagger a_i(x)|h(P)\rangle$$

Corresponds to the number of partons (probabilistic interpretation)

Gauge invariance is saved by the insertion of the **gauge link**:

$$[y, x|\Gamma] = \mathcal{P}\exp\left[-ig\int_{x[\Gamma]}^{y} dz_\mu A_a^\mu(z) t_a\right]$$

so that

$$\hat{q}_{i/h}(x) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-, \mathbf{0}_\perp)[\xi^-, 0^-]\gamma^+\psi_i(0^-, \mathbf{0}_\perp)|h(P)\rangle$$

Renormalization properties are described by the DGLAP equation:

$$\mu\frac{d}{d\mu}\hat{q}_{i/h}(x,\mu) = \sum_j \int_x^1 \frac{dz}{z}\, P_{ij}\left(\frac{x}{z}\right)\hat{q}_{j/h}(x,\mu)$$

where $P_{ij}\left(\frac{x}{z}\right)$ is the DGLAP integral kernel, which controls the dependence from the UV scale μ

Formally, this approach can be applied to the more complicated case—uPDF's (unintegrated, or TMD)...

Unintegrated (TMD) PDF's

"Naive" definition:

$$f_i(x, \boldsymbol{k}_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \; \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp}.$$

$$\cdot \langle p | \bar{\psi}_i(\xi^-, \xi_\perp) [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp;]^\dagger \gamma^+ [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp] \psi_i(0^-, 0_\perp) | p \rangle \; |_{\xi^+ = 0}$$

Formally:

$$\int d^2k_\perp f_i(x, \boldsymbol{k}_\perp) = q_i(x)$$

However: this definition suffers from several shortcomings.

- **extra divergences** associated with light-cone gauge, or light-like Wilson lines (*in the integrated case, these divergences cancel*)
- **gauge invariance** is not complete: in the light-cone gauge, dependence on the pole prescription in the gluon propagator still takes place
- **reduction to the integrated case**: formal integration doesn't produce correct result

Looking for the solution:

- **gauge invariance** is completely restored by means of the additional transverse Wilson path integral at the light-cone infinity (Belitsky, Ji, Yuan). This gauge link contribute only in the light-cone gauge, and cancel pole-prescription dependence

- **extra divergences** can be avoided by using non-light-like gauge connectors in the covariant gauges (Collins, Soper), or non-light-cone axial gauge. Thus, additional variable introduced (rapidity cutoff); calculations become more complicated; problems with factorization.

- **generalized renormalization** for the light-like Wilson lines (Collins, Hautmann): extra divergences cancel by the additional "soft" factor, defined by the vacuum average of special Wilson lines (demonstrated explicitly in the covariant gauge, in the 1-loop order)

Towards the “completely correct” definition:

- starting from the requirement of the **gauge invariance**, one formulates the anomalous dimensions sum rule:

- calculate the **anomalous dimension** of TMD PDF in the light-cone gauge and identify extra divergences in terms of the *defect* of anomalous dimension

- perform the **generalized renormalization** of TMD PDF, in analogue to the renormalization of the Wilson contours with cusps or self-intersection

In the **tree** approximation, the TMD PDF reads

$$f^{(0)}(x, k_\perp) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle p| \bar{\psi}(\xi^-, \xi_\perp) \gamma^+ \psi(0^-, 0_\perp) |p\rangle =$$

$$= \delta(1-x)\delta^{(2)}(\boldsymbol{k}_\perp)$$

The one-gluon exchanges, contributing to the UV-divergences, are described by the diagrams:

