Resummation approach in (F)APT
How many loops do we need to calculate?

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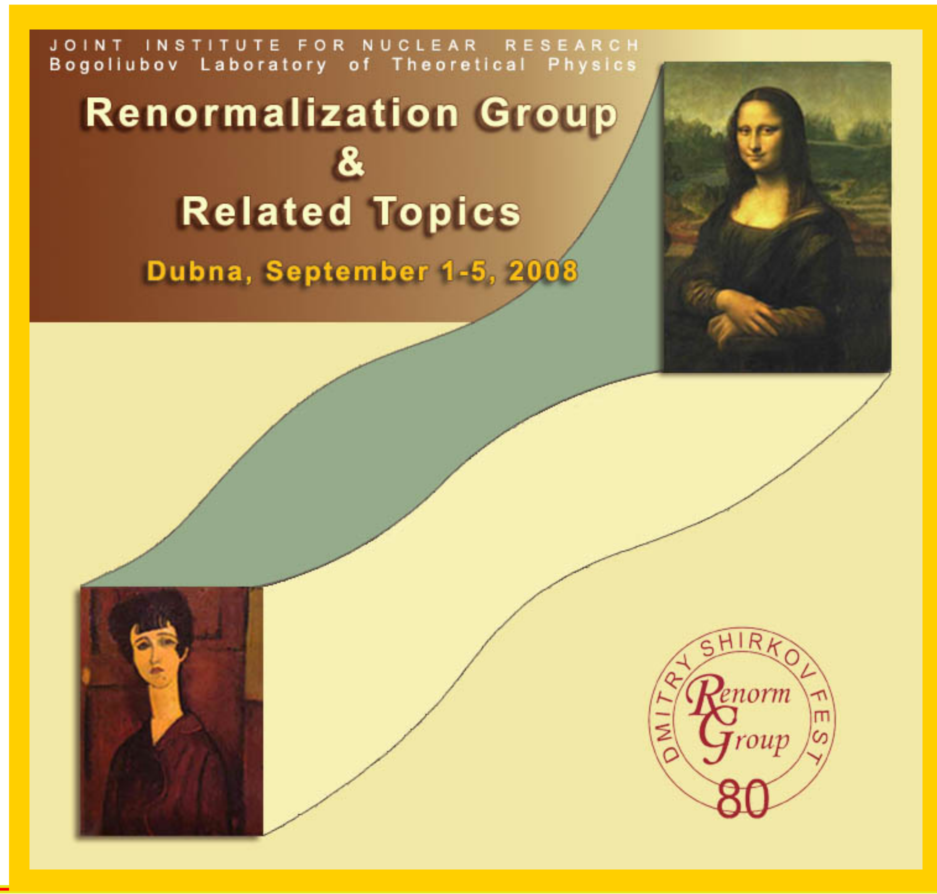

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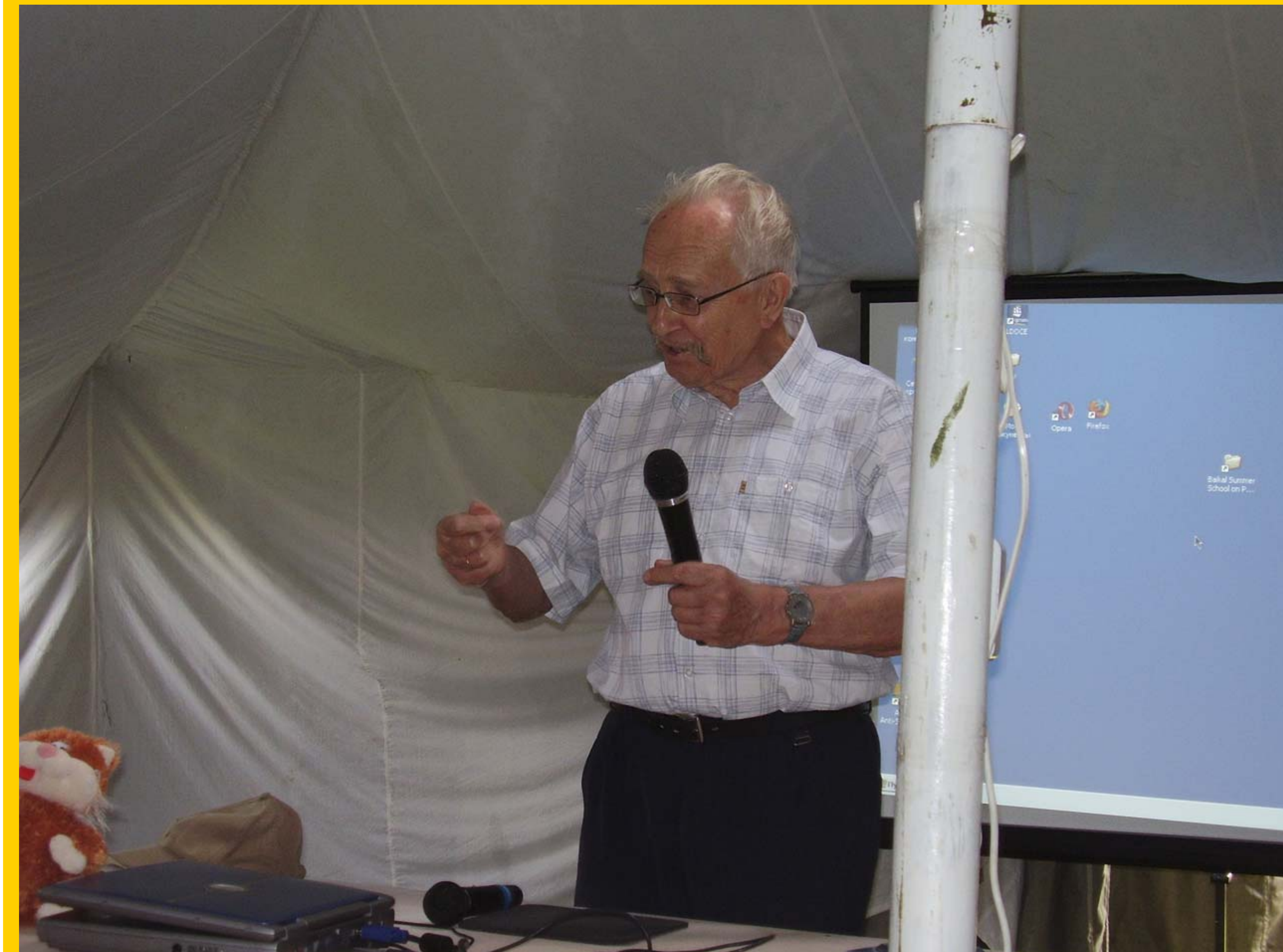

D. V. Shirkov in BLTPh and outside

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**
- Applications: Higgs decay $H^0 \to b\bar{b}$
- Applications: Adler function $D(Q^2)$ and ratio $R(s)$ in $N_f = 4$ region
- Conclusions

Collaborators & Publications

Collaborators:

S. Mikhailov (Dubna), N. Stefanis (Bochum), and
A. Karanikas (Athens)

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Publications:

- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014**
- A. B., Karanikas, Stefanis — **PRD 72 (2005) 074015**
- A. B., Mikhailov, Stefanis — **PRD 75 (2007) 056005**
- A. B.&Mikhailov — “Resummation in (F)APT”, **arXiv:0803.3013 [hep-ph]**
- A. B. — “Global FAPT in QCD with Selected Applications”, **arXiv:0805.0829 [hep-ph]**

Analytic Perturbation Theory

in

QCD

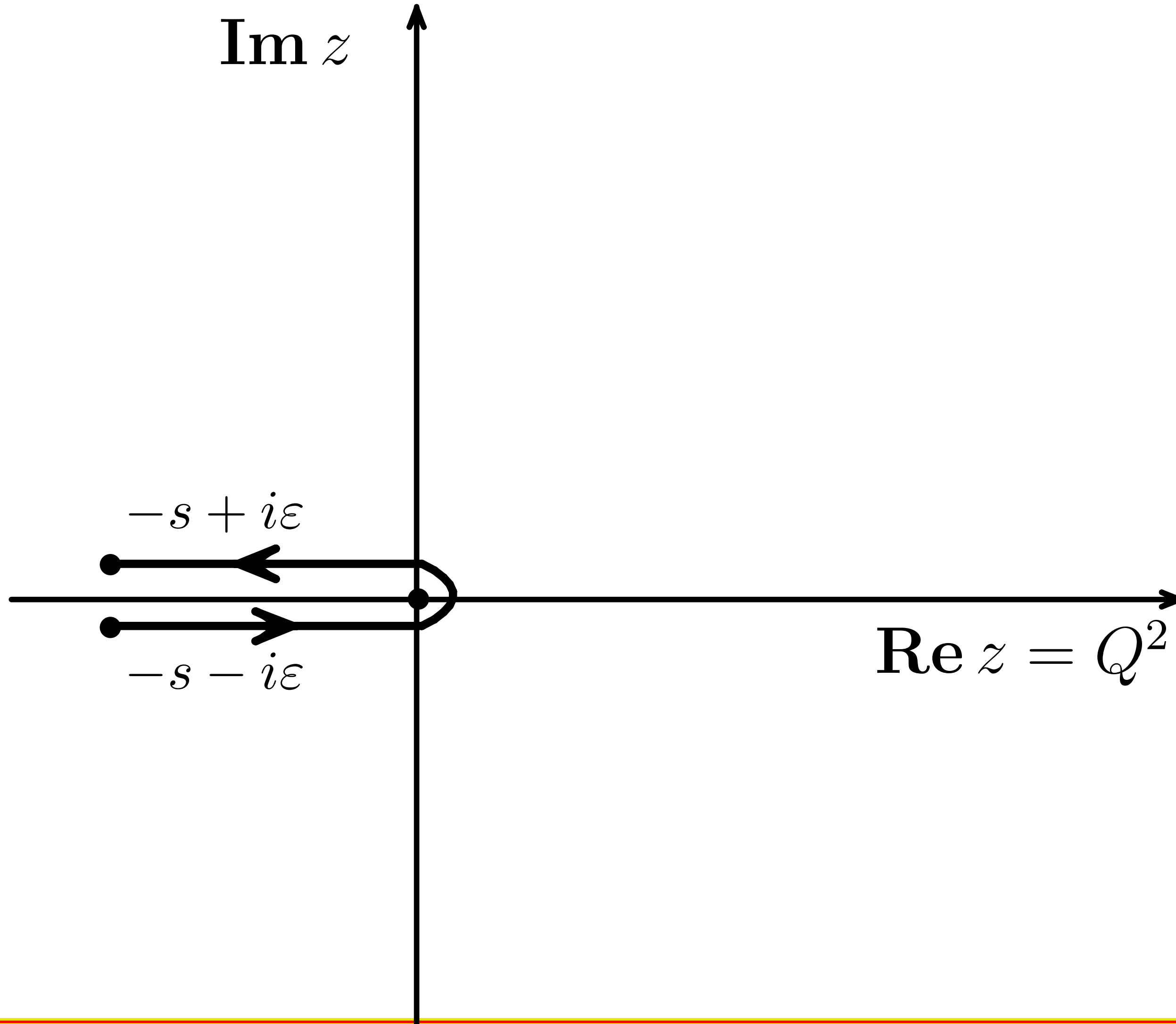
Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
- 1-loop solution generates Landau pole singularity: $a_s[L] = 1/L$
- 2-loop solution generates square-root singularity: $a_s[L] \sim 1/\sqrt{L + c_1 \ln c_1}$
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$
- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to

