Cosmological models with nonlocal scalar fields

Sergey Yu. Vernov

SINP MSU

based on the following papers

I.Ya. Aref'eva, L.V. Joukovskaya, S.V.,
J. Phys A **41** (2008) 304003, arXiv:0711.1364
S.V., Class. Quant. Grav. **27** (2010) 035006, arXiv:0907.0468
S.V., arXiv:1005.0372
S.V., arXiv:1005.5007

Papers about cosmological models with nonlocal fields:

I.Ya. Aref'eva, Nonlocal String Tachyon as a Model for Cosmological Dark Energy, astro-ph/0410443, 2004.

I.Ya. Aref'eva and L.V. Joukovskaya, 2005;

I.Ya. Aref'eva and A.S. Koshelev, 2006; 2008;

I.Ya. Aref'eva and I.V. Volovich, 2006; 2007;

I.Ya. Aref'eva, 2007; A.S. Koshelev, 2007;

L.V. Joukovskaya, 2007; 2008; 2009

I.Ya. Aref'eva, L.V. Joukovskaya, S.Yu.V., 2007

J.E. Lidsey, 2007;

G. Calcagni, 2006; G. Calcagni, M. Montobbio and G. Nardelli, 2007;

G. Calcagni and G. Nardelli, 2007; 2009; 2010

N. Barnaby, T. Biswas and J.M. Cline, 2006; N. Barnaby and J.M. Cline, 2007; N. Barnaby and N. Kamran, 2007; 2008; N. Barnaby, 2008; 2010;

D.J. Mulryne, N.J. Nunes, 2008;

B. Dragovich, 2008;

A.S. Koshelev, S.Yu.V., 2009; 2010

The SFT inspired nonlocal cosmological models

From the Witten action of bosonic cubic string field theory, considering only tachyon scalar field $\phi(x)$ one obtains:

$$S = \frac{1}{g_o^2}\int d^{26}x \left[\frac{\alpha'}{2}\phi(x)\Box\phi(x) + \frac{1}{2}\phi^2(x) - \frac{1}{3}\gamma^3\Phi^3(x) - \tilde{\Lambda}\right], \qquad (1)$$

where

$$\Phi = e^{k\Box}\phi, \qquad k = \alpha'\ln(\gamma), \qquad \gamma = \frac{4}{3\sqrt{3}}. \qquad (2)$$

g_o **is the open string coupling constant,** α' **is the string length squared and** $\tilde{\Lambda} = \frac{1}{6}\gamma^{-6}$ **is added to the potential to set the local minimum of the potential to zero. The action (1) leads to equation of motion**

$$(\alpha'\Box + 1)e^{-2k\Box}\Phi = \gamma^3\Phi^2. \qquad (3)$$

In the majority of the SFT inspired nonlocal gravitation models the action is introduced by hand as a sum of the SFT action of tachyon field and gravity part of the action:

$$S = \frac{1}{g_o^2} \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R + \frac{1}{2}\phi \Box_g \phi + \frac{1}{2}\phi^2 - \frac{1}{3}\gamma^3 \Phi^3 - \Lambda \right), \qquad (4)$$

Action (4) includes a nonlocal potential. Using a suitable redefinition of the fields, one can made the potential local, at that the kinetic term becomes nonlocal.

This nonstandard kinetic term leads to a nonlocal field behavior similar to the behavior of a phantom field, and it can be approximated with a phantom kinetic term.

The behavior of an open string tachyon can be effectively simulated by a scalar field with a phantom kinetic term.

Another type of the SFT inspired models includes nonlocal modification of gravity.

Recently G. Calcagni and G. Nardelli have considered nonlocal gravity with nonlocal scalar field (arXiv: 1004.5144).

Nonlocal action in the general form

We consider a general class of gravitational models with a non-local scalar field, which are described by the following action:

$$S = \int d^4x \sqrt{-g}\alpha' \left(\frac{R}{16\pi G_N} + \frac{1}{g_o^2} \left(\frac{1}{2}\phi \mathcal{F}(\Box_g)\phi - V(\phi) \right) - \Lambda \right), \qquad (5)$$

G_N **is the Newtonian constant:** $8\pi G_N = 1/M_P^2$**,**
M_P **is the Planck mass.**
We use the signature $(-,+,+,+)$**,**
$g_{\mu\nu}$ **is the metric tensor,**
R **is the scalar curvature,**
Λ **is the cosmological constant.**

Hereafter the d'Alembertian $\Box_g$ **is applied to scalar functions and can be written as follows**

$$\Box_g = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \,. \qquad (6)$$

The function $\mathcal{F}(\Box_g)$ is assumed to be an analytic function:

$$\mathcal{F}(\Box_g)=\sum_{n=0}^{\infty}f_n\Box_g^{\,n}. \tag{7}$$

Note that the term $\phi\mathcal{F}(\Box_g)\phi$ include not only terms with derivatives, but also $f_0\phi^2$.

