Renormdynamics, multiparticle production, negative binomial distribution and Riemann zeta function

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In the Universe, matter has manly two geometric structures, homogeneous, [Weinberg,1972] and hierarchical, [Okun, 1982].

The homogeneous structures are naturally described by real numbers with an infinite number of digits in the fractional part and usual archimedean metrics. The hierarchical structures are described with p-adic numbers with an infinite number of digits in the integer part and non-archimedean metrics, [Koblitz, 1977].

A discrete, finite, regularized, version of the homogenous structures are homogeneous lattices with constant steps and distance rising as arithmetic progression. The discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.

There is an opinion that present day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [Collins, 1984].
If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics.
The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).
We can invent scale variable λ and consider QFT on $D+1+1$ dimensional space-time-scale. For the scale variable $\lambda \in (0,1]$ it is natural to consider q-discretization, $0 < q < 1,\ \lambda_n = q^n,\ n = 0, 1, 2,$ ... and p - adic, nonarchimedian metric, with $q^{-1} = p$ - prime integer number.
The field variable $\varphi(x,t,\lambda)$ is complex function of the real, x, t, and p - adic, λ, variables. The solution of the UV renormdynamic problem means, to find evolution from finite to small scales with respect to the scale time $\tau = \ln \lambda/\lambda_0 \in (0, -\infty)$. Solution of the IR renormdynamic problem means to find evolution from finite to the large scales, $\tau = \ln \lambda/\lambda_0 \in (0, \infty)$.

This evolution is determined by Renormdynamic motion equations with respect to the scale-time.
As a concrete model, we take a relativistic scalar field model with lagrangian (see e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n,\ \mu = 0, 1, ..., D-1 \qquad (1)$$

The mass dimension of the coupling constant is

$$[g] = d_g = D - n\frac{D-2}{2} = D + n - \frac{nD}{2}. \qquad (2)$$

In the case

$$n = \frac{2D}{D-2} = 2 + \frac{4}{D-2} = 2 + \epsilon(D)$$
$$D = \frac{2n}{n-2} = 2 + \frac{4}{n-2} = 2 + \epsilon(n) \qquad (3)$$

the coupling constant g is dimensionless, and the model is renormalizable.
We take the euklidean form of the QFT which unifies quantum and statistical physics problems. In the case of the QFT, we can return (in)to minkowsky space by transformation: $p_D = ip_0,\ x_D = -ix_0$.

The main objects of the theory are Green functions - correlation functions - correlators,

$$G_m(x_1, x_2, ..., x_m) =< \varphi(x_1)\varphi(x_2)...\varphi(x_m) >$$
$$= Z_0^{-1} \int d\varphi(x)\varphi(x_1)\varphi(x_2)...\varphi(x_m)e^{-S(\varphi)} \quad (4)$$

where $d\varphi$ is an invariant measure,

$$d(\varphi + a) = d\varphi. \quad (5)$$

For gaussian actions,

$$S = S_2 = \frac{1}{2} \int dxdy\phi(x)A(x,y)\phi(y) = \varphi \cdot A \cdot \varphi \quad (6)$$

the QFT is solvable,

$$G_m(x_1, ..., x_m) = \frac{\delta^m}{\delta J(x_1)...J(x_m)} lnZ_J|_{J=0},$$
$$Z_J = \int d\varphi e^{-S_2 + J\cdot\varphi} = \exp(\frac{1}{2}\int dxdyJ(x)A^{-1}(x,y)J(y))$$
$$= \exp(\frac{1}{2}J \cdot A^{-1} \cdot J) \quad (7)$$

Nontrivial problem is to calculate correlators for non gaussian QFT.

Perturbative series have the following qualitative form

$$f(g) = f_0 + f_1 g + \dots + f_n g^n + \dots, \; f_n = n!P(n)$$
$$f(x) = \sum_{n \geq 0} P(n) n! x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \; \delta = x\frac{d}{dx} \quad (8)$$

In usual sense these series are divergent, but with proper nomalization of the expansion parametre g, the coefficients of the series are rational numbers and if experimental dates indicates for some rational value for g, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0\dots} \quad (9)$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \; f_n = n!P(n), \; p = 137,$$
$$|f|_p \leq \sum |f_n|_p p^n \quad (10)$$

In the Youkava theory of strong interections (see e.g. [Bogoliubov,1959]), we take $g = 13$,

$$f(g) = \sum f_n p^n, \ f_n = n!P(n), \ p = 13, \\ |f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1 - p^{-1}} \tag{11}$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity. Note also, that the inverse coupling expansions, e.g. in lattice(gauge) theories,

$$f(\beta) = \sum r_n \beta^n, \tag{12}$$

are also p-adically convergent for $\beta = p^k$. We can take the following scenery. We fix coupling constants and masses, e.g in QED or QCD, in low order perturbative expansions. Than put the models on lattice and calculate observable quantities as inverse coupling expansions, e.g.

