A RENORMALIZATION GROUP STUDY OF JUMP PHENOMENA IN NON-LINEAR OSCILLATORS

Dhruba Banerjee

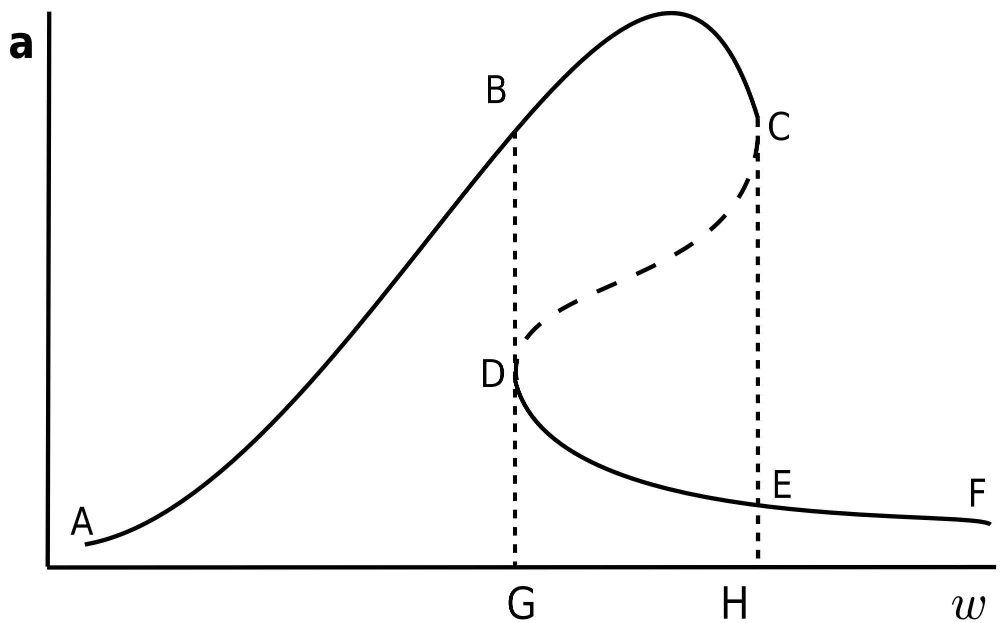
St. Xavier's College, Kolkata, India.

Jayanta K Bhattacharjee

S N Bose National Centre for Basic Sciences, Kolkata, India.

Inclusion of non-linear effects in periodically driven non-linear oscillators yield interesting results which had been studied by prominent Russian physicists (Landau, Bogoliubov, Mitropolsky etc.) several decades back. In the near resonance situation a plot of amplitude versus the difference of frequencies between the forcing and the natural frequency of the oscillator takes a peculiar structure much similar to the well known hysteresis phenomenon in ferromagnetism. This is a jump phenomenon where the amplitude shows sudden jumps at certain threshold values of the frequency. This jump phenomenon has been studied using the renormalization group approach.

As the phenomenon under investigation is very well known in the non-linear dynamics literature, allow me to start with the end result.



$\omega =$ difference between forcing and natural frequencies
$a =$ amplitude

The system under investigation is a DUFFING oscillator subjected to a damping and a periodic forcing. The governing equation is:

$$\ddot{x} + \omega_0^2 x + k\dot{x} - \beta x^3 = F \cos \Omega t \tag{1}$$

There is a non-linearity carried by the β term that makes the system interesting.

Putting $\Omega t = \tau$ and dividing by Ω^2 makes Eq.(1)

$$\ddot{x} + \omega_r^2 x + k_r\dot{x} - \lambda x^3 = \overline{F} \cos \Omega t \tag{2}$$

where the rescaled terms have their obvious meanings.

Our goal is to study the NEAR RESONANCE scenario. For this purpose we introduce THREE PERTURBATION PARAMETERS.

$$\begin{aligned}\omega_r^2 &= 1+\phi\omega \\ \overline{F} &= \phi f \\ k_r &= \epsilon\chi\end{aligned}$$

and λ. The functions of these parameters are independent of each other. ϵ takes care of the damping, λ the non-linearity and ϕ the forcing. THIS IS AN IMMENSE FLEXIBILITY OFFERED BY RG IN THE PRESENT STUDY.