In an arbitrary metric the energy-momentum tensor

$$T_{\mu\nu}=\,-\,\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}=\frac{1}{g_o^2}\Big(E_{\mu\nu}+E_{\nu\mu}-g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma}+W\right)\Big), \tag{8}$$

$$E_{\mu\nu}\equiv\frac{1}{2}\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\partial_\mu\Box_g^l\phi\partial_\nu\Box_g^{n-1-l}\phi, \tag{9}$$

$$W\equiv\frac{1}{2}\sum_{n=2}^{\infty}f_n\sum_{l=1}^{n-1}\Box_g^l\phi\Box_g^{n-l}\phi-\frac{f_0}{2}\phi^2+V(\phi). \tag{10}$$

From action (5) we obtain the following equations

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \Lambda g_{\mu\nu}\right), \qquad (11)$$

$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \qquad (12)$$

where $G_{\mu\nu}$ **is the Einstein tensor.**

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$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \tag{14}$$

where $G_{\mu\nu}$ is the Einstein tensor.

It is a system of nonlocal nonlinear equations !!!

HOW CAN WE FIND A SOLUTION?

The Ostragradski representation.

- **M. Ostrogradski**, *Mémoire sur les équations differentielles relatives aux problèmes des isoperimétres*, Mem. St. Petersbourg VI Series, V. 4 (1850) 385–517
- **A. Pais and G.E. Uhlenbeck**, *On Field Theories with Nonlocalized Action*, Phys. Rev. 79 (1950) 145–165

Let $\mathcal{F}$ is a polynomial:

$$\mathcal{F}(\Box) = \mathcal{F}_1(\Box) \equiv \prod_{j=1}^{N} \left(1 + \frac{\Box}{\omega_j^2}\right), \qquad (15)$$

all roots, which are equal to $-\omega_j^2$, are simple.

We want to get the Ostrogradski representation for

$$\mathcal{L}_F = \phi \mathcal{F}_1(\Box) \phi. \qquad (16)$$

We should find such numbers c_j, that the Lagrangian $\mathcal{L}_F$ can

be written in the following form

$$L_l = \sum_{j=1}^{N} c_j \phi_j (\Box + \omega_j^2) \phi_j. \tag{17}$$

$$\phi_j = \prod_{k=1, k\neq j}^{N} \left(1 + \frac{1}{\omega_k^2}\Box\right) \phi, \qquad \Rightarrow \qquad \left(\Box + \omega_j^2\right) \phi_j = 0. \tag{18}$$

Substituting ϕ_j in L_l, we get

$$L_l \cong \mathcal{L}_F \quad \Leftrightarrow \quad \sum_{k=1}^{N} \frac{c_k \omega_k^4}{\omega_k^2 + \Box} = \frac{1}{\mathcal{F}_1(\Box)}. \tag{19}$$

All roots of $\mathcal{F}_1(\Box)$ are simple, hence, we can perform a partial fraction decomposition of $1/\mathcal{F}_1(\Box)$.

$$c_k = \frac{\mathcal{F}_1'(-\omega_k^2)}{\omega_k^4}, \quad \textbf{where} \quad \mathcal{F}_1(-\omega_k^2)' \equiv \frac{d\mathcal{F}_1}{d\Box}|_{\Box=-\omega_k^2}. \tag{20}$$

Let $\mathcal{F}_1(\Box)$ has two real simple roots. $\mathcal{F}_1' > 0$ in one and only one root. We get model with one phantom and one real root.

An algorithm of localization in the case of an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$.

$$V_{eff} = \left(C_2 - \frac{f_0}{2}\right)\phi^2 + C_1\phi + C_0 + \Lambda. \tag{21}$$

We can change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$.