$$Z \sim a^{\nu}[L]\Big|_{\nu = \nu_0 \equiv \gamma_0/(2b_0)}$$

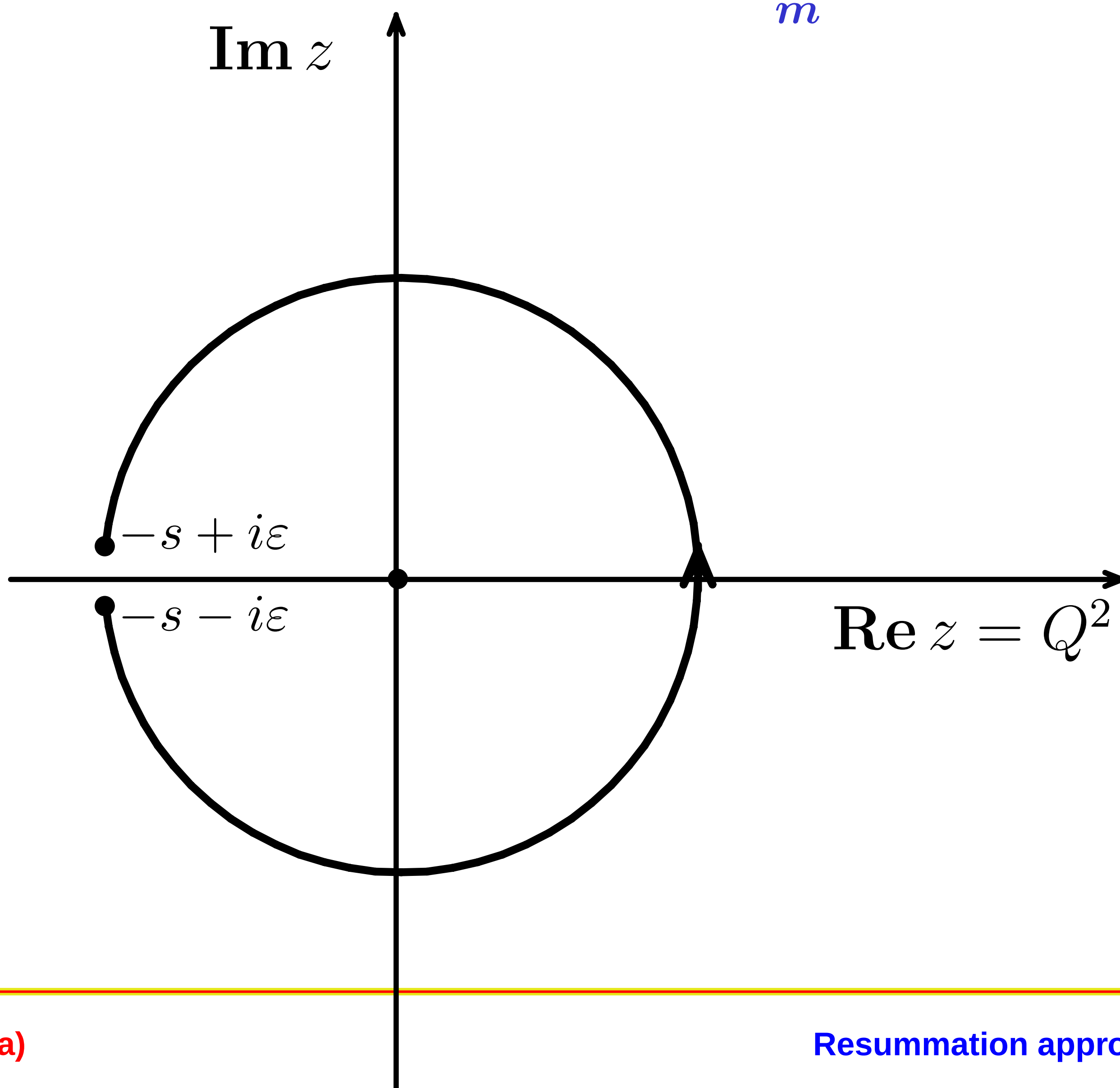
Problem in QCD PT: Minkowski region?

Quantities in Minkowski region $= \oint f(z)D(z)dz$.



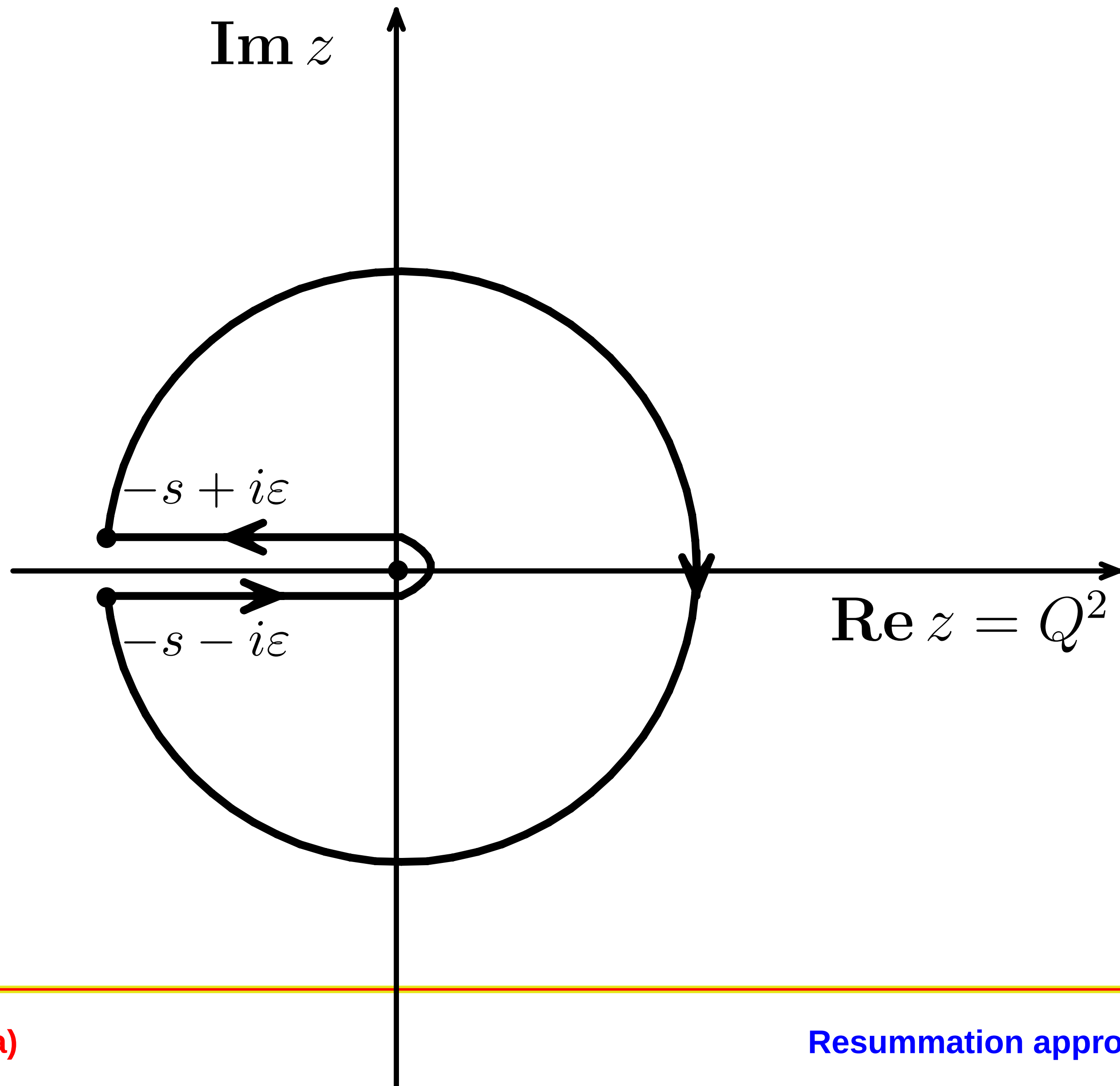
Problem in QCD PT: Minkowski region?

In $\oint f(z) D(z) dz$ one uses $D(z) = \sum_m d_m \alpha_s^m(z)$.



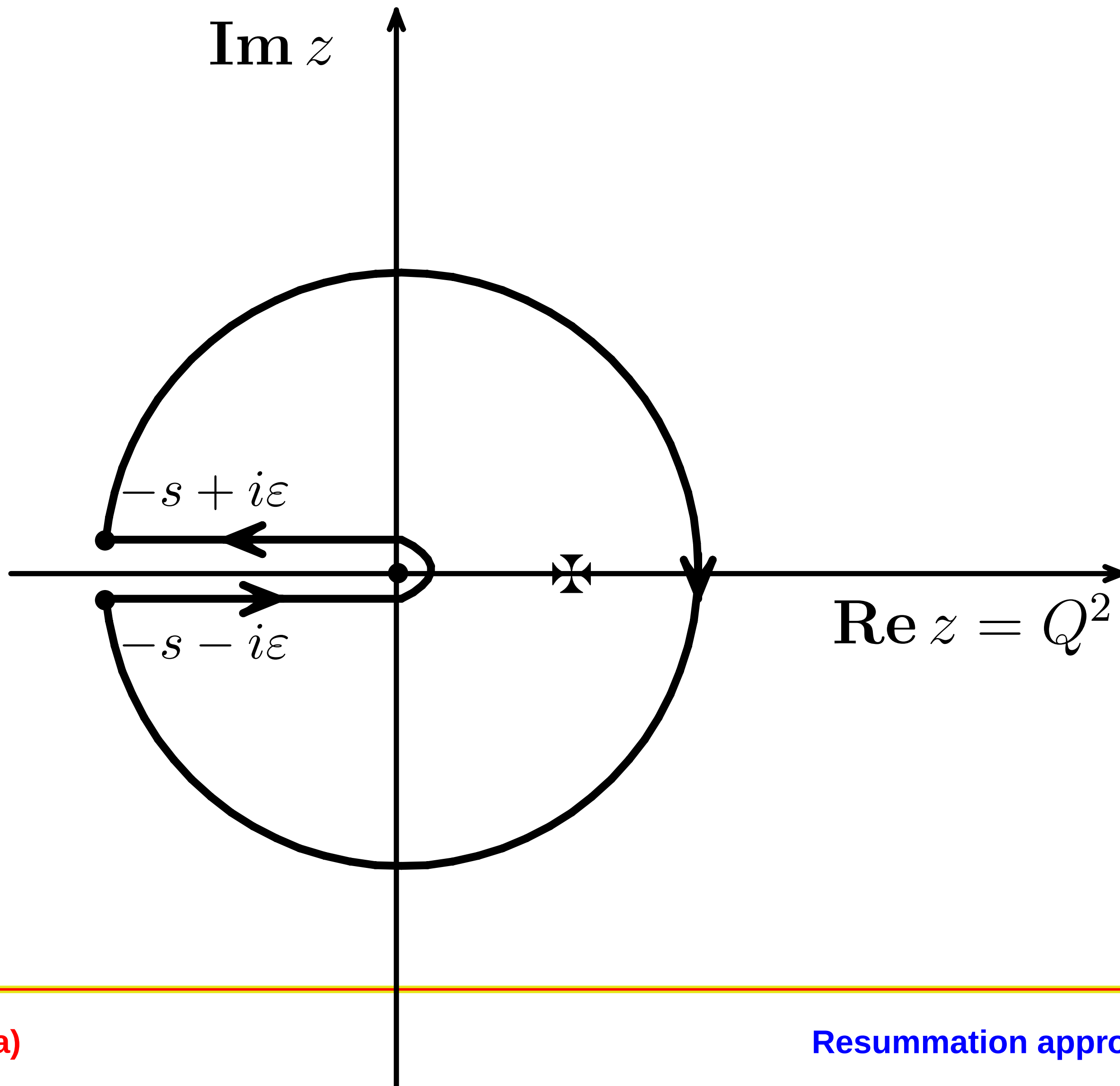
Problem in QCD PT: Minkowski region?

This change of integration contour is legitimate if $D(z)f(z)$ is analytic inside



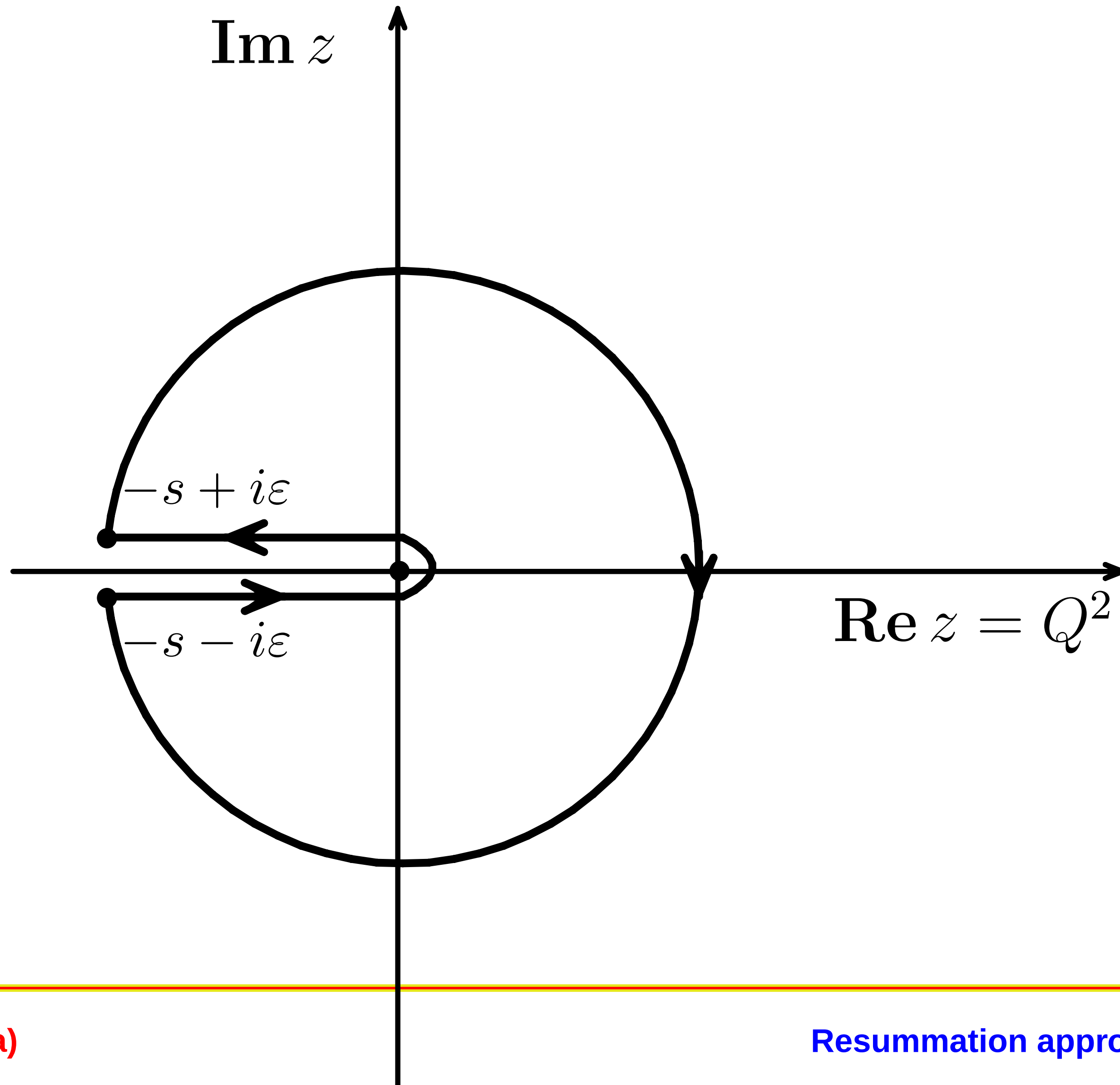
Problem in QCD PT: Minkowski region?