All these considerations bring us to our starting point of our study. The equation is:

$$\ddot{x} + (1 + \phi\omega)x + \epsilon\chi - \lambda x^3 = \phi f \cos\tau \qquad (3)$$

Accordingly we expand x in terms of these three parameters.

$$x = x_0 + \epsilon x_{1\epsilon} + \lambda x_{1\lambda} + \phi x_{1\phi} \qquad (4)$$

Equating the first powers of the expansion parameters separately, we get the following equations:

$$\begin{aligned} \text{0th order:} \quad \ddot{x_0} + x_0 &= 0 && (5) \\ \text{1st order in} \quad \epsilon : \ddot{x_{1\epsilon}} + x_{1\epsilon} &= -\chi \dot{x_0} && (6) \\ \text{1st order in} \quad \lambda : \ddot{x_{1\lambda}} + x_{1\lambda} &= x_0^3 && (7) \\ \text{1st order in} \quad \phi : \ddot{x_{1\phi}} + x_{1\phi} &= -\omega x_0 + f \cos \tau && (8) \end{aligned}$$

Solution of Eq.(5)

$$\ddot{x_0} + x_0 = 0$$

yields

$$\begin{aligned} x_0 &= A\cos\tau + B\sin\tau \\ &= a\cos(\tau + \theta) \end{aligned} \tag{9}$$

Wronskian is AB. Accordingly we can find out the other solutions in straight forward manner. They go like this:

Rewriting Eq.(6) as

$$\ddot{x_{1\epsilon}} + x_{1\epsilon} = \chi \sin(\tau + \theta)$$

yields

$$x_{1\epsilon} = \frac{1}{2}\chi a[-\tau \cos(\tau + \theta) + \frac{1}{2} \sin(\tau + \theta)] \quad (10)$$

The first term is divergent and with increasing time would give an unbounded solution.

Rewriting Eq.(7) as

$$
\begin{aligned}
\ddot{x_{1\lambda}} + x_{1\lambda} &= a^3 \cos^3(\tau + \theta) \\
&= \frac{a^3}{4}[\cos 3(\tau + \theta) + 3\cos(\tau + \theta)]
\end{aligned}
$$

yields

$$
x_{1\lambda} = -\frac{a^3}{32}\cos 3(\tau + \theta) + \frac{3a^3}{16}\cos(\tau + \theta) + \tau\frac{3a^3}{8}\sin(\tau + \theta) \quad (11)
$$

Here the last term is divergent.

Similarly the ϕ equation yields

$$x_{1\phi} = -\frac{\omega a}{4}\cos(\tau+\theta) - \tau\frac{\omega a}{2}\sin(\tau+\theta) + \frac{f}{4}\cos\tau + \tau\frac{f}{2}\sin\tau \quad (12)$$

Here the 2nd and 4th terms are divergent.

Whatever be the initial time (here $\tau = 0$), these divergences would appear.

GOAL IS TO CAPITALIZE ON THIS FREEDOM IN CHOICE OF THE INITIAL TIME.

Thus in the final expression for x (upto first order) there are four divergent terms. One in ϵ, one in λ and two in ϕ expressions.

We have

$$
\begin{aligned}
x &= x_0 + \epsilon x_{1\epsilon} + \lambda x_{1\lambda} + \phi x_{1\phi} \\
&= a\cos(\tau+\theta) \\
&+ \epsilon[\frac{1}{2}\chi a\{-\tau\cos(\tau+\theta) + \frac{1}{2}\sin(\tau+\theta)\}] \\
&+ \lambda[-\frac{a^3}{32}\cos 3(\tau+\theta) + \frac{3a^3}{16}\cos(\tau+\theta) + \tau\frac{3a^3}{8}\sin(\tau+\theta)] \\
&+ \phi[-\frac{\omega a}{4}\cos(\tau+\theta) - \tau\frac{\omega a}{2}\sin(\tau+\theta) + \frac{f}{4}\cos\tau + \tau\frac{f}{2}\sin\tau]
\end{aligned}
\tag{13}
$$

To deal with the divergent terms we reparametrize a and θ by introducing renormalizing constants Z_1 and Z_2. They will be calculated perturbatively as follows:

$$\begin{aligned} a &= a(\mu)Z_1(\mu) \quad \text{(Multiplicative)} \\ &= a(\mu)(1+\epsilon Z_{1\epsilon}+\lambda Z_{1\lambda}+\phi Z_{1\phi}+...) \end{aligned} \tag{14}$$

$$\begin{aligned} \theta &= \theta(\mu)+Z_2(\mu) \quad \text{(Additive)} \\ &= \theta(\mu)+\epsilon Z_{2\epsilon}+\lambda Z_{2\lambda}+\phi Z_{2\phi}+... \end{aligned} \tag{15}$$

What is μ anyway ? It is an ARBITRARY TIMESCALE.