In other words, we put $C_2 = 0$ and $C_0 = 0$.

There exist 3 cases:

- $C_1 = 0$
- $C_1 \neq 0$ **and** $f_0 \neq 0$
- $C_1 \neq 0$ **and** $f_0 = 0$

I will speak about the case $C_1 = 0$. Cases $C_1 \neq 0$ have been considered in S.V., arXiv:1005.0372.

Let us consider the case $C_1 = 0$ and the equation

$$\mathcal{F}(\Box_g)\phi = 0. \tag{22}$$

We seek a particular solution of (14) in the following form

$$\phi_0 = \sum_{i=1}^{N_1} \phi_i + \sum_{k=1}^{N_2} \tilde{\phi}_k. \tag{23}$$

$$(\Box_g - J_i)\phi_i = 0, \tag{24}$$

J_i **are simple roots of the characteristic equation $\mathcal{F}(J) = 0$.** $\tilde{J}_k$ **are double roots. The fourth order differential equation**

$$(\Box - \tilde{J}_k)^2 \tilde{\phi}_k = 0 \tag{25}$$

is equivalent to the following system of equations:

$$(\Box - \tilde{J}_k)\tilde{\phi}_k = \varphi_k, \qquad (\Box - \tilde{J}_k)\varphi_k = 0. \tag{26}$$

Energy–momentum tensor for special solutions

If we have *one simple root* ϕ_1 such that $\Box_g\phi_1 = J_1\phi_1$, then

$$E_{\mu\nu}(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n-1}\partial_\mu\phi_1\partial_\nu\phi_1 = \frac{\mathcal{F}'(J_1)}{2}\partial_\mu\phi_1\partial_\nu\phi_1.$$

$$W(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n}\phi_1^2 = \frac{J_1}{2}\sum_{n=1}^{\infty} f_n n J_1^{n-1}\phi_1^2 = \frac{J_1\mathcal{F}'(J_1)}{2}\phi_1^2.$$

In the case of *two simple roots* ϕ_1 and ϕ_2 we have

$$E_{\mu\nu}(\phi_1+\phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) + E^{cr}_{\mu\nu}(\phi_1,\phi_2), \tag{27}$$

where the cross term

$$E^{cr}_{\mu\nu}(\phi_1,\phi_2) = A_1\partial_\mu\phi_1\partial_\nu\phi_2 + A_2\partial_\mu\phi_2\partial_\nu\phi_1. \tag{28}$$

$$A_1 = \frac{1}{2}\sum_{n=1}^{\infty} f_n J_1^{n-1}\sum_{l=0}^{n-1}\left(\frac{J_2}{J_1}\right)^l = \frac{\mathcal{F}(J_1)-\mathcal{F}(J_2)}{2(J_1-J_2)} = 0, \tag{29}$$

$$A_2 = 0. \tag{30}$$

So, the cross term $E^{cr}_{\mu\nu}(\phi_1, \phi_2) = 0$ and

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) \tag{31}$$

Similar calculations shows

$$W(\phi_1 + \phi_2) = W(\phi_1) + W(\phi_2). \tag{32}$$

In the case of N *simple roots* the following formula has been obtained:

$$T_{\mu\nu} = \sum_{k=1}^{N} \mathcal{F}'(J_k) \left(\partial_\mu \phi_k \partial_\nu \phi_k - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \phi_k \partial_\sigma \phi_k + J_k \phi_k^2 \right) \right). \tag{33}$$

Note that the last formula is exactly the energy-momentum tensor of many free massive scalar fields. If $\mathcal{F}(J)$ has simple real roots, then positive and negative values of $\mathcal{F}'(J_i)$ alternate, so we can obtain phantom fields.

Let $\tilde{J}_1$ is a double root. The fourth order differential equation $(\Box - \tilde{J}_1)^2\tilde{\phi}_1 = 0$ is equivalent to the following system of equations:

$$(\Box - \tilde{J}_1)\tilde{\phi}_1 = \varphi_1, \qquad (\Box - \tilde{J}_1)\varphi_1 = 0. \tag{34}$$