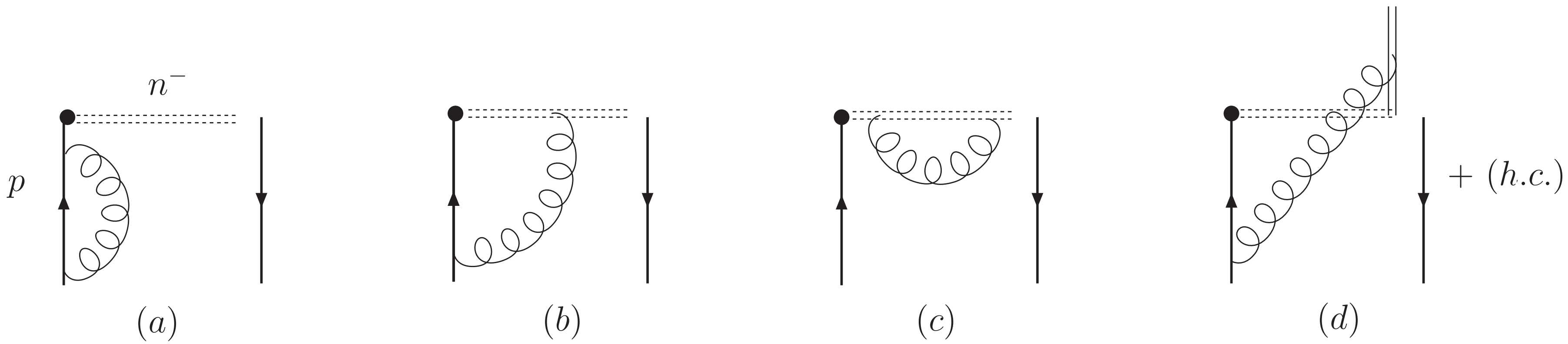

Figure 1: One-gluon exchanges for the TMD PDF: diagrams producing UV divergences. Only (a) and (d) contribute in the light-cone gauge

Source of the uncertainties and extra divergences: pole in the gluon propagator

$$D^{\mu\nu}_{\rm LC}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

Possible pole prescriptions:

$$d^{\mu\nu}_{\rm PV}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

$$d^{\mu\nu}_{\rm Adv/Ret}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

In what follows, we keep η small, but finite. Dimensional regularization is used to control UV singularities.

The **UV divergent** part read:

$$\Sigma^{UV}_{\text{left}}(p, \alpha_s; \epsilon) =$$

$$= -\frac{\alpha_s}{\pi} c_F \, \frac{1}{\epsilon} \left[-\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} + i\pi \, C_\infty \right] + \alpha_s c_F \, \frac{1}{\epsilon} \left[i C_\infty \right] =$$

$$= -\frac{\alpha_s}{\pi} c_F \, \frac{1}{\epsilon} \left[-\frac{3}{4} - \ln \frac{\eta}{p^+} + \frac{i\pi}{2} \right]$$

Imaginary part of AD — gluons in the Glauber regime (`Idilbi, Majumder`)

prescription dependence is canceled

$$C_\infty = \begin{cases} 0 \, , & \text{Advanced} : \frac{1}{[q^+]} = \frac{1}{q^+ - i\eta} \\ -1 \, , & \text{Retarded} : \frac{1}{[q^+]} = \frac{1}{q^+ + i\eta} \\ -\frac{1}{2} \, , & \text{PrincipalValue} : \frac{1}{[q^+]} = \frac{1}{2} \left(\frac{1}{q^+ - i\eta} + \frac{1}{q^+ + i\eta} \right) \end{cases}$$

Taking into account $(h.c.)$ contributions, one gets total real UV divergent part:

$$\Sigma_{\rm tot}(p, \alpha_s(\mu); \epsilon) = \Sigma_{\rm left} + \Sigma_{\rm right} =$$

$$= -\frac{\alpha_s}{4\pi} c_F \, \frac{2}{\epsilon} \left(-3 - 4 \ln \frac{\eta}{p^+} \right)$$

Dependence on η remains:

- gauge invariance is not complete
- AD doesn't coincide with AD$_{2q}$

One-loop **anomalous dimension** is defined via the renormalization constant

$$\gamma = \frac{1}{2}\,\frac{1}{Z^{(1)}}\,\mu\frac{\partial\alpha_s(\mu)}{\partial\mu}\,\frac{\partial Z^{(1)}(\mu,\alpha_s(\mu);\epsilon)}{\partial\alpha_s}$$

and reads

$$\gamma_{\rm LC} = \gamma_{\rm smooth} - \delta\gamma \ \ , \ \ \gamma_{\rm smooth} = \frac{3}{4}\frac{\alpha_s}{\pi}c_F + O(\alpha_s^2)$$

Defect of anomalous dimension

$$\delta\gamma = -\frac{\alpha_s}{\pi}c_F\ \ln\frac{\eta}{p^+}$$

contains undesirable p^+-dependent term which should be removed by a consistent procedure.