But $\alpha_s(z)$ and hence $D(z)f(z)$ have Landau pole singularity just inside!



Problem in QCD PT: Minkowski region?

In **APT** effective couplings $\mathcal{A}_n(z)$ are analytic functions $\Rightarrow$ Problem does not appear! Equivalence to CIPT for $\boldsymbol{R(s)}$.



Basics of APT

- Different couplings in Minkowskian **(Radyushkin, Krasnikov&Pivovarov; 1982)** and Euclidean **(Shirkov&Solovtsov; 1996)** regions

- Based on RG $\oplus$ Causality

  RG $\Downarrow$ UV asymptotics

  Causality $\Downarrow$ Spectrality

- Euclidean: $-q^2 = Q^2, \; L = \ln Q^2/\Lambda^2, \; \{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$

  Minkowskian: $q^2 = s, \; L_s = \ln s/\Lambda^2, \; \{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$

- PT $\sum_m d_m a_s^m(Q^2) \Rightarrow \sum_m d_m \mathcal{A}_m(Q^2)$ APT

  m – power $\Rightarrow$ m – index

Spectral representation

By **analytization** we mean “Källen–Lehman” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \textbf{Im}\left[f(-\sigma)\right]/\pi$.

Spectral representation

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$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then

$$\rho(\sigma) = \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}$$

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$$\begin{aligned}
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$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

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$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

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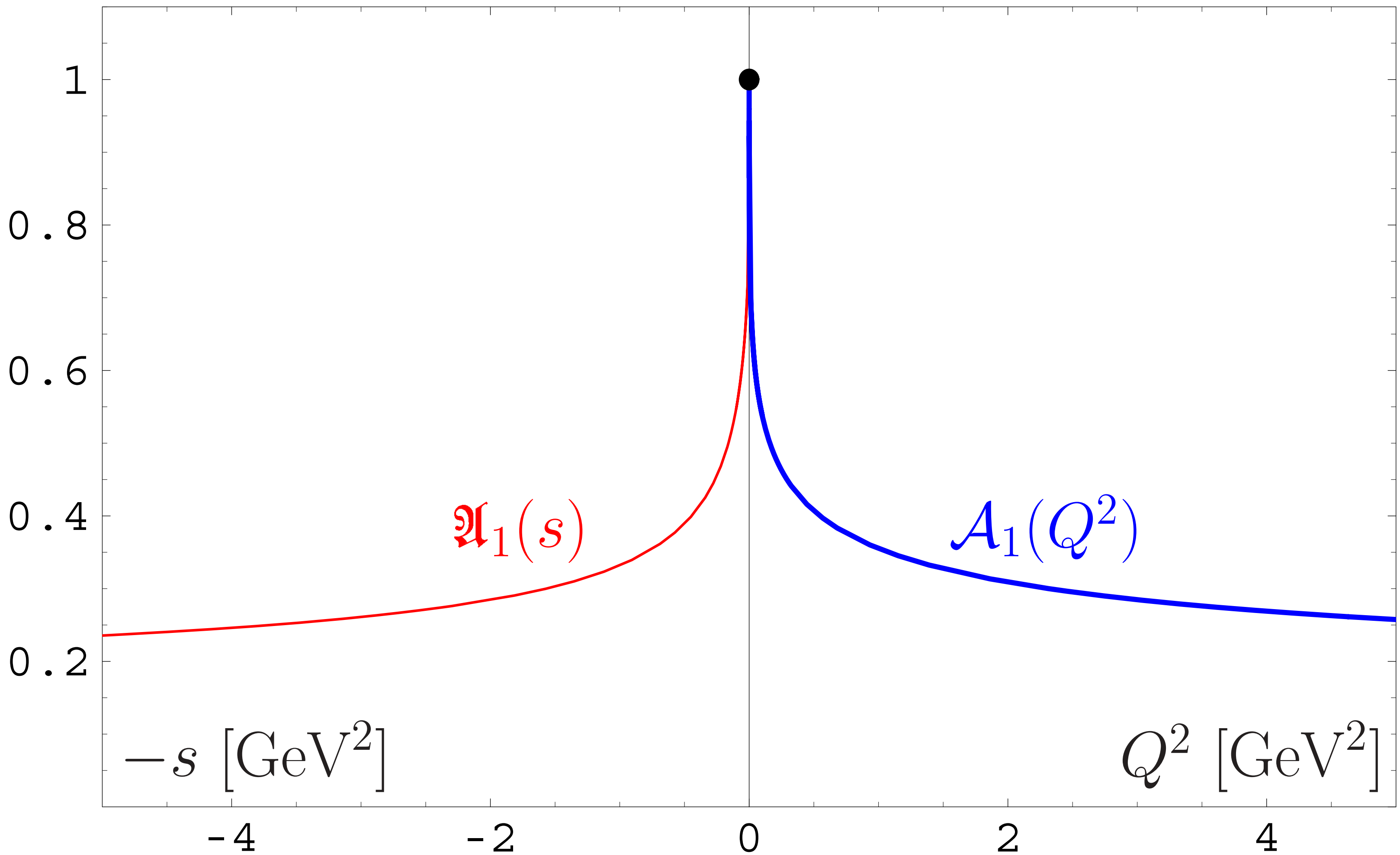
$$a_s^n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

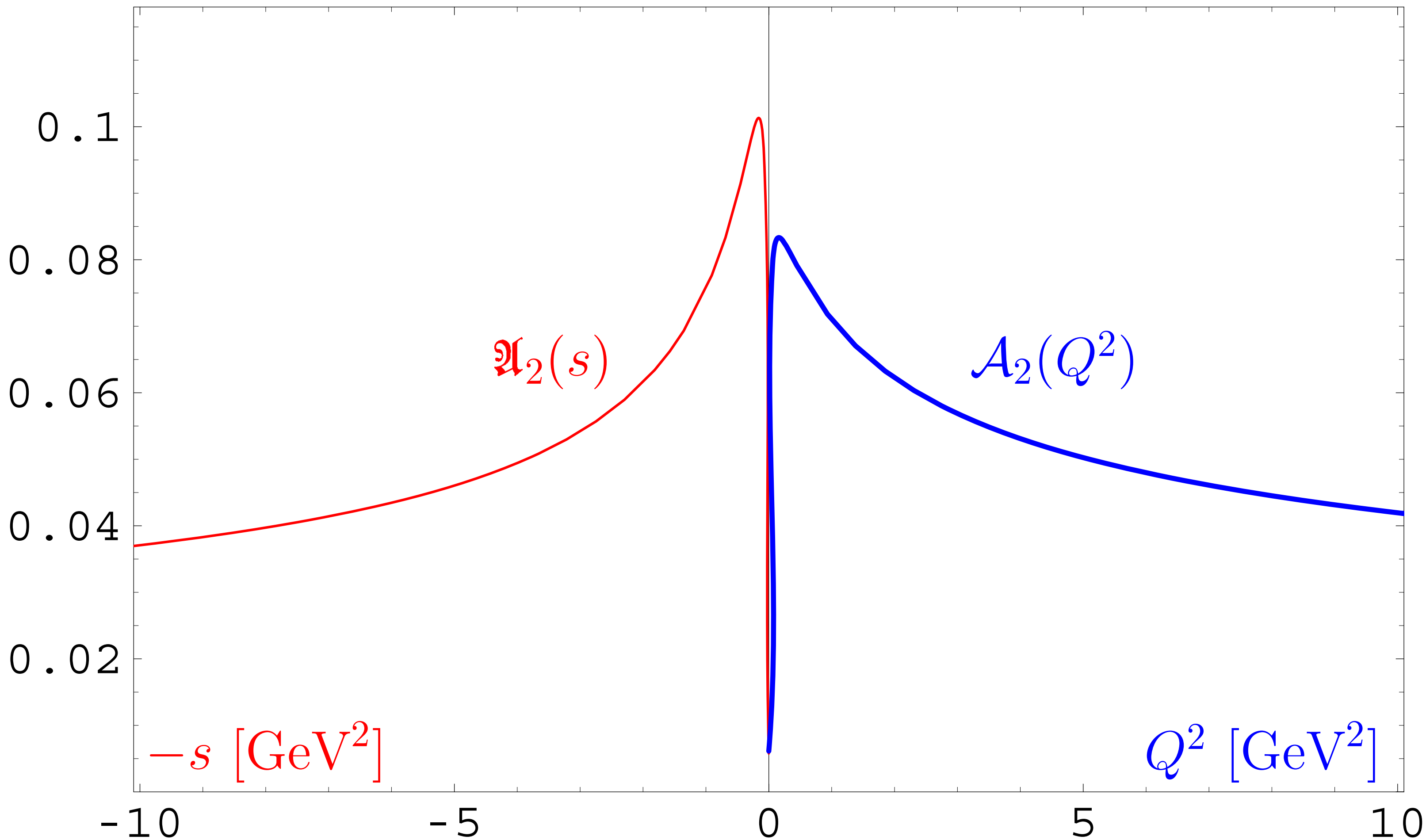
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



Problems of APT.
Resolution:
Fractional APT

Problems of APT

Open Questions

- "Analytization" of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as factorization or renormalization scale
[Karanikas&Stefanis – PLB 504 (2001) 225]

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Problems of APT

Open Questions

- “Analytization” of multi-scale amplitudes beyond LO of pQCD: additional logs depending on scale that serves as factorization or renormalization scale **[Karanikas&Stefanis – PLB 504 (2001) 225]**
- Evolution induces some non-integer, fractional, powers of coupling constant
- Resummation of gluonic corrections, giving rise to Sudakov factors, under “Analytization” difficult task **[Stefanis, Schroers, Kim – PLB 449 (1999) 299; EPJC 18 (2000) 137]**

Problems of APT

In standard QCD PT we have not only power series $F[L] = \sum_m f_m \, a_s^m[L]$, but also:

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- RG-improvment to account for higher-orders $\rightarrow$

$$Z[L] = \textbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)} \, da\right\} \xrightarrow{\textbf{1-loop}} [a_s[L]]^{\gamma_0/(2\beta_0)}$$

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- Factorization $\longrightarrow$ $[a_s[L]]^n \, L^m$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recursive relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

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and the same in Minkowski domain

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We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν.

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Here we need only elementary functions.