It is convenient to write $\Box^l\tilde{\phi}_1$ in terms of the $\tilde{\phi}_1$ and φ_1:

$$\Box^l\tilde{\phi}_1 = \tilde{J}_1^l\tilde{\phi}_1 + l\tilde{J}_1^{l-1}\varphi_1. \tag{35}$$

$$E_{\mu\nu}(\tilde{\phi}_1) = B_1\partial_\mu\tilde{\phi}_1\partial_\nu\tilde{\phi}_1 + B_2\partial_\mu\tilde{\phi}_1\partial_\nu\varphi_1 + B_3\partial_\mu\phi_1\partial_\nu\tilde{\varphi}_1 + B_4\partial_\mu\varphi_1\partial_\nu\varphi_1, \tag{36}$$

where

$$B_1 = \frac{\mathcal{F}'(\tilde{J}_1)}{2} = 0, \qquad B_2 = B_3 = \frac{\mathcal{F}''(\tilde{J}_1)}{4}, \qquad B_4 = \frac{\mathcal{F}'''(\tilde{J}_1)}{12}.$$

Thus, for one double root we obtain the following result:

$$E_{\mu\nu}(\tilde{\phi}_1) = \frac{\mathcal{F}''(\tilde{J}_1)}{4}(\partial_\mu\tilde{\phi}_1\partial_\nu\varphi_1 + \partial_\mu\phi_1\partial_\nu\tilde{\varphi}_1) + \frac{\mathcal{F}'''(\tilde{J}_1)}{12}\partial_\mu\varphi_1\partial_\nu\varphi_1.$$

Similar calculations gives

$$W(\tilde{\phi}_1) = \frac{\tilde{J}_1\mathcal{F}''(\tilde{J}_1)}{2}\tilde{\phi}_1\varphi_1 + \left(\frac{\tilde{J}_1\mathcal{F}'''(\tilde{J}_1)}{12} + \frac{\mathcal{F}''(\tilde{J}_1)}{4}\right)\varphi_1^2. \tag{37}$$

For any analytical function $\mathcal{F}(J)$, which has simple roots J_i and double roots $\tilde{J}_k$, the energy–momentum tensor

$$T_{\mu\nu}\left(\phi_0\right) = T_{\mu\nu}\left(\sum_{i=1}^{N_1}\phi_i + \sum_{k=1}^{N_2}\tilde{\phi}_k\right) = \sum_{i=1}^{N_1} T_{\mu\nu}(\phi_i) + \sum_{k=1}^{N_2} T_{\mu\nu}(\tilde{\phi}_k), \qquad (38)$$

where

$$T_{\mu\nu} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \qquad (39)$$

$$E_{\mu\nu}(\phi_i) = \frac{\mathcal{F}'(J_i)}{2}\partial_\mu\phi_i\partial_\nu\phi_i, \qquad W(\phi_i) = \frac{J_i\mathcal{F}'(J_i)}{2}\phi_i^2, \qquad \mathcal{F}' \equiv \frac{d\mathcal{F}}{dJ} \qquad (40)$$

$$E_{\mu\nu}(\tilde{\phi}_k) = \frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k + \partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right) + \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k, \qquad (41)$$

$$W(\tilde{\phi}_k) = \frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2. \qquad (42)$$

Consider the following local action

$$S_{loc}=\int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N}-\Lambda\right)+\sum_{i=1}^{N_1}S_i+\sum_{k=1}^{N_2}\tilde{S}_k, \qquad (43)$$

where

$$S_i=-\frac{1}{g_o^2}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i+J_i\phi_i^2\right),$$

$$\tilde{S}_k=-\frac{1}{g_o^2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\left(\frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k+\partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right)+\right.\right.$$
$$\left.\left.+\ \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k\right)+\frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k+\left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12}+\frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2\right).$$

Remark 1. If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models. In this case the initial nonlocal action (5) generates infinity number of local actions (43).

Remark 2. **We should prove that the way of localization is self-consistent. To construct local action (43) we assume that equations (24) are satisfied. Therefore, the method of localization is correct only if these equations can be obtained from the local action S_{loc}. The straightforward calculations show that**

$$\frac{\delta S_{loc}}{\delta \phi_i} = 0 \;\Leftrightarrow\; \Box_g \phi_i = J_i \phi_i; \qquad \frac{\delta S_{loc}}{\delta \tilde{\phi}_k} = 0 \;\Leftrightarrow\; \Box_g \varphi_k = \tilde{J}_k \varphi_k. \tag{44}$$

$$\frac{\delta S_{loc}}{\delta \varphi_k} = 0 \qquad \Leftrightarrow \qquad \Box_g \tilde{\phi}_k = \tilde{J}_k \tilde{\phi}_k + \varphi_k. \tag{45}$$

We obtain from S_{loc} the Einstein equations as well:

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu}(\phi_0) - \Lambda g_{\mu\nu}\right), \tag{46}$$

where ϕ_0 is given by (23) and $T_{\mu\nu}(\phi_0)$ can be calculated by (38).