Note, that $\delta\gamma$ is nothing else, but the cusp anomalous dimension:

$$p^+ = (p \cdot n^-) \sim \cosh \chi$$

defines an angle χ between the direction of the quark momentum p_μ and the light-like vector n^-. In the large χ limit:

$$\ln p^+ \to \chi, \ \chi \to \infty$$

Renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle (`Korchemsky, Radyushkin`)

$$Z_\chi = \left[\langle 0 | \mathcal{P} \exp \left[ig \int_\chi d\zeta^\mu \ \hat{A}^a_\mu(\zeta) \right] | 0 \rangle \right]^{-1}$$

Generalized renormalization:

$$\mathcal{O}_{\rm ren}(\chi, ...) = Z_\chi Z_{\rm R} \ \mathcal{O}(\chi, ...)$$

Choose the integration path as follows

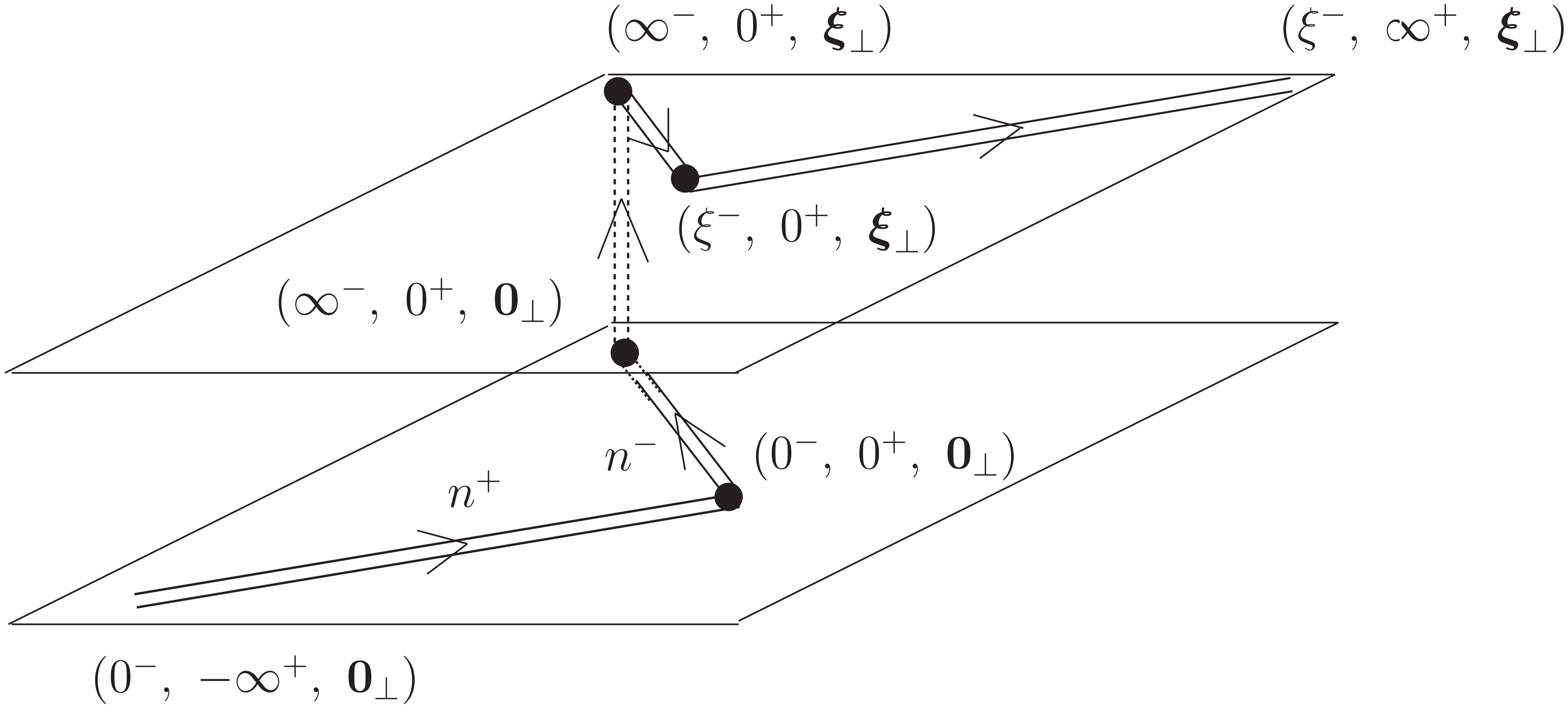

Figure 2: Space-time picture: integration trajectory for the additional cusp-dependent renormalization factor