FAPT(M): Properties of $\mathfrak{A}_{\nu}[L]$

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Here we need only elementary functions. Properties:

- $\mathfrak{A}_{0}[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L$;
- $\mathfrak{A}_{-2}[L] = L^2 - \frac{\pi^2}{3}$, $\mathfrak{A}_{-3}[L] = L\left(L^2 - \pi^2\right), \ \ldots$;
- $\mathfrak{A}_{m}[L] = (-1)^m \mathfrak{A}_{m}[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
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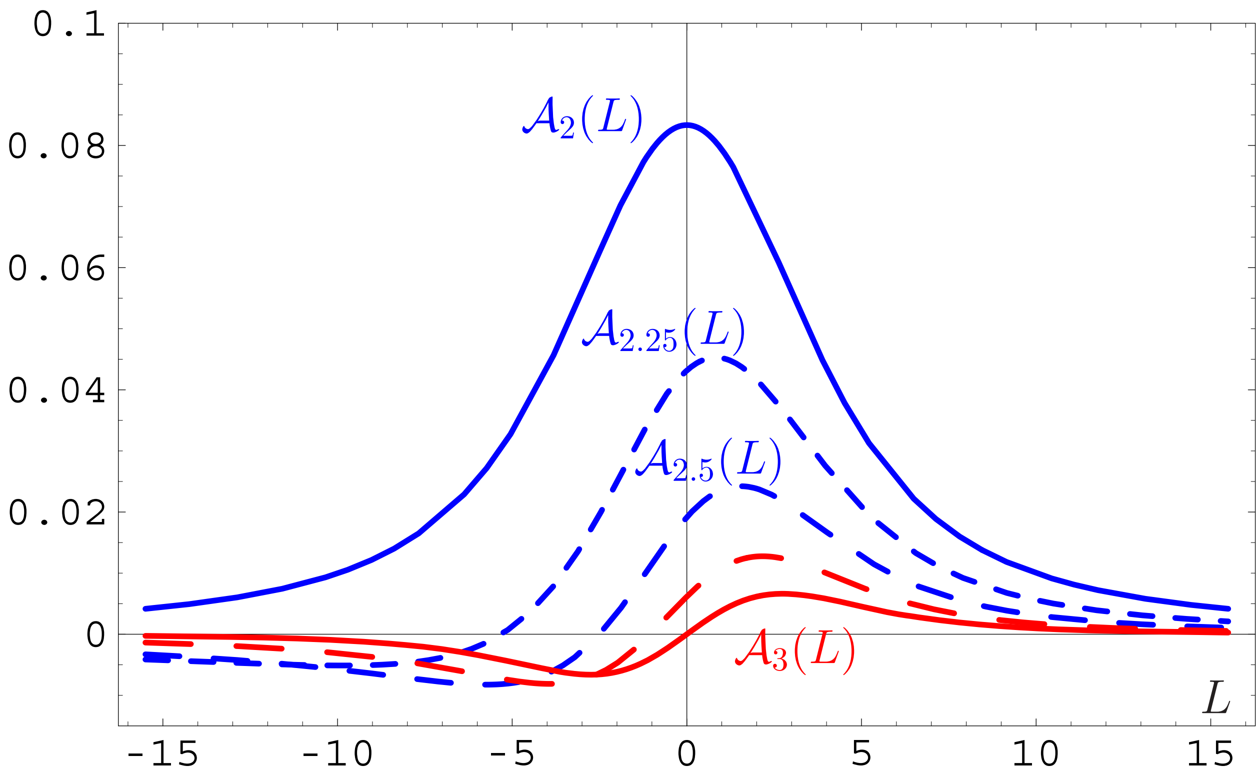
FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

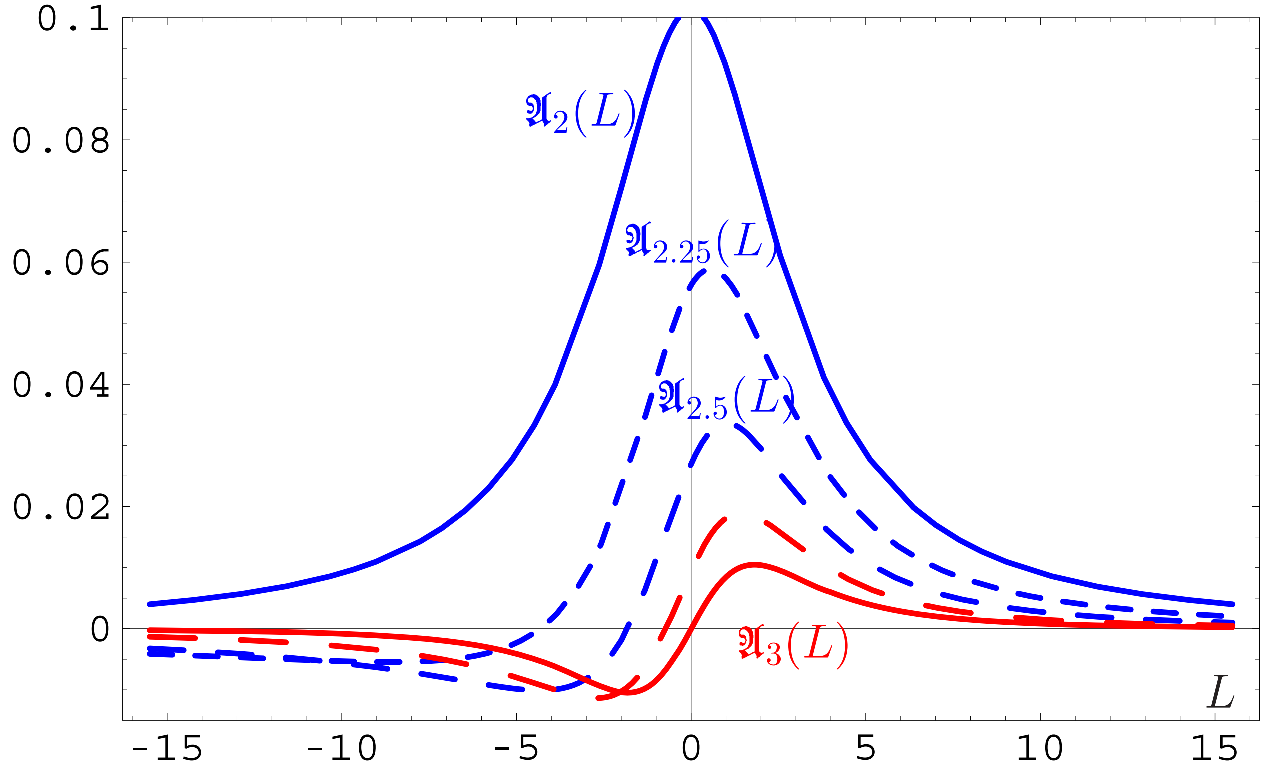
Graphics for fractional $\nu \in [2, 3]$:



FAPT(M): Graphics of $\mathfrak{A}_\nu[L]$ *vs.* L

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

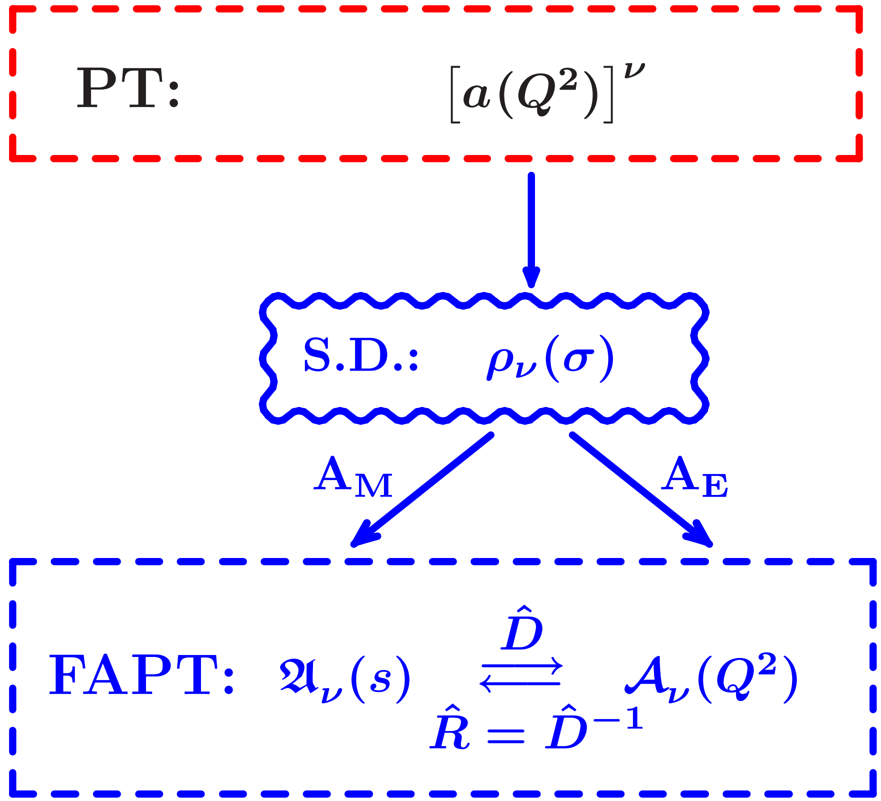
Compare with graphics in Minkowskian region :



Development of FAPT:

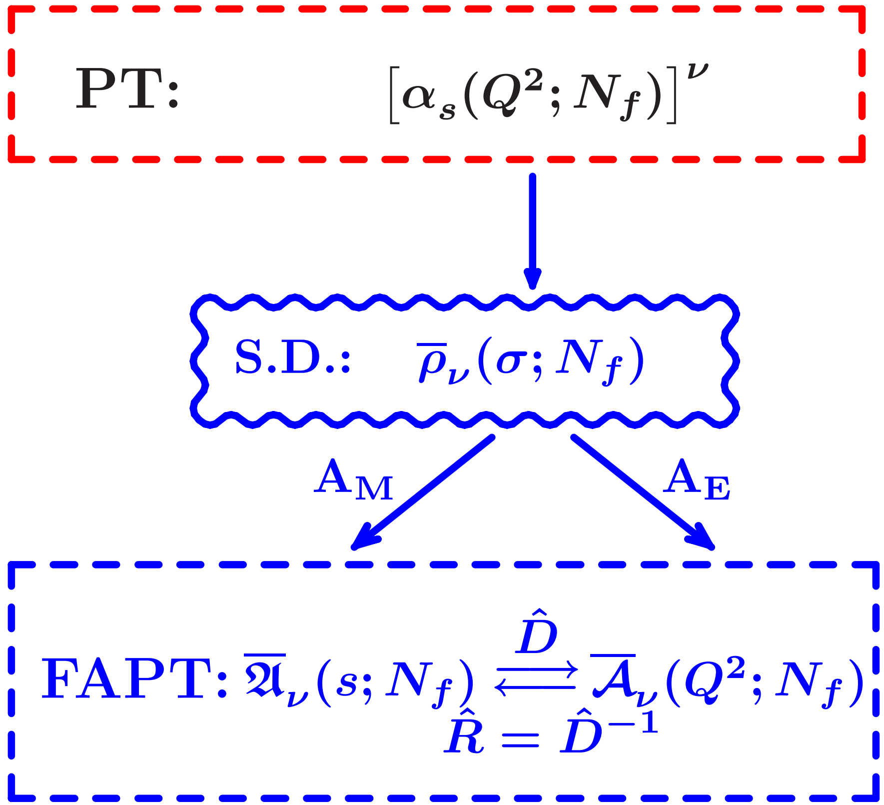
Heavy-Quark Thresholds

Conceptual scheme of ***FAPT***



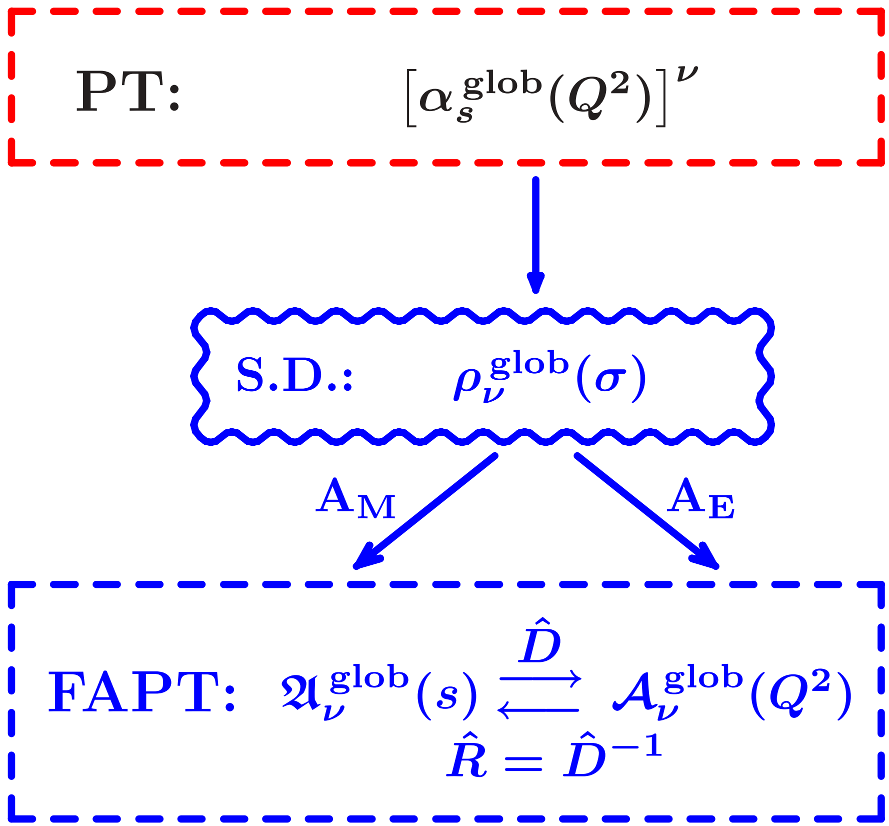
Here N_f is fixed and factorized out.