Any solutions of system (44)–(46) are particular solutions of the initial nonlocal system (13)–(14).

To clarify physical interpretation of local fields $\tilde{\phi}_k$ and φ_k, we diagonalize the kinetic terms of these scalar fields in S_{loc}.

Expressing $\tilde{\phi}_k$ and φ_k in terms of new fields:

$$\tilde{\phi}_k = \frac{1}{2\mathcal{F}''(\tilde{J}_k)}\left(\left(\mathcal{F}''(\tilde{J}_k) - \frac{1}{3}\mathcal{F}'''(\tilde{J}_k)\right)\xi_k - \left(\mathcal{F}''(\tilde{J}_k) + \frac{1}{3}\mathcal{F}'''(\tilde{J}_k)\right)\chi_k\right),$$

$$\varphi_k = \xi_k + \chi_k,$$

we obtain the corresponding $\tilde{S}_k$ in the following form:

$$\tilde{S}_k = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\frac{\mathcal{F}''(\tilde{J}_k)}{4}\Bigl(\partial_\mu\xi_k\partial_\nu\xi_k - \partial_\nu\chi_k\partial_\mu\chi_k\Bigr) + \right.$$
$$+\frac{\tilde{J}_k}{4}\left((\mathcal{F}''(\tilde{J}_k) - \frac{1}{3}\mathcal{F}'''(\tilde{J}_k))\xi_k - (\mathcal{F}''(\tilde{J}_k) + \frac{1}{3}\mathcal{F}'''(\tilde{J}_k))\chi_k\right)(\xi_k + \chi_k) +$$
$$\left. + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)(\xi_k + \chi_k)^2\right).$$

For a quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$ ***there exists the following algorithm of localization*:**

- **Change values of** f_0 **and** Λ **such that the potential takes the form** $V(\phi) = C_1\phi$**.**
- **Find roots of the function** $\mathcal{F}(J)$ **and calculate orders of them.**
- **Select an finite number of simple and double roots.**
- **Construct the corresponding local action. In the case** $C_1 = 0$ **one should use formula (43).**
- **Vary the obtained local action and get a system of the Einstein equations and equations of motion.**
- **Seek solutions of the obtained local system.**

Conclusions 1

We have studied the SFT inspired nonlocal models with quadratic potentials and obtained:

- **The Ostrogradski representations for nonlocal Lagrangians in an arbitrary metric.**
- **The algorithm of localization.**
- **Local and nonlocal Einstein equations have one and the same solutions.**
- **Nonlocality arises in the case of $\mathcal{F}(\Box_g)$ with an infinite number of roots.**
- **One system of nonlocal Einstein equations $\Leftrightarrow$ Infinity number of systems of local Einstein equations.**

SOLUTIONS FOR EQUATIONS OF MOTION

(S.V. arXiv:1005.5007)

Let us consider nonlocal Klein–Gordon equation in the case of an arbitrary potential:

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \tag{47}$$

where prime is a derivative with respect to ϕ. A particular solution of (47) is a solution of the following system:

$$\sum_{n=0}^{N-1} f_n \Box_g^{\,n}\phi = V'(\phi) - C, \qquad f_N \Box_g^{\,N}\phi = C, \tag{48}$$

where $N-1$ is a natural number, C is an arbitrary constant.

In the case $f_1 \neq 0$ we can choose $N = 2$.