In the expression for x in Eq.(13) all the divergent terms are either of the form $\tau \cos(\tau + \theta)$ or $\tau \sin(\tau + \theta)$ excepting one that occurs in the ϕ portion and has the form $\tau \sin \tau$. We split this expression as:

$$\begin{aligned}
\tau \sin \tau &= \tau \sin(\tau + \theta - \theta) \\
&= \sin(\tau + \theta) \cos \theta - \cos(\tau + \theta) \sin \theta \\
&= \sin(\tau + \theta(\mu)) \cos \theta(\mu) - \cos(\tau + \theta(\mu)) \sin \theta(\mu) \\
&+ \text{higher order terms} \qquad (16)
\end{aligned}$$

where in the last line we have invoked Eqs.(14-15).

Using these expansions for a and θ, we get the expression for x upto the first order in the perturbation parameters as:

$$
\begin{aligned}
x &= a(\mu)\cos(\tau+\theta(\mu)) \\
&+ \epsilon\Bigg[\left\{-\tau\frac{1}{2}\chi a(\mu)+a(\mu)Z_{1\epsilon}\right\}\cos(\tau+\theta(\mu)) \\
&\quad +\left\{\frac{1}{4}\chi a(\mu)-a(\mu)Z_{2\epsilon}\right\}\sin(\tau+\theta(\mu))\Bigg] \\
&+ \lambda\Bigg[\left\{\frac{3a^3(\mu)}{16}+a(\mu)Z_{1\lambda}\right\}\cos(\tau+\theta(\mu)) \\
&\quad +\left\{\tau\frac{3a^3(\mu)}{8}-a(\mu)Z_{2\lambda}\right\}\sin(\tau+\theta(\mu))-\frac{a^3(\mu)}{32}\cos 3(\tau+\theta(\mu))\Bigg] \\
&+ \phi\Bigg[\left\{-\tau\frac{f}{2}\sin\theta(\mu)+a(\mu)Z_{1\phi}\right\}\cos(\tau+\theta(\mu))-\frac{\omega a(\mu)}{4}\cos(\tau+\theta(\mu)) \\
&\quad +\left\{\tau\frac{f}{2}\cos\theta(\mu)-\tau\frac{\omega a(\mu)}{2}-a(\mu)Z_{2\phi}\right\}\sin(\tau+\theta(\mu))+\frac{f}{4}\cos\tau\Bigg]
\end{aligned}
\tag{17}
$$

where we have rearranged the first order terms spawned by the first line of Eq.(13) into the respective brackets and have clustered the cos and sin terms accordingly.

Explicitly,

$$
\begin{aligned}
a \cos(\tau+\theta) &= a(\mu)(1+\epsilon Z_{1\epsilon}+\lambda Z_{1\lambda}+\phi Z_{1\phi}) \\
&\quad \times \cos(\tau+\theta(\mu)+\epsilon Z_{2\epsilon}+\lambda Z_{2\lambda}+\phi Z_{2\phi}) \\
&= a(\mu)(1+\epsilon Z_{1\epsilon}+\lambda Z_{1\lambda}+\phi Z_{1\phi}) \cos(\tau+\theta(\mu)) \\
&\quad - a(\mu)(\epsilon Z_{2\epsilon}+\lambda Z_{2\lambda}+\phi Z_{2\phi}) \sin(\tau+\theta(\mu)) \\
&\quad + \text{higher order terms}
\end{aligned}
$$

The curly brackets in Eq.(17) include only those coefficients of cos and sin through which we can define the Z terms.

As μ is an arbitrary time scale, in all the divergent terms (where τ occurs in the coefficients) we replace τ as

$$\tau = \tau - \mu + \mu \tag{18}$$

The divergences are now carried by the μ terms. THE FREEDOM OF CHOICE IN THE INITIAL CONDITIONS LEADS US TO CAST THE PROBLEM IN TERMS OF RG FLOW.

Our next aim is to REMOVE THESE DIVERGENCES. Looking back at Eq.(17), and using Eq.(18) we get the following set of equations WHICH FIX THE RENORMALIZATION CONSTANTS.