Conceptual scheme of FAPT



Here N_f is fixed, but not factorized out.

Conceptual scheme of ***FAPT***



Here we see how “analytization” takes into account N_f-dependence.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln\left(m_c^2/\Lambda_3^2\right)$ and $\lambda_4 = \ln\left(\Lambda_3^2/\Lambda_4^2\right)$.

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Then:

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta\left(L < L_4\right)\left[\overline{\mathfrak{A}}_\nu[L;3] - \overline{\mathfrak{A}}_\nu[L_4;3] + \overline{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right] + \theta\left(L \geq L_4\right)\overline{\mathfrak{A}}_\nu[L+\lambda_4;4]$$

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and

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \int\limits_{-\infty}^{L_4} \frac{\overline{\rho}_\nu\,[L_\sigma;3] - \overline{\rho}_\nu\,[L_\sigma+\lambda_4;4]}{1+e^{L-L_\sigma}}\,dL_\sigma$$

Resummation in one-loop APT and FAPT

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_n[L]$

Resummation in one-loop APT

Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n \, \mathcal{A}_n[L]$

Let exist the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^{\infty} P(t) \, t^{n-1} dt \quad \text{with} \quad \int_0^{\infty} P(t) \, dt = 1 \, .$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^{\infty} f(t) \, P(t) \, dt \, .$$

Then coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Resummation in one-loop APT

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We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L] \, .$$

Resummation in one-loop APT

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in one-loop APT

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and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \left\langle\!\left\langle \mathfrak{A}_1[L-t] \right\rangle\!\right\rangle_{P(t)}$$

Resummation in Global Minkowskian APT

Consider series $\mathcal{R}[L] = d_0 + \sum_{n=1}^{\infty} d_n\, \mathfrak{A}_n^{\text{glob}}[L]$

with coefficients $d_n = d_1 \,\langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Result:

$$\mathcal{R}[L] = d_0 \;+\; d_1 \langle\langle \theta\,(L < L_4) \Big[\Delta_4 \overline{\mathfrak{A}}_1[t] + \overline{\mathfrak{A}}_1 \Big[L - \frac{t}{\beta_3}; 3\Big]\Big] \rangle\rangle_{P(t)}$$

$$+\; d_1 \langle\langle \theta\,(L \geq L_4) \overline{\mathfrak{A}}_1 \Big[L + \lambda_4 - \frac{t}{\beta_4}; 4\Big] \rangle\rangle_{P(t)}\,.$$

where

$$\Delta_4 \overline{\mathfrak{A}}_1[t] = \overline{\mathfrak{A}}_1\Big[L_4 + \lambda_4 - \frac{t}{\beta_4}; 4\Big] - \overline{\mathfrak{A}}_1\Big[L_3 - \frac{t}{\beta_3}; 3\Big]\,.$$

Resummation in Global Euclidean APT

In Euclidean domain the result is more complicated:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \int\limits_{-\infty}^{L_4} \frac{\overline{\rho}_1\left[L_\sigma; 3\right] dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_3}} \rangle\rangle_{P(t)}$$

$$+ \langle\langle \Delta_4[L,t] \rangle\rangle_{P(t)} + d_1 \langle\langle \int\limits_{L_4}^{\infty} \frac{\overline{\rho}_1\left[L_\sigma + \lambda_4; 4\right] dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_4}} \rangle\rangle_{P(t)} \cdot$$

where

$$\Delta_4[L,t] = \int\limits_0^1 \frac{\overline{\rho}_1\left[L_4 + \lambda_4 - tx/\beta_4; 4\right] t}{\beta_4\left[1 + e^{L - L_4 - t\bar{x}/\beta_4}\right]} dx$$

$$- \int\limits_0^1 \frac{\overline{\rho}_1\left[L_3 - tx/\beta_3; 3\right] t}{\beta_3\left[1 + e^{L - L_4 - t\bar{x}/\beta_3}\right]} dx.$$

Resummation in FAPT

Consider seria $$\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}[L]$$

and $$\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu[L] + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}[L]$$

with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Result:

$$\begin{aligned} \mathcal{R}_\nu[L] &= d_0\,\mathfrak{A}_\nu[L] + d_1\,\langle\langle \mathfrak{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,; \\ \mathcal{D}_\nu[L] &= d_0\,\mathcal{A}_\nu[L] + d_1\,\langle\langle \mathcal{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,. \end{aligned}$$

where $P_\nu(t) = \int\limits_0^1 P\left(\frac{t}{1-z}\right)\nu\, z^{\nu-1}\frac{dz}{1-z}\,.$

Resummation in Global Minkowskian FAPT

Consider series $$\mathcal{R}_\nu[L] = d_0\,\mathfrak{A}_\nu^{\textbf{glob}} + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_{n+\nu}^{\textbf{glob}}[L]$$

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Then result is complete analog of the Global APT(M) result with natural substitutions:

$$\overline{\mathfrak{A}}_1[L] \to \overline{\mathfrak{A}}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

with $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right) \nu\, z^{\nu-1} \frac{dz}{1-z}\,.$

Resummation in Global Euclidean FAPT

Consider series $\mathcal{D}_\nu[L] = d_0\,\mathcal{A}_\nu^{\text{glob}} + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_{n+\nu}^{\text{glob}}[L]$

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Then result is complete analog of the Global APT(E) result with natural substitutions:

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with $P_\nu(t) = \int_0^1 P\left(\frac{t}{1-z}\right)\nu\, z^{\nu-1}\frac{dz}{1-z}\,.$

Higgs boson decay

$$H^0 \to b\bar{b}$$

Higgs boson decay into $b\bar{b}$*-pair*

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\mathsf{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\mathsf{S}}(x) J_{\mathsf{S}}(0)\,]\,|0\rangle$$

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in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \mathbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma(\mathbf{H} \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathsf{H}}\, m_b^2(M_{\mathsf{H}})\, R_{\mathsf{S}}(s = M_{\mathsf{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[\frac{\alpha_s(Q^2)}{\pi}\right]^{\nu_0} \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

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$$\left[3\,\hat{m}_b^2\right]^{-1} \widetilde{D}_{\mathbf{S}}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{\nu_0} + \sum_{m>0} d_m \left(\frac{\alpha_s(Q^2)}{\pi}\right)^{m+\nu_0}$$

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In FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathbf{S}}^{(l);N}[L] = \frac{3\hat{m}^2}{\pi^{\nu_0}} \left[\mathfrak{A}_{\nu_0}^{(l);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m^{(l)}}{\pi^m} \mathfrak{A}_{m+\nu_0}^{(l);\text{glob}}[L]\right]$$

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
pQCD	1	7.42	62.3		—

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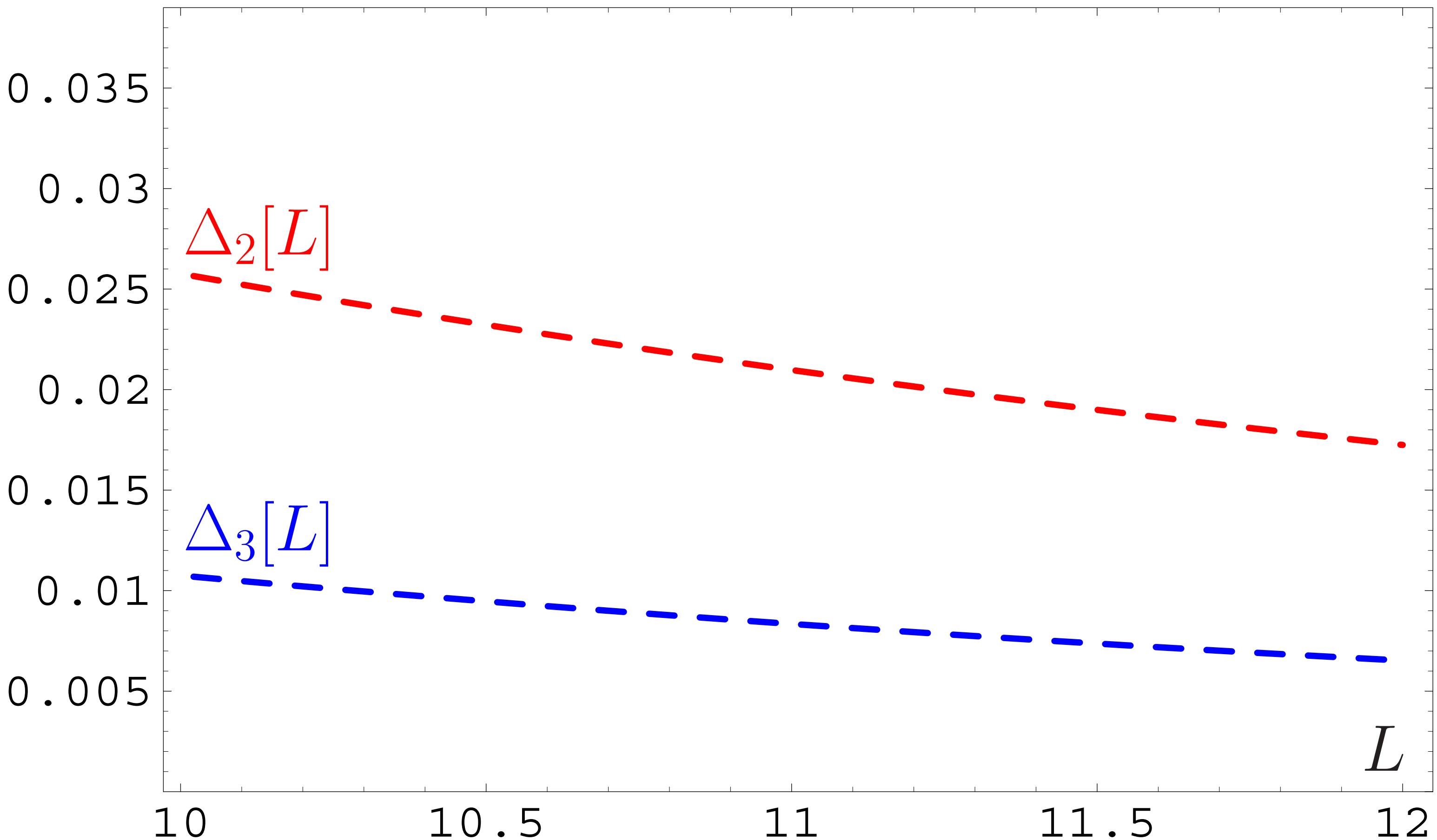
FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L] / \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



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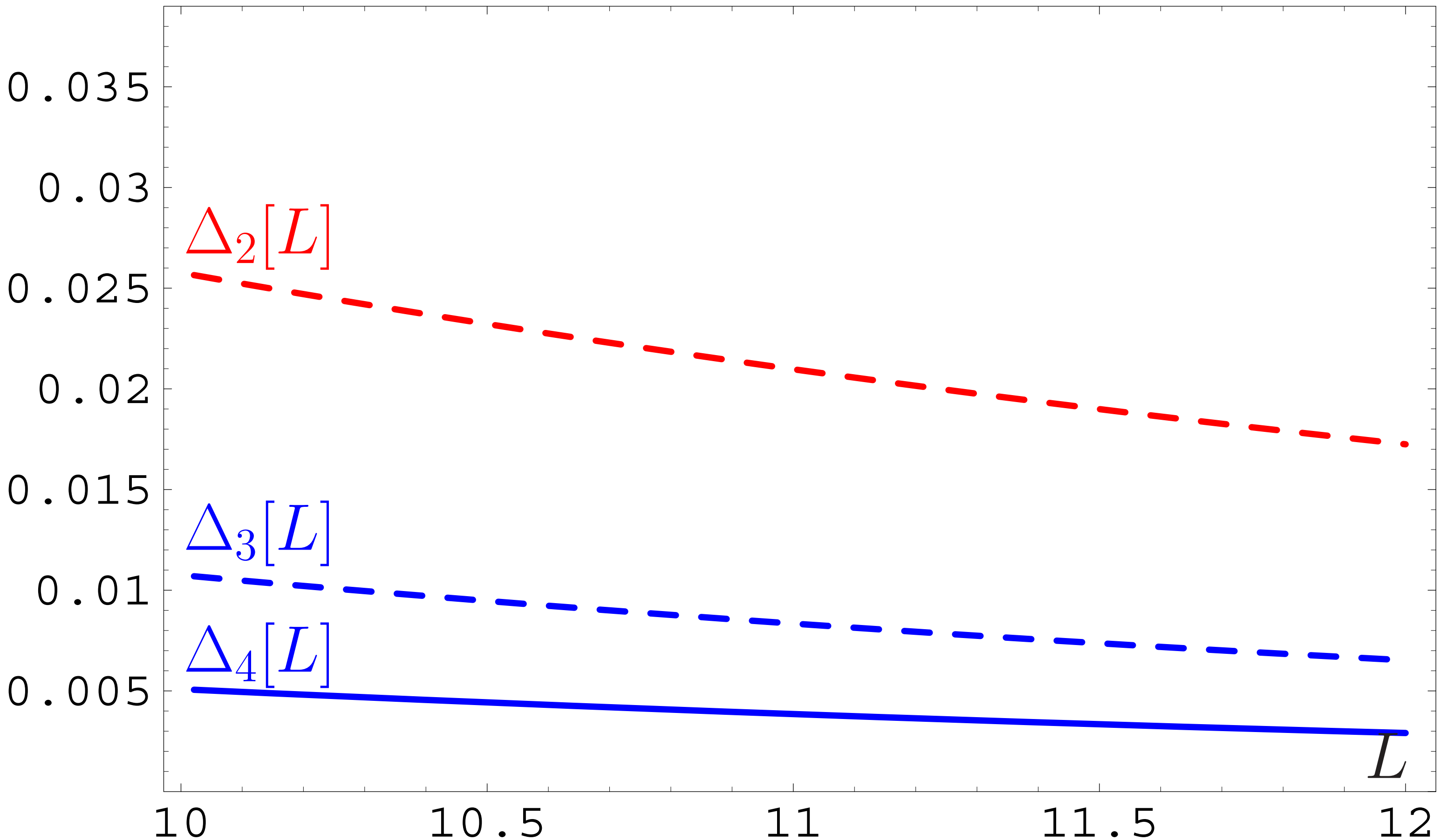