In the spatially flat FRW metric with the interval:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{49}$$

where $a(t)$ is the scale factor, we obtain from (48):

$$f_1 \Box_g \phi = -f_1\left(\ddot{\phi} + 3H\dot{\phi}\right) = V'(\phi) - f_0\phi - C, \qquad f_2 \Box_g^2 \phi = C. \tag{50}$$

The Hubble parameter

$$H = -\frac{1}{3\dot{\phi}}\left(\ddot{\phi} + \tilde{V}'(\phi) - \frac{C}{f_1}\right), \tag{51}$$

where

$$\tilde{V}'(\phi) \equiv \frac{1}{f_1}\left(V'(\phi) - f_0\phi\right). \tag{52}$$

Equation

$$(\partial_t^2 + 3H\partial_t)\left(\ddot{\phi} + 3H\dot{\phi}\right) = \frac{C}{f_2}, \tag{53}$$

is as follows

$$(\partial_t^2 + 3H\partial_t)\tilde{V}' = \tilde{V}'''\dot{\phi}^2 + \tilde{V}''(\ddot{\phi} + 3H\dot{\phi}) = -\frac{C}{f_2}. \tag{54}$$

We eliminate H and obtain

$$\dot{\phi}^2 = \frac{1}{\tilde{V}'''}\left(\tilde{V}''\tilde{V}' - \frac{C}{f_1}\tilde{V}'' - \frac{C}{f_2}\right). \qquad (55)$$

The obtained equation can be solved in quadratures. Its general solution depend on two arbitrary parameters C and t_0, which corresponds to the time shift.

It allows to find solutions for an arbitrary potential $V(\phi)$, with the exception of linear and quadratic potentials.

Note that we do not consider other Einstein equations. In distinguish to the localization method, which allows to localize all Einstein equations, this method solves only the field equation, whereas the obtained solutions maybe do not solve other equations.

The adding of other type of matter can give an exact solution of the system of all Einstein equations.

CUBIC POTENTIAL

The case of cubic potential is is connected with the bosonic string field theory:

$$V(\phi) = B_3\phi^3 + B_2\phi^2 + B_1\phi + B_0, \tag{56}$$

where B_0**,** B_1**,** B_2**, and** B_3 **are arbitrary constants, but** $B_3 \neq 0$**. For this potential we get (55) in the following form**

$$\dot{\phi}^2 = 4C_3\phi^3 + 6C_2\phi^2 + 4C_1\phi + C_0, \tag{57}$$

$$C_0 = \frac{(B_1 - C)(2B_2 - f_0)}{6f_1B_3} - \frac{Cf_1^2}{6f_1f_2B_3}, \qquad C_2 = \frac{2B_2 - f_0}{4f_1}, \tag{58}$$

$$C_1 = \frac{6B_3(B_1 - C) + (2B_2 - f_0)^2}{24f_1B_3}, \qquad C_3 = \frac{3B_3}{4f_1}. \tag{59}$$

Note, that $C_3 \neq 0$ **since** $B_3 \neq 0$**. Using the transformation**

$$\phi = \frac{1}{2C_3}(2\xi - C_2), \quad \Rightarrow \quad \dot{\xi}^2 = 4\xi^3 - g_2\xi - g_3, \tag{60}$$

where

$$g_2 = \frac{(2B_2 - f_0)^2 - 12B_3(B_1 - C)}{16f_1^2}, \quad g_3 = -\frac{3B_3C}{32f_2f_1}.$$

A solution of equation (60) is the Weierstrass elliptic function

$$\xi(t) = \wp(t - t_0, g_2, g_3) \tag{61}$$

or a degenerate elliptic function.

Let us consider degenerated cases. At $g_2 = 0$ **and** $g_3 = 0$

$$\phi_1 = \frac{4f_1}{3B_3(t - t_0)^2} - \frac{2B_2 - f_0}{6B_3}, \qquad H_1 = \frac{5}{3(t - t_0)}. \tag{62}$$

We have also obtained a bounded solution, which tends to a finite limit at $t \to \infty$**:**

$$\phi_2 = D_2 \tanh(\beta(t - t_0))^2 + D_0, \tag{63}$$

$$D_2 = \frac{4}{3B_3} f_1 \beta^2, \qquad D_0 = \frac{1}{18B_3}\left(3(f_0 - 2B_2) - 16 f_1 \beta^2\right), \tag{64}$$

where β **is a root of the following equation**

$$1024 f_2 f_1 \beta^6 + 576 f_1^2 \beta^4 + 324 B_3 B_1 - 27(2B_2 - f_0)^2 = 0. \tag{65}$$

The solution ϕ_2 **exists at**

$$C = \frac{1}{36B_3}\left(64 f_1^2 \beta^4 - 3(2B_2 - f_0)^2 + 36 B_3 B_1\right). \tag{66}$$

Cosmological model with a nonlocal scalar field and a k-essence field

Let us consider the k-essence cosmological model with a non-local scalar field:

$$S_2=\int d^4x\sqrt{-g}\alpha'\left(\frac{R}{16\pi G_N}+\frac{1}{g_o^2}\left(\frac{1}{2}\phi\mathcal{F}(\Box_g)\phi-V(\phi)\right)-\mathcal{P}(\Psi,X)-\Lambda\right), \tag{67}$$

where

$$X\equiv\,-g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi. \tag{68}$$

In the FRW metric $X=\dot{\Psi}^2$.