$$f(\alpha) = \sum r_n \alpha^{-n}, \\ \alpha_{QED}(0) = 1/137; \ \alpha_{QCD}(m_Z) = 0.11... = 1/3^2 \tag{13}$$

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [Gross,Wilczek,1973, Politzer,1973, 't Hooft,1972].
The MS-scheme ['t Hooft,1972] belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge

$$\begin{aligned}
L &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n \\
&-\frac{1}{2\xi}(\partial A) + \partial^\mu \bar{c}^a(\partial_\mu c^a + g f^{abc}A^b_\mu c^c) \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\
(D_\mu)_{kl} &= \delta_{kl}\partial_\mu - i g t^a_{kl} A^a_\mu,
\end{aligned} \tag{14}$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; c^a are ghost fields; ξ is gauge parameter; t^a are generators of fundamental representation and f^{abc} are structure constants of the Lie algebra

$$[t^a, t^b] = i f^{abc} t^c, \tag{15}$$

we will consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.

The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = -\beta_2 a^2 - \beta_3 a^3 - \beta_4 a^4 - \beta_5 a^5 + O(a^6),$$
$$a = \alpha_s/\pi = \frac{g^2}{4\pi^2},\ g(t), t = \mu^2,$$
$$\int_{a_0}^{a} \frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu}{\mu_0}, \qquad (16)$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme. To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge.

The expression of the β-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),$$
$$\beta(a) = a\frac{d}{da}(aZ_{(1)}) \tag{17}$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a) \tag{18}$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in Z expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \tag{19}$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = -\gamma_1 a - \gamma_2 a^2 - \gamma_3 a^3 - \gamma_4 a^4 + O(a^5),$$
$$b = \ln m_q,$$
$$b(t) = b_0 + \int_{t_0}^{t} dt \gamma(a(t)) = b_0 + \int_{a_0}^{a} da \gamma(a)/\beta(a). \tag{20}$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant Z_m of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot})$$
$$\Rightarrow \gamma(a) = -\frac{d \ln Z_m}{dt}$$
$$= -\frac{d \ln Z_m}{da}\frac{da}{dt} = -\frac{d \ln Z_m}{da}(-\varepsilon a + \beta(a)) = a\frac{dZ_m^{(1)}}{da}, \tag{21}$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \tag{22}$$

and $Z_m^{(1)}$ is the coefficient of the first pole in the $\varepsilon-$expantion of the Z_m in MS-scheme

$$Z_m(\varepsilon, g) = 1 + \frac{Z_m^{(1)}(g)}{\varepsilon} + \frac{Z_m^{(2)}(g)}{\varepsilon^2} + ... \quad (23)$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \tag{24}$$

can be reparametrized,

$$\begin{aligned}
&a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... \\
&\dot{A} = b_1 A + b_2 A^2 + ..., \\
&(b_1 A + b_2 A^2 + ...)(1 + 2 f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) \\
&+ \beta_2 (A^2 + 2 f_2 A^3 + ...) + ... + \beta_n (A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ...
\end{aligned} \tag{25}$$

$$\begin{aligned}
&b_1 = \beta_1, \\
&b_2 = \beta_2 + f_2 \beta_1 - 2 f_2 b_1 = \beta_2 - f_2 \beta_1, \\
&b_3 = \beta_3 + 2 f_2 \beta_2 + f_3 \beta_1 - 2 f_2 b_2 - 3 f_3 b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1, ... \\
&b_n = \beta_n + ... + \beta_1 f_n - 2 f_2 b_{n-1} - ... - n f_n b_1 \\
&= \beta_n + ... + (1-n)\beta_1 f_n - 2 f_2 b_{n-1} - ... - (n-1) f_{n-1} b_2
\end{aligned} \tag{26}$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1}, \; f_3 = \frac{\beta_3}{2\beta_1} + f_2^2, \; \ldots, f_n = \frac{\beta_n + \ldots}{(n-1)\beta_1}, \ldots \tag{27}$$

In this case we have exact classical dynamics in the (external) space-time and simple scale dynamics,

$$\begin{aligned} &g = (\mu/\mu_0)^{-2\varepsilon} g_0 = e^{-2\varepsilon\tau} g_0; \\ &\varphi(\tau, t, x) = e^{-(D-2)/2\tau} \varphi_0(t, x), \\ &\psi(\tau, t, x) = e^{-(D-1)/2\tau} \psi_0(t, x) \end{aligned} \tag{28}$$

We will consider in applications the case when only one of higher coefficient is nonzero.
In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3. If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [Kazakov,Shirkov,1980]) than we can construct reparametrization function (25) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [Faddeev, Takhtajan, 1987]).