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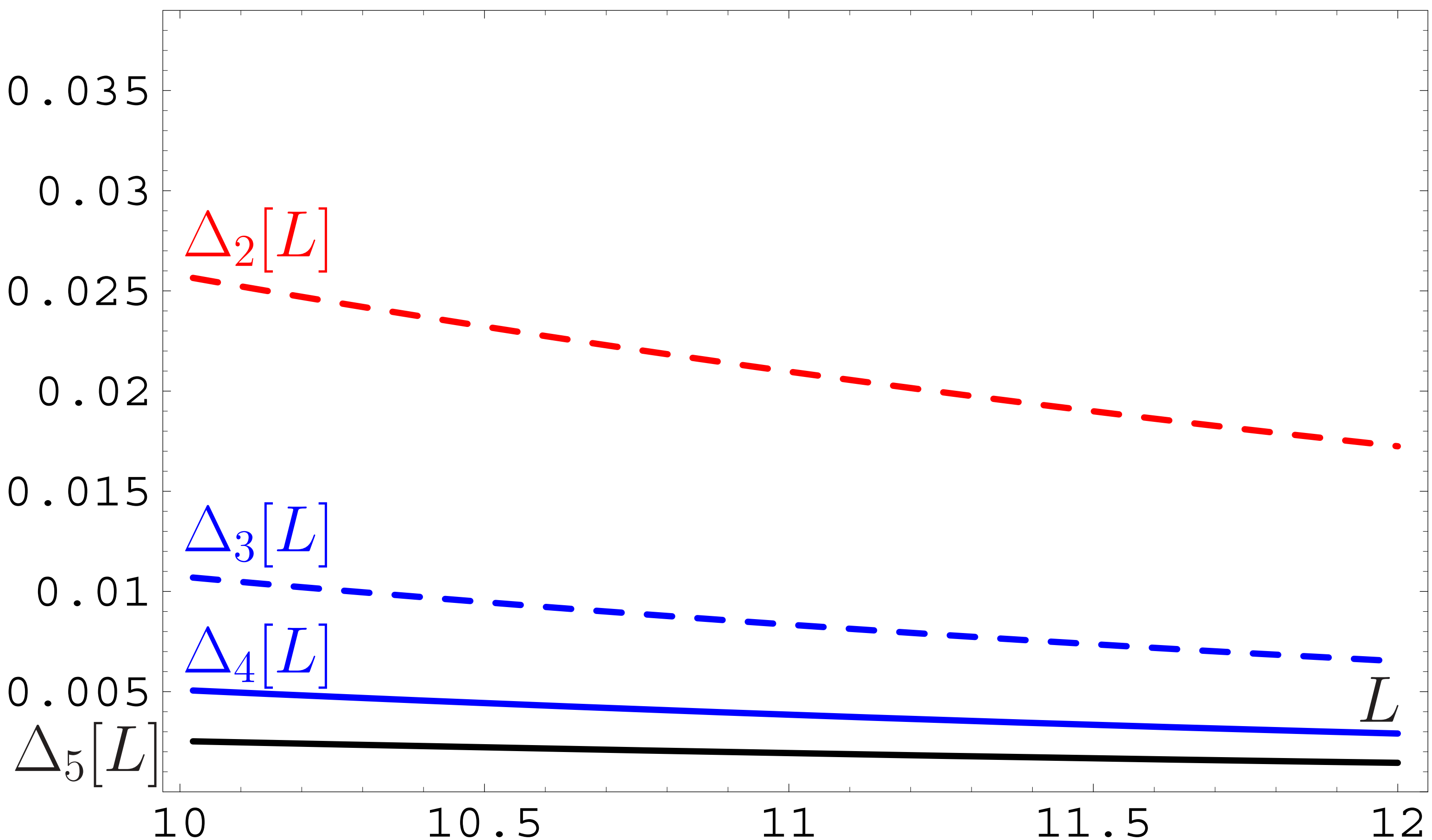

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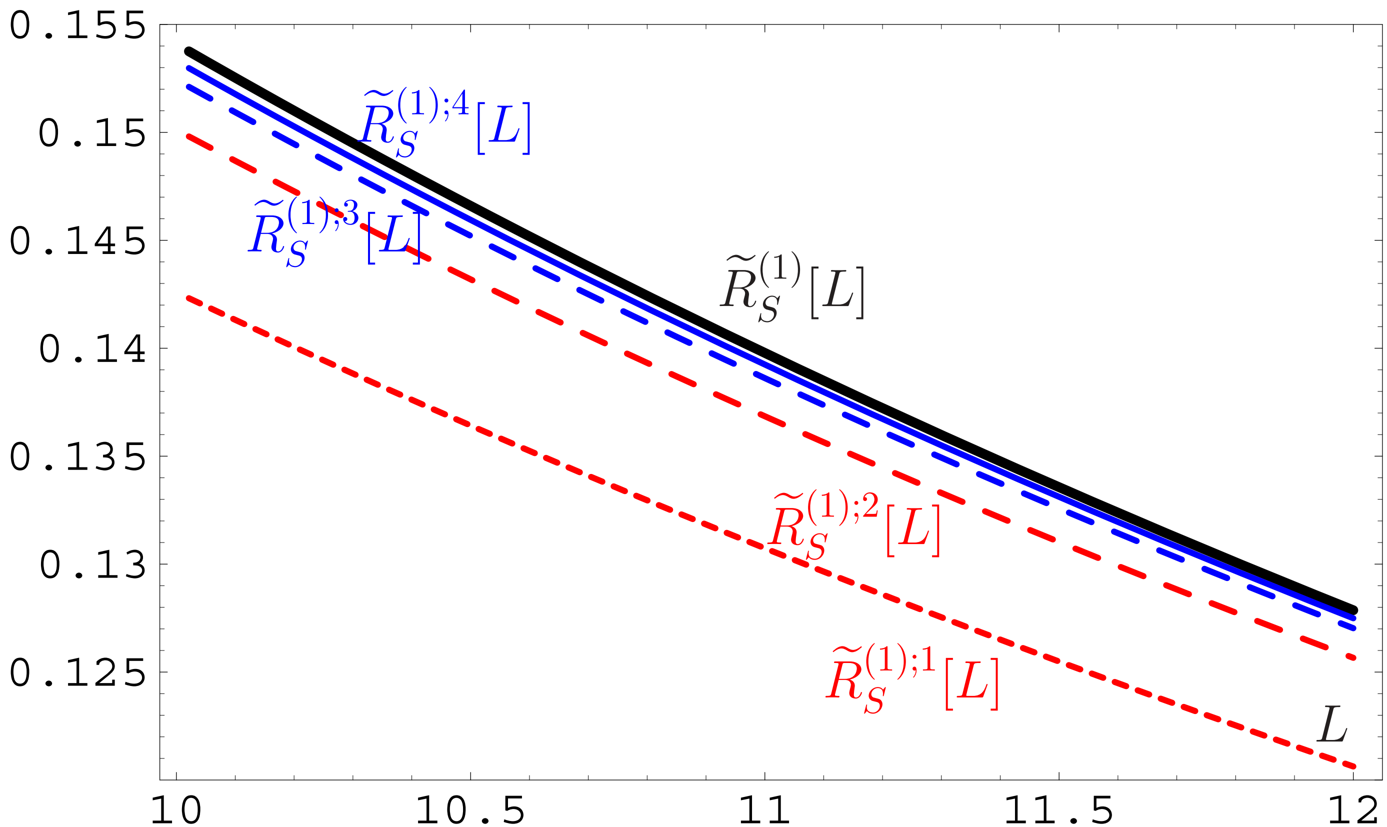
But profit will be tiny — instead of $\mathbf{0.5}\%$ one'll obtain $\mathbf{0.3}\%$!



Adler function $D(Q^2)$ and ratio $R(s)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_{\mathsf{V}}(Q^2) = \frac{(4\pi)^2}{3q^2}\, i \int dx\, e^{iqx} \langle 0|\, T[\, J_\mu(x) J^\mu(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{V}}(s) = \frac{1}{\pi}\, \mathrm{Im}\, \Pi_{\mathsf{V}}(-s - i\epsilon)\,,$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_{\mathsf{V}}(\sigma)}{(\sigma + Q^2)^2}\, d\sigma\,.$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

$$D(Q^2) = 1 + \sum_{m>0} \frac{d_m}{\pi^m} \left(\frac{\alpha_s(Q^2)}{\pi} \right)^m .$$

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Model for perturbative coefficients

Let us have a look to coefficients d_m of the PT series.

Model	d_1	d_2	d_3	d_4	d_5
pQCD results with $N_f = 4$	1	1.52	2.59		—

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$c = 3.467,\ \beta = 1.325$	1	1.50	2.62		

We use model $d_n^{\text{mod}} = \dfrac{c^{n-1}(\beta^{n+1} - n)}{\beta^2 - 1}\,\Gamma(n)$

with parameters β and c estimated by known $\tilde{d}_n$ and with use of **Lipatov** asymptotics.