The standard variant of the k-essence field Lagrangian

$$\mathcal{P}(\Psi,X)=\frac{1}{2}(p_q(\Psi)-\varrho_q(\Psi))+\frac{1}{2}(p_q(\Psi)+\varrho_q(\Psi))X+\frac{1}{2}M^4(\Psi)(X-1)^2. \tag{69}$$

Here $p_q(\Phi)$, $\varrho_q(\Phi)$, and $M^4(\Phi)$ are arbitrary differentiable functions. The energy density is

$$\mathcal{E}(\Psi,X)=(p_q(\Psi)+\varrho_q(\Psi))X+2M^4(\Psi)(X^2-X)-\mathcal{P}(\Psi,X). \tag{70}$$

The Einstein equations are

$$3H^2 = 8\pi G_N(\varrho + \mathcal{E} + \Lambda), \quad (71)$$

$$2\dot{H} + 3H^2 = -8\pi G_N(p + \mathcal{P} - \Lambda). \quad (72)$$

From S_2 we also have equation

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \quad (73)$$

and

$$\dot{\mathcal{E}} = -3H(\mathcal{E} + \mathcal{P}). \quad (74)$$

A k-essence model (without an additional field) has one important property. For any real differentiable function $H_0(t)$, there exist such real differentiable functions $\varrho_q(\Phi)$ and $p_q(\Phi)$ that the functions $H_0(t)$ and $\Psi(t) = t$ are a particular solution for the system of the Einstein equations.

This property can be generalized on the model with the action S_2.

If $\Psi(t) = t$, then

$$\mathcal{E} = \varrho_q(\Psi) = \varrho_q(t), \qquad \mathcal{P} = p_q(\Psi) = p_q(t). \tag{75}$$

Substituting ϱ_q and p_q in (71)–(74), we get

$$\varrho_q(\Psi) = \varrho_q(t) = \frac{3}{8\pi G_N} H_0^2(t) - \varrho(t) - \Lambda, \tag{76}$$

$$p_q(\Psi) = p_q(t) = \, - \varrho_q(t) - \varrho(t) - p(t) - \frac{1}{4\pi G_N}\dot{H}(t). \tag{77}$$

Using $f_2\Box_g^2\phi_2 = C$, one can get the energy–momentum tensor for $\phi = \phi_2$:

$$E_{\mu\nu}(\phi_2) = \frac{1}{2}\left(f_1\partial_\mu\phi\partial_\nu\phi + f_2(\partial_\mu\Box_g\phi\partial_\nu\phi + \partial_\mu\phi\partial_\nu\Box_g\phi) + f_3\partial_\mu\Box_g\phi\partial_\nu\Box_g\phi\right),$$

$$W(\phi_2) = \frac{1}{2}\left(f_2\Box_g\phi^2 + 2\frac{f_3C}{f_2}\Box_g\phi + \frac{f_4C^2}{f_2^2}\right) - \frac{f_0}{2}\phi^2 + V(\phi).$$

In the FRW metric

$$\varrho = E_{00} + W, \qquad p = E_{00} - W. \tag{78}$$

Conclusions 2

So, we can propose the following algorithm to construct exact solvable k-essence cosmological models with a nonlocal scalar fields and an arbitrary $V(\phi)$, except linear and quadratic potentials:

- **For given potential $V(\phi)$ find $H(t)$ and $\phi(t)$ as a particular solution for**

$$\mathcal{F}(\Box_g)\phi = V'(\phi). \tag{79}$$

- **Calculate p and ϱ for the obtained solution.**
- **Add k-essence field in the action.**
- **Using the Einstein equations, calculate $\varrho_q(\Psi)$ and $p_q(\Psi)$. The exact solution corresponds to $\Psi(t) = t$.**