$\underline{\epsilon \text{ expression}}$:

$$\text{coefficient of } \cos(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{1\epsilon} = \frac{1}{2}\mu\chi$$

$$\text{coefficient of } \sin(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{2\epsilon} = 0$$

$\underline{\lambda \text{ expression}}$:

$$\text{coefficient of } \cos(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{1\lambda} = 0$$

$$\text{coefficient of } \sin(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{2\lambda} = \mu\frac{3a^2(\mu)}{8}$$

$\underline{\phi \text{ expression}}$:

$$\text{coefficient of } \cos(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{1\phi} = \frac{\mu f}{2}\frac{\sin\theta(\mu)}{a(\mu)}$$

$$\text{coefficient of } \sin(\tau + \theta(\mu)) \quad \Rightarrow \quad Z_{2\phi} = \frac{\mu f}{2}\frac{\cos\theta(\mu)}{a(\mu)} - \frac{\omega\mu}{2}$$

Absorbing $(\tau-\mu)$ terms in renormalized $a(\mu)$ and $\theta(\mu)$ yields:

$$\begin{aligned} x &= a(\mu)\cos(\tau+\theta(\mu)) \\ &+ \epsilon\left[-\frac{\chi a}{2}(\tau-\mu)\cos(\tau+\theta(\mu))+\frac{\chi a}{4}\sin(\tau+\theta(\mu))\right] \\ &+ \lambda\left[\frac{3a^3(\mu)}{8}(\tau-\mu)\sin(\tau+\theta(\mu))+\frac{3a^3(\mu)}{16}\cos(\tau+\theta(\mu))\right. \\ &\quad \left.-\frac{a^3(\mu)}{32}\cos 3(\tau+\theta(\mu))\right] \\ &+ \phi\left[-(\tau-\mu)\frac{f}{2}\sin\theta(\mu)\cos(\tau+\theta(\mu))\right. \\ &\quad +\frac{1}{2}(\tau-\mu)(f\cos\theta(\mu)-\omega a(\mu))\sin(\tau+\theta(\mu)) \\ &\quad \left.-\frac{\omega a(\mu)}{4}\cos(\tau+\theta(\mu))+\frac{f}{4}\cos\tau\right] \end{aligned} \tag{19}$$

Now we require that, μ BEING ARBITRARY, x SHOULD BE INDEPENDENT OF μ. THIS BRINGS US TO THE EQUATION FOR RG FLOW. By retaining derivatives only upto first order we get,

$$\begin{aligned}
\frac{dx}{d\mu} = 0 \;\Rightarrow\; & \frac{da}{d\mu}\cos(\tau+\theta(\mu)) - a(\mu)\frac{d\theta}{d\mu}\sin(\tau+\theta(\mu)) \\
+ \; & \epsilon\left[\frac{\chi a}{2}\cos(\tau+\theta(\mu))\right] + \lambda\left[-\frac{3a^3(\mu)}{8}\sin(\tau+\theta(\mu))\right] \\
+ \; & \phi\left[\frac{f\sin\theta(\mu)}{2}\cos(\tau+\theta(\mu))\right. \\
& \left. + \frac{1}{2}(\omega a(\mu) - f\cos\theta(\mu))\sin(\tau+\theta(\mu))\right] \\
= \; & 0 \qquad (20)
\end{aligned}$$

Equating the coefficients of $\cos(\tau+\theta(\mu)$ and $\sin(\tau+\theta(\mu)$ separately to zero,

$$\frac{da}{d\mu} = -\frac{1}{2}\epsilon\chi a - \frac{1}{2}\phi f \sin\theta(\mu) = F_1(a,\theta) \tag{21}$$

$$\frac{d\theta}{d\mu} = -\lambda\frac{3a^2(\mu)}{8} + \phi\frac{\omega}{2} - \phi\frac{f\cos\theta(\mu)}{2a(\mu)} = F_2(a,\theta) \tag{22}$$

These are the RG equations whose fixed points would determine the stability of the solutions.

For determining the fixed points we put

$$\left[\frac{da}{d\mu}\right]_{a_0,\theta_0} = \left[\frac{d\theta}{d\mu}\right]_{a_0,\theta_0} = 0 \tag{23}$$

Here $(a_0,\ \theta_0)$ are the fixed points. How many ?