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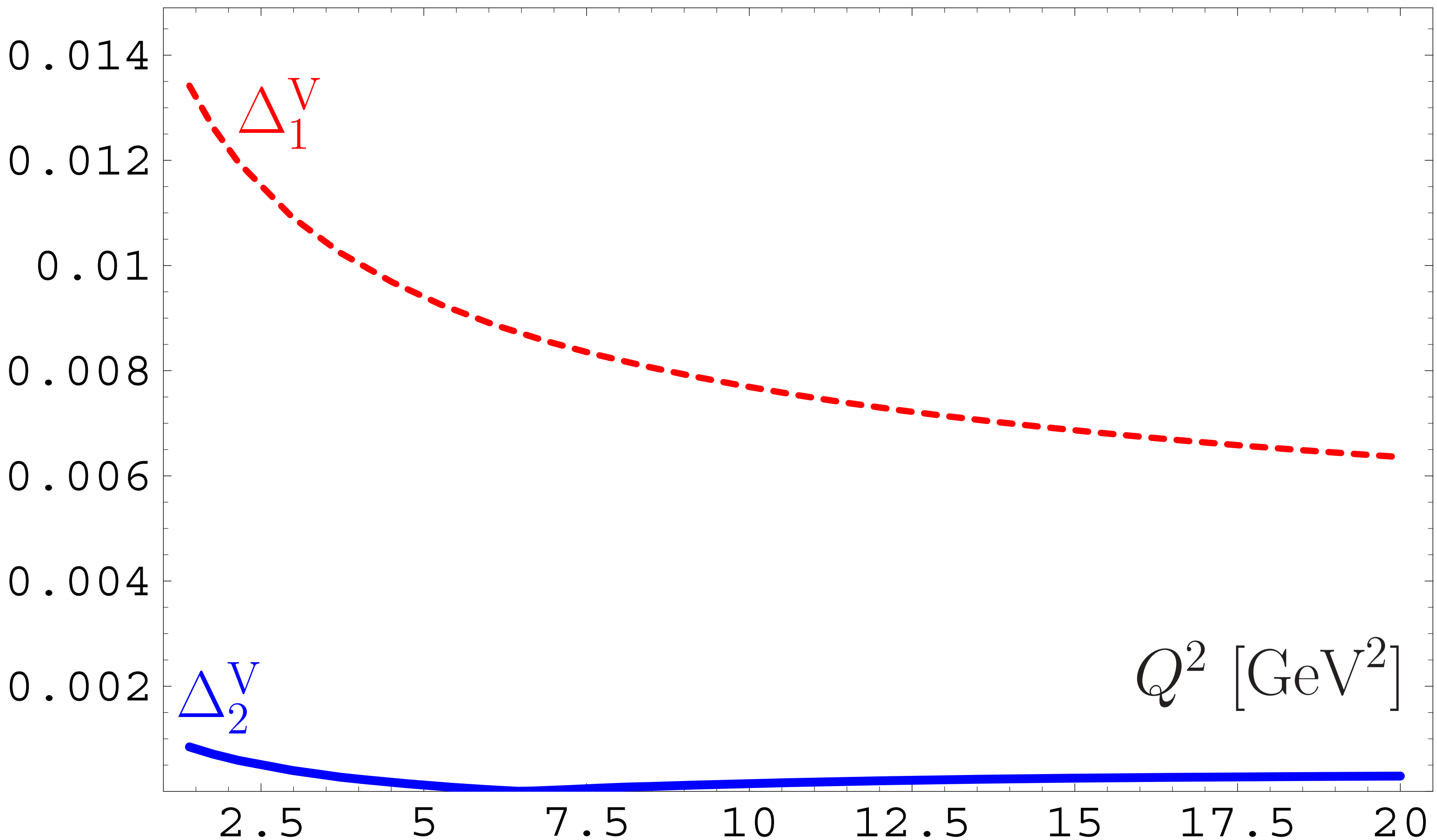
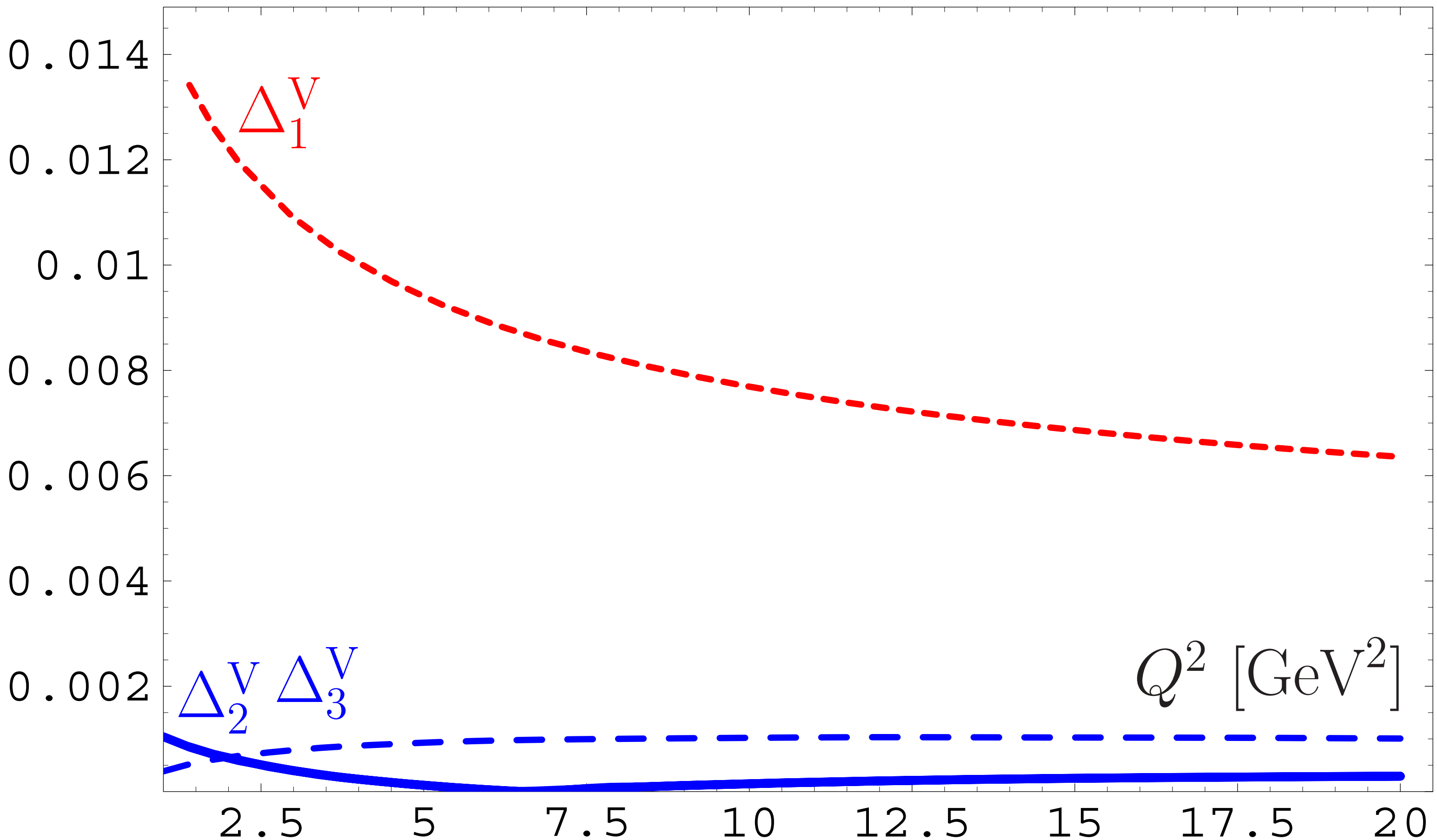
We define relative errors of series truncation at Nth term:

$$\Delta_N^{\mathrm{V}}[L] = 1 - \mathcal{D}_N[L]/\mathcal{D}_\infty[L]$$



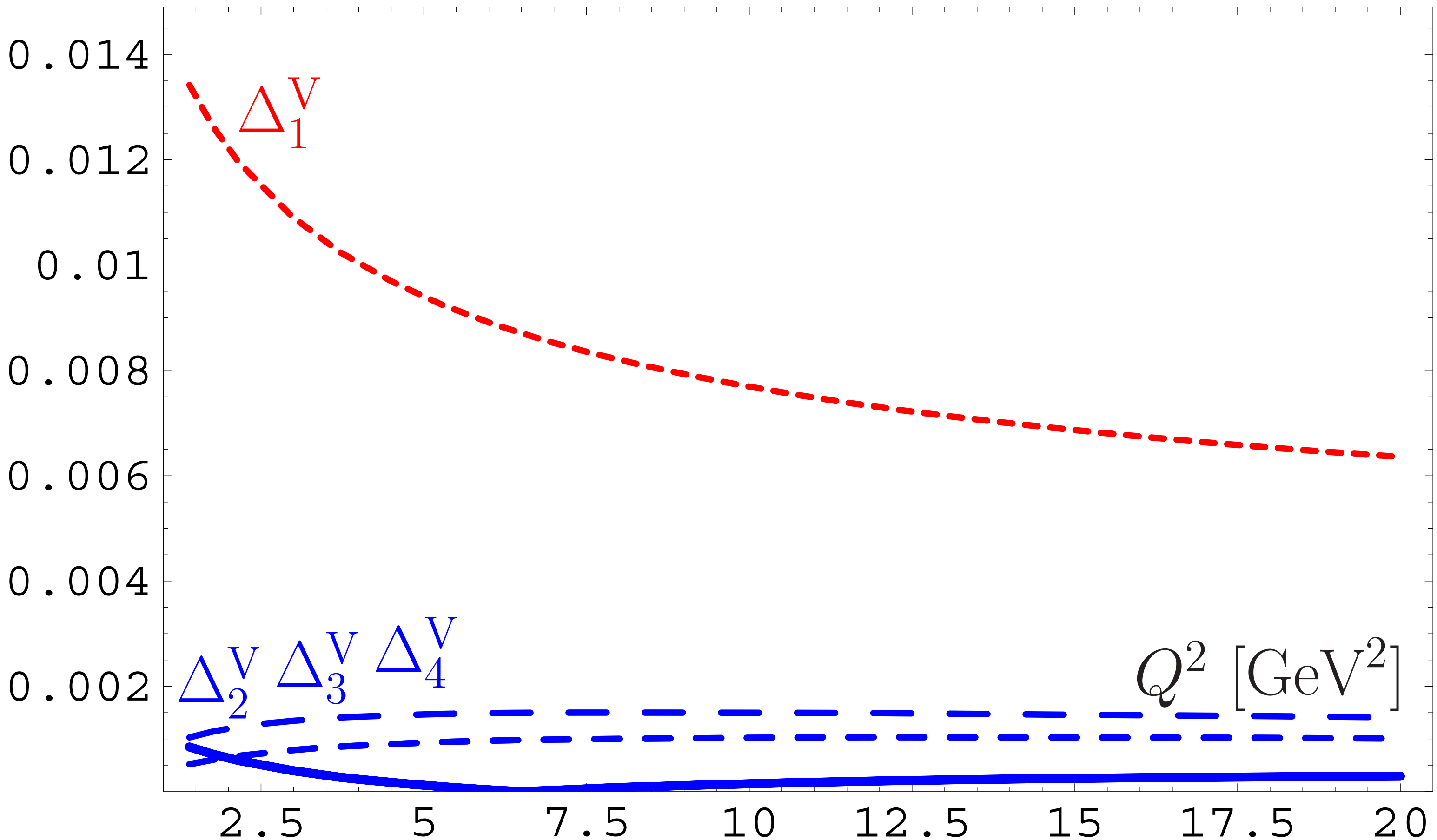
APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: The best accuracy (better than 0.1%) is achieved for **N^{2}LO** approximation.



APT(E) for $\mathcal{D}(Q^2)$: Truncation errors

Conclusion: If we add more terms **N^{3}LO** — truncation error increases.



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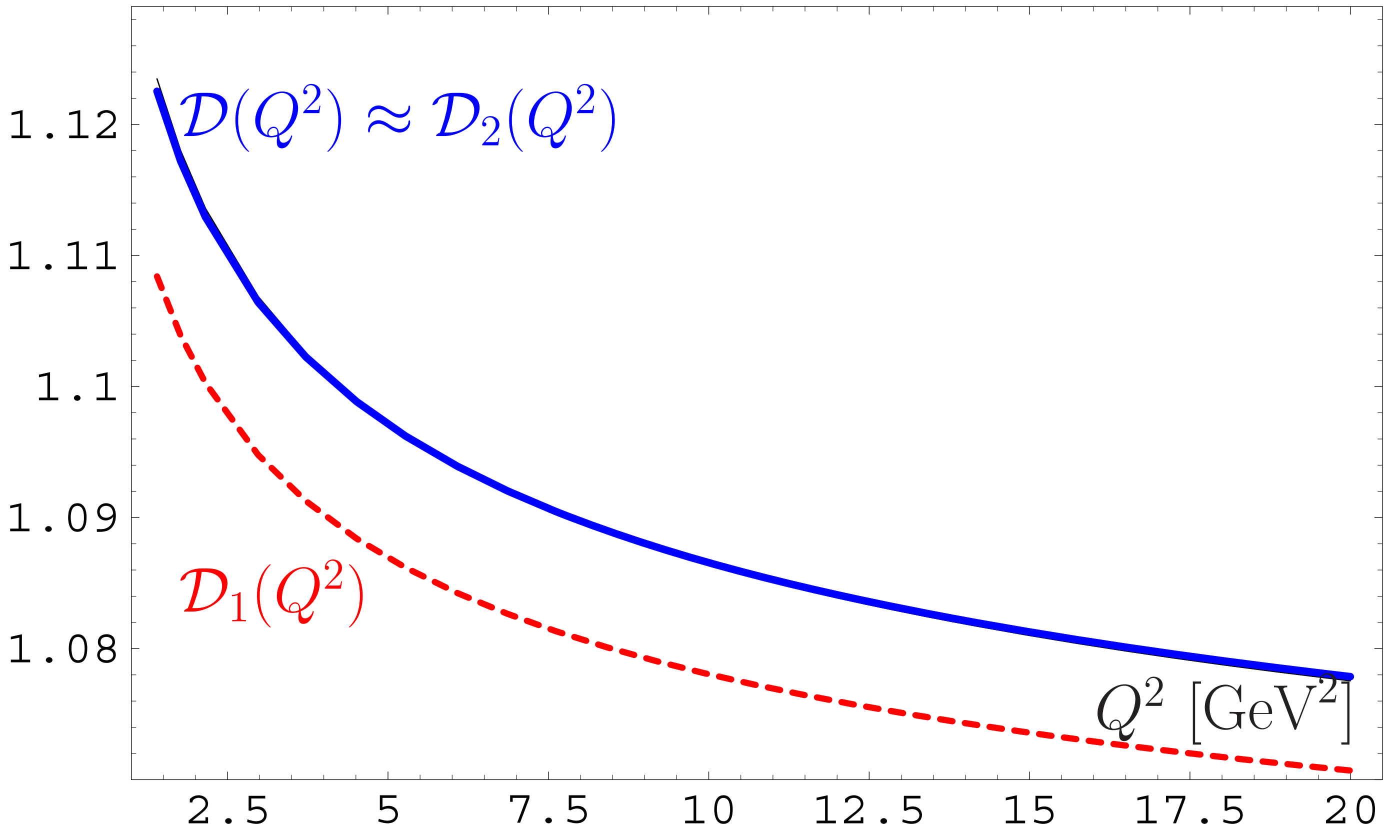