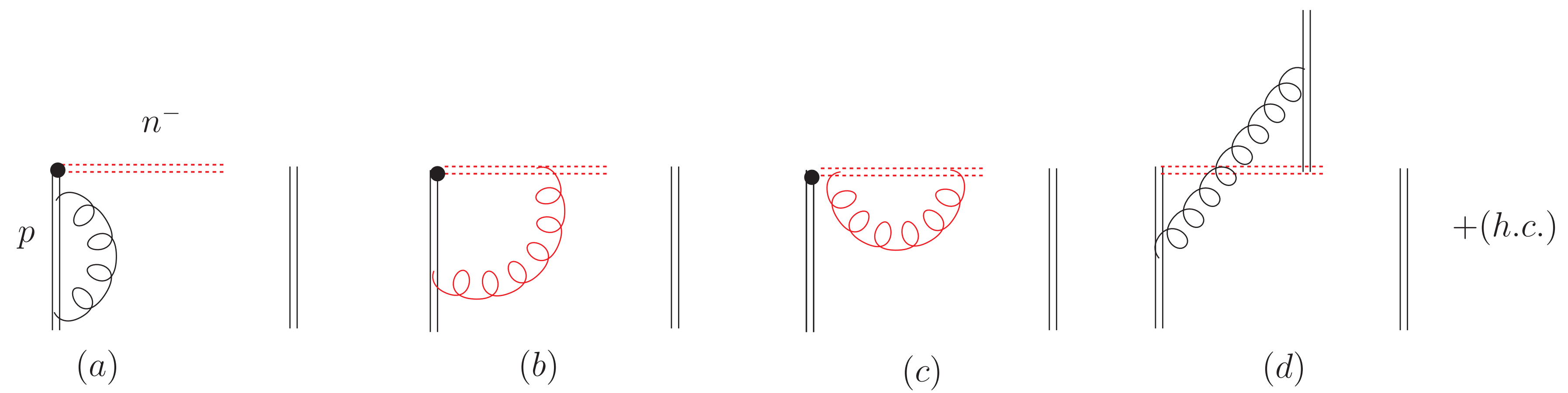

Figure 3: One-gluon exchanges for the generalized multiplicative renormalization factor

The generalized **renormalization constant** reads

$$\hat{Z}_{\rm mod} = 1 + \frac{\alpha_s}{4\pi} c_F \, \frac{2}{\epsilon} \left(-3 - 4 \ln \frac{\eta}{p^+} + 4 \ln \frac{\eta}{p^+} \right) = 1 - \frac{3\alpha_s}{4\pi} c_F \, \frac{2}{\epsilon}$$

so that

$$\frac{1}{2} \mu \frac{d}{d\mu} \ln \hat{Z}_{\rm mod}(\mu, \alpha_s, p^+) = \frac{3}{4} \frac{\alpha_s}{\pi} c_F + O(\alpha_s^2)$$

i.e., equal to the anomalous dimension of the corresponding operator with the **smooth gauge connector**, according to the anomalous dimensions sum rule.

ADSR can be formulated in the following form

$$\mathbf{AD}\ \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp}\ \langle p|\bar{\psi}(\xi)\gamma^+[\xi, 0]_{\text{direct link}}\psi(0)|p\rangle =$$

$$= \mathbf{AD}\ \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp}$$

$$\cdot\langle p|\bar{\Psi}(\xi|\infty)\gamma^+\Psi(0|\infty)|p\rangle \cdot \Phi(p^+, n^-|0^-, \mathbf{0}_\perp)\Phi^\dagger(p^+, n^-|\xi^-, \boldsymbol{\xi}_\perp)$$

Generalized definition of TMD PDF:

$$
\begin{aligned}
&f_{q/q}^{\rm mod}\,(x,\boldsymbol{k}_\perp;\mu,\eta) = \\
&\frac{1}{2}\int\frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2}\mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp}\,\langle q(p)|\bar{\psi}(\xi^-,\boldsymbol{k}_\perp)[\xi^-,\boldsymbol{\xi}_\perp;\infty^-,\boldsymbol{\xi}_\perp]^\dagger \\
&\times[\infty^-,\boldsymbol{\xi}_\perp;\infty^-,\boldsymbol{\infty}_\perp]^\dagger\gamma^+[\infty^-,\boldsymbol{\infty}_\perp;\infty^-,\boldsymbol{0}_\perp][\infty^-,\boldsymbol{0}_\perp;0^-,\boldsymbol{0}_\perp] \\
&\times\psi(0^-,\boldsymbol{0}_\perp)|q(p)\rangle\left[\Phi(p^+,n^-|0^-,\boldsymbol{0}_\perp)\Phi^\dagger(p^+,n^-|\xi^-,\boldsymbol{\xi}_\perp)\right]
\end{aligned}
$$