Combining Eqs(21) and (22) WE GET THE AMPLITUDE EQUATION (cubic equation in a_0^2). Putting the perturbation parameters equal to 1, we get,

$$a_0^6 - \frac{8\omega}{3}a_0^4 + \frac{16}{9}(\omega^2 + \chi^2)a_0^2 - \frac{16}{9}f^2 = 0 \tag{24}$$

There are certain standard consequences of this equation. One can treat this equation as a quadratic in ω to obtain

$$\omega = \frac{3a_0^2}{4} \pm \sqrt{\frac{f^2}{a_0^2} - \chi^2}$$

For $f > a_0\chi$ we get double root of ω. One can also differentiate the cubic w.r.t ω and then put $\frac{d\omega}{da_0} = 0$ to obtain

$$a_0^2 = \frac{8\omega}{9} \pm \frac{4}{9}\sqrt{\omega^2 - 3\chi^2}$$

and conclude that $\omega^2 = 3\chi^2$ (hence $a_0^2 = 8\omega/9$) correspond to a critical value of f above which the tilt in the amplitude-frequency curve ensues.

Testing the stability of the three roots of the cubic means solving the determinant (all derivatives being taken at $a = a_0$ and $\theta = \theta_0$)

$$\begin{vmatrix} (\partial F_1/\partial a)_0 - \eta & (\partial F_1/\partial \theta)_0 \\ (\partial F_2/\partial a)_0 & (\partial F_2/\partial \theta)_0 - \eta \end{vmatrix} = \begin{vmatrix} -\frac{\chi}{2} - \eta & \frac{f}{2}\cos\theta_0 \\ -\frac{3a_0}{4} - \frac{f\cos\theta_0}{2a_0^2} & \frac{f\sin\theta_0}{2a_0} - \eta \end{vmatrix} = 0$$

We write this as,

$$\eta^2 - 2s\eta + d = 0 \tag{25}$$

where

$$s = \frac{1}{4}\left[\frac{f}{a_0}\sin\theta_0 - \chi\right] \tag{26}$$

$$d = \frac{1}{4}\left[\frac{f^2}{a_0^2}\cos^2\theta_0 - \frac{f\chi}{a_0}\sin\theta_0 - \frac{3fa_0}{2}\cos\theta_0\right] \tag{27}$$

From the condition $\left[\frac{da}{d\mu}\right]_{a_0,\theta_0} = 0$ given by Eq.(23) we get

$$\frac{f}{a_0}\sin\theta_0 = -\chi \tag{28}$$

This with Eq.(26) above, yields a useful relation and that is

$$s = -\frac{1}{2}\chi < 0 \tag{29}$$

Thus in the two roots of Eq.(25) given by $\eta = s \pm \sqrt{s^2 - d}$, we have s ALWAYS NEGATIVE for all the three roots of the cubic.

Thus the sign and magnitude of d decides the stability of the fixed points.

$$\begin{aligned} 0 < d < s^2 \;&\Rightarrow\; \text{Stable fixed point} \\ d < 0 \;&\Rightarrow\; \text{Unstable fixed point} \end{aligned} \tag{30}$$

Solving the cubic and then testing the stability is what we should call the direct approach. This is, however, cumbersome. THE TWO RG EQUATIONS OFFER USEFUL HINT TO GO THE OTHER WAY ROUND. How ?

Using the criteria of Eq.(30) in the expression for d in Eq.(27), we shall examine what bounds it entails on θ_0 (and hence on a_0). Writing Eq.(27) as

$$
\begin{aligned}
4d &= \frac{f^2}{a_0^2}\cos^2\theta_0 - \frac{f\chi}{a_0}\sin\theta_0 - \frac{3fa_0}{2}\cos\theta_0 \\
&= \left(\frac{f}{a_0}\cos\theta_0 - \frac{3a_0^2}{4}\right)^2 + \chi^2 - \frac{9a_0^4}{16} \\
&= A^2 + \chi^2 - \frac{9a_0^4}{16} \qquad (31)
\end{aligned}
$$

with

$$A^2 = \left(\frac{f}{a_0}\cos\theta_0 - \frac{3a_0^2}{4}\right)^2 \tag{32}$$

Now $d < s^2$ means $A^2 + \chi^2 - \frac{9a_0^4}{16} < \chi^2$, i.e., $A^2 < \frac{9a_0^4}{16}$, yielding

$$f\cos\theta_0 \left[\frac{f}{a_0^2}\cos\theta_0 - \frac{3a_0}{2}\right] < 0 \tag{33}$$

This gives us the first bound on θ_0 as

$$0 < \cos\theta_0 < \frac{3a_0^3}{2f} \tag{34}$$

From Eq.(28) we learnt that $\sin\theta_0$ is always negative and here we see that $\cos\theta_0$ is always positive. This implies that for all the three solutions of the cubic, θ_0 lies in the FOURTH QUADRANT and here $\sin\theta_0$ is monotonically increasing. Now let us see what the other condition for stability, viz., $d > 0$ implies.