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We define relative errors of series truncation at Nth term:

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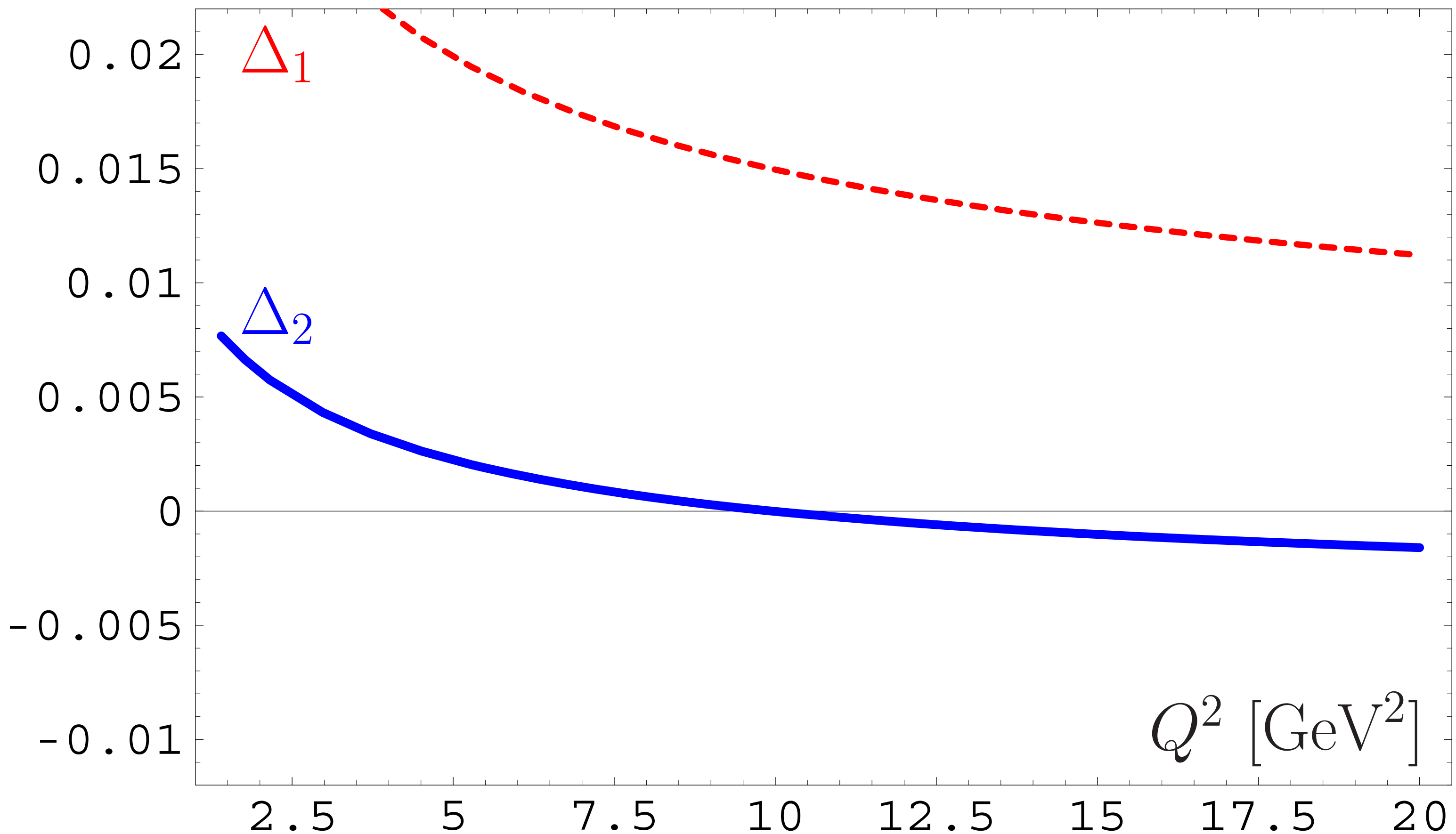

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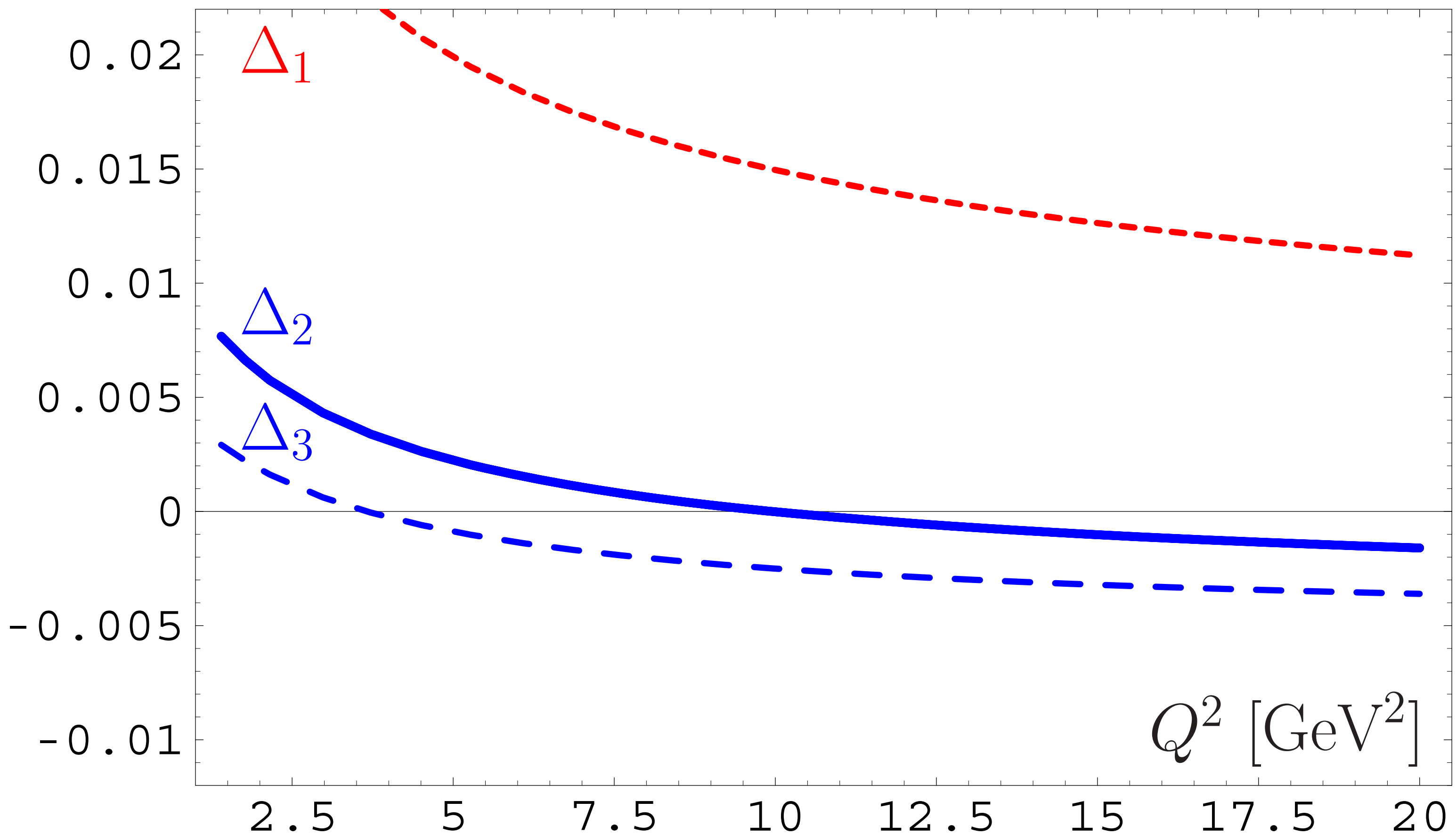
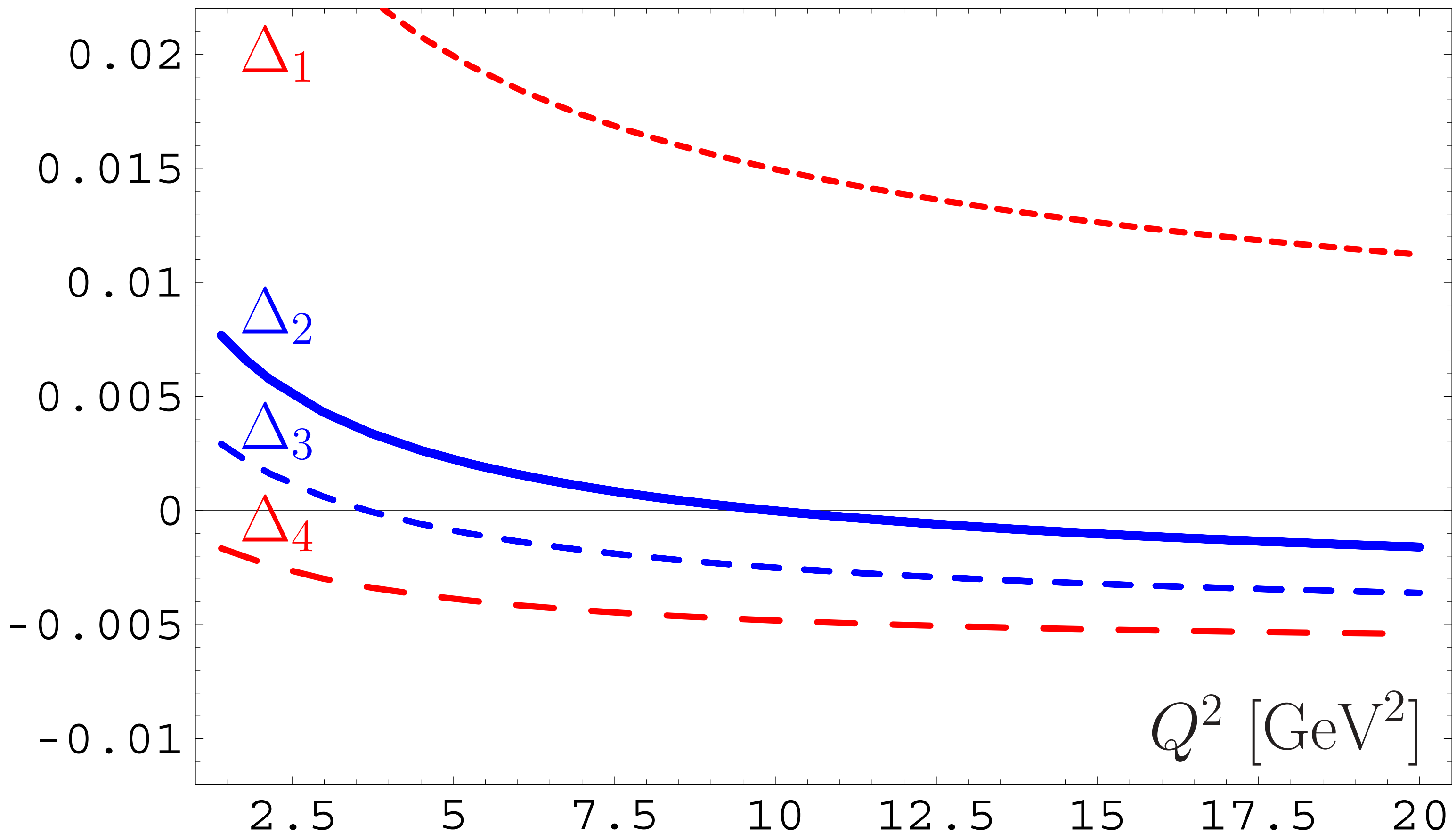
Conclusion: The best accuracy (of the order of 0.1%) is achieved for **N^{2}LO** approximation for $s \geq 7$ **GeV^2**.



APT(M) for $\mathcal{R}(s)$: Truncation errors

Conclusion: The best accuracy (of the order of 0.1%) is achieved for **N^{3}LO** approximation for $s \in [2.5, 7]$ **GeV^2**.



CONCLUSIONS

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Do not calculate higher-order corrections!
Use instead APT and FAPT!