Soft factor:

$$
\Phi(p^+,n^-|0) = \left\langle 0\left|\mathcal{P}\exp\left[ig\int_{\mathcal{C}_{\rm cusp}} d\zeta^\mu\, t^a A_\mu^a(\zeta)\right]\right|0\right\rangle
$$

$$
\Phi^\dagger(p^+,n^-|\xi) = \left\langle 0\left|\mathcal{P}\exp\left[-ig\int_{\mathcal{C}_{\rm cusp}} d\zeta^\mu\, t^a A_\mu^a(\xi+\zeta)\right]\right|0\right\rangle
$$

Evolution equation:

$$\eta \frac{d}{d\eta} f_{q/q}^{\rm mod}(x, \boldsymbol{k}_\perp; \mu, \eta) = [\mathcal{K}(\mu) + \mathcal{G}(\mu)] \otimes f_{q/q}^{\rm mod}(x, \boldsymbol{k}_\perp; \mu, \eta)$$

$$\mathcal{K}(\mu) + \mathcal{G}(\mu) =$$

$$= \frac{\alpha_s}{\pi} C_{\rm F}\, \delta(1-x) \left[\delta^{(2)}(\boldsymbol{k}_\perp) \left(\ln \frac{\mu^2}{p^2} - \ln \frac{\mu^2}{\lambda^2} \right) - \frac{1}{\pi} \left(\frac{1}{\boldsymbol{k}_\perp^2 + x\lambda^2 - x(1-x)p^2} + \frac{1}{\boldsymbol{k}_\perp^2 + \lambda^2} \right) \right]$$

The renormalization-group behavior of the functions $\mathcal{K}(\mu)$ and $\mathcal{G}(\mu)$ is determined by the universal cusp anomalous dimension

$$\frac{1}{2} \mu \frac{d}{d\mu} \ln \mathcal{K}(\mu) = -\frac{1}{2} \mu \frac{d}{d\mu} \ln \mathcal{G}(\mu) = \gamma_{\rm cusp} = \frac{\alpha_s}{\pi} C_{\rm F} + O(\alpha_s^2)$$

Dependence on the dimensional regularization scale μ of the re-defined TMD PDF:

$$\frac{1}{2}\mu\frac{d}{d\mu}\ln f_{q/q}^{\mathrm{mod}}(x,\boldsymbol{k}_\perp;\mu,\eta) = \frac{3}{4}\frac{\alpha_s}{\pi}C_{\mathrm{F}} + O(\alpha_s^2)$$

Consistency equation:

$$\mu\frac{d}{d\mu}\left[\eta\frac{d}{d\eta}f_{q/q}^{\mathrm{mod}}(x,\boldsymbol{k}_\perp;\mu,\eta)\right] = 0$$

Reduction to the integrated PDF:

$$\int d^{\omega-2}\boldsymbol{k}_\perp f_{i/a}^{\mathrm{mod}}(x, \boldsymbol{k}_\perp; \mu, \eta) = f_{i/a}(x, \mu)$$

DGLAP equation

$$\mu \frac{d}{d\mu} f_{q/q}(x, \mu) = \sum_j \int_x^1 \frac{dz}{z} P_{qi}\left(\frac{x}{z}\right) f_{i/q}(z, \mu)$$

Probabilistic interpretation

The distribution functions cannot be calculated from first principles, but its **evolution** can. The RG properties define the necessary condition for the unintegrated PDF to be a number density.

The requirement that the off-the-light-cone two-quark matrix element should have an anomalous dimension equal to that of the corresponding quantity with the **smooth gauge connector** in order to respect the probabilistic interpretation, is tantamount to the **anomalous dimensions sum rule**.

The generalized TMD PDF obeys this condition.

Conclusions

- The anomalous dimension of the TMD PDF in the **light-cone gauge** is calculated in 1-loop order; the contribution of the cusp anomalous dimension is obtained.

- The generalized completely gauge invariant **definition of TMD PDF** is proposed.

- The direct connection to the **integrated PDF** is established.

- The **parton interpretation** of the distribution functions in terms of the RG properties is proposed.

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