This would mean $A^2 > \frac{9a_0^4}{16} - \chi^2$. Using the expression for A^2 we get a quadratic inequation in $\cos\theta_0$ as,

$$
\begin{aligned}
& \frac{f^2}{a_0^2}\cos^2\theta_0 - \frac{3fa_0}{2}\cos\theta_0 + \chi^2 > 0 \\
\Rightarrow \quad & \cos\theta_0 \le \frac{3a_0^3}{2f} - \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} \\
\text{or} \quad & \cos\theta_0 \ge \frac{3a_0^3}{2f} + \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} \qquad (35)
\end{aligned}
$$

Both the roots of the quadratic "equation" are positive. One should note that for $\chi = 0$, this limit imposed by Eq.(35) reduces to the former one, i.e., Eq.(34).

Thus the stability condition, i.e., $0 < d < s^2$ confines the value of $\cos\theta_0$ within two separate regions, viz.,

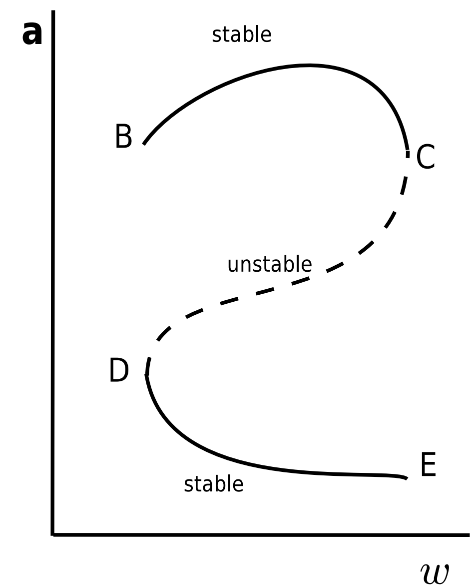
$$0 < \cos\theta_0 < \frac{3a_0^3}{2f} - \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} \tag{36}$$

and
$$\frac{3a_0^3}{2f} + \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} < \cos\theta_0 < \frac{3a_0^3}{2f} \tag{37}$$

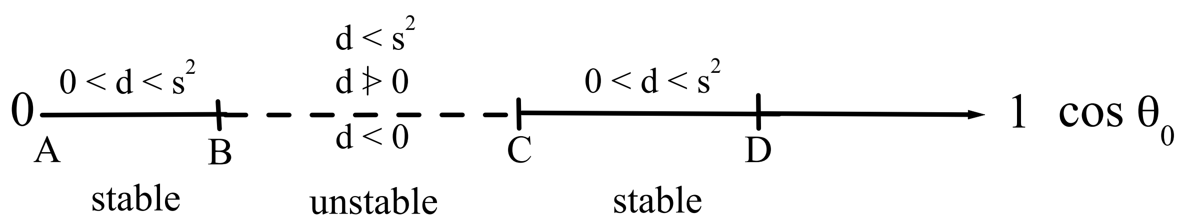
The intermediate region, i.e.,

$$\frac{3a_0^3}{2f} - \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} < \cos\theta_0 < \frac{3a_0^3}{2f} + \frac{a_0^2}{2f}\sqrt{\frac{9a_0^2}{4} - \frac{4\chi^2}{a_0^2}} \tag{38}$$

corresponds to the condition $d < s^2$ and $d \not> 0$ which is equivalent to saying $d < 0$. From Eq.(30), this implies instability.



Middle root of θ_0 unstable implies middle root of a_0 unstable, because Eq.(28) implies $\sin\theta_0 = -\frac{\chi a_0}{f}$, and this is monotonically increasing in the fourth quadrant.



SUMMARY

1) RG has been used to study the problem of a forced Duffing Oscillator.

2) To the first order in perturbation, secular terms appear. The freedom in choice of initial conditions motivates the use of RG in this context.

3) In the secular terms τ divergences have been removed by writing $\tau = \tau - \mu + \mu$, where μ is an arbitrary timescale. The amplitude and phase terms are renormalized accordingly.

4) The μ-independence of the dynamics leads to RG flow equations.

5) The RG equations thus derived provide a very simple way to analyze the fixed points of the (cubic) amplitude equation.

6) This method of applying the RG has the immense flexibility of dealing with several perturbation parameters where multiple scale analysis leads to cumbersome calculations and often misses hidden scales.

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