In the case of several integrals of motion, $H_n,\ 1 \leq n \leq N$, we can formulate Renormdynamics as Nambu - Poisson dynamics (see e.g. [Makhaldiani,2007])

$$\dot{\varphi}(x) = [\varphi(x), H_1, H_2, ..., H_N], \tag{29}$$

where φ is an observable as a function of the coupling constants $x_m,\ 1 \leq m \leq M$.
In the case of Standard model [Weinberg,1995], we have three coupling constants, $M = 3$.

The renormdynamic motion equations

$$\dot{g}_n = \beta_n(g),\ 1 \le n \le N \tag{30}$$

where $g_n,\ 1 \le n \le N$, are coupling constants, can be presented as nonlinear part of a hamiltonian system with linear part

$$\dot{\Psi}_n = -\frac{\partial \beta_m}{\partial g_n} \Psi_m, \tag{31}$$

hamiltonian and canonical Poisson bracket as

$$H = \sum_{n=1}^{N} \beta(g)_n \Psi_n,\ \{g_n, \Psi_m\} = \delta_{nm} \tag{32}$$

In this extended version, we can define optimal control theory approach [Pontryagin, 1983] to the unified field theories. We can start from the unified value of the coupling constant, e.g. $\alpha^{-1}(M) = 29.0...$ at the scale of unification M, put the aim to reach the SM scale with values of the coupling constants measured in experiments, and find optimal threshold corrections to the RD coefficients.

The fundamental quark and gluon degrees of freedom are the relevant ones at high temperatures and/or densities. Since these degrees of freedom are confined in the low temperature and density regime there must be a quark and/or gluon (de)confinement phase transition.
It is difficult to describe the phase transition because there is not known a local parameter which can be linked to confinement. We consider the fractal dimension of the hadronic/quark-gluon space as order parameter of (de)confinement phase transition. It has value less than 3 in the abelian, hadronic, phase, and more than 3, in nonabelian, quark-gluon, phase.

Let us consider $l-$particle semi-inclusive distribution

$$F_l(n,q) = \frac{d^l\sigma_n}{\bar{d}q_1 ... \bar{d}q_l} = \frac{1}{n!}\int \prod_{i=1}^{n} \bar{d}q_i'\delta(p_1 + p_2 - \Sigma_{i=1}^{l} q_i - \Sigma_{i=1}^{n} q_i')$$
$$\cdot |M_{n+l+2}(p_1, p_2, q_1, ..., q_l, q_1', ..., q_n'; g(\mu), m(\mu)), \mu)|^2,$$
$$\bar{d}p \equiv \frac{d^3p}{E(p)},\ E(p) = \sqrt{p^2 + m^2}. \quad (33)$$

From the renormdynamic equation

$$DM_{n+l+2} = \frac{\gamma}{2}(n+l+2)M_{n+l+2}, \tag{34}$$

we obtain

$$\begin{aligned}
&DF_l(n,q) = \gamma(n+l+2)F_l(n,q),\\
&DF_l(q) = \gamma(< n > +l+2)F_l(q),\\
&D < n^k(q) >= \gamma(< n^{k+1}(q) > - < n^k(q) >< n(q) >),\\
&DC_k = \gamma < n(q) > (C_{k+1} - C_k(1 + k(C_2 - 1)))\\
&F_l(q) \equiv \frac{d^l\sigma}{\bar{d}q_1 \ldots \bar{d}q_l} = \sum_n \frac{d^l\sigma_n}{\bar{d}q_1 \ldots \bar{d}q_l},\ < n^k(q) >= \frac{\sum_n n^k d^l\sigma_n/\bar{d}q^l}{\sum_n d^l\sigma_n/\bar{d}q^l}\\
&C_k = \frac{< n^k(q) >}{< n(q) >^k}
\end{aligned} \tag{35}$$

From dimensional considerations, the following combination of cross sections [Koba et al, 1972] must be universal function

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \tag{36}$$

Corresponding relation for the inclusive cross sections is [Matveev et al, 1976].

$$< n(p) > \frac{d\sigma_n}{\bar{d}p} / \frac{d\sigma}{\bar{d}p} = \Psi(\frac{n}{< n(p) >}). \tag{37}$$

Indeed, let us define $n-$dimension of observables [Makhaldiani, 1980]

$$[n] = 1,\ [\sigma_n] = -1, \sigma = \Sigma_n \sigma_n,\ [\sigma] = 0,\ [< n >] = 1. \tag{38}$$

The following expression does not depend on any dimensional quantities and must have a corresponding universal form

$$P_n =< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \tag{39}$$

Let us find an explicit form of the universal functions using renormdynamic equations.

From the definition of the moments we have

$$C_k = \int_0^{\infty} dx x^k \Psi(x), \tag{40}$$

so they are universal parameters,

$$DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow$$
$$C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \tag{41}$$

Now we can invert momentum transform and find (see [Makhaldiani, 1980] and appendix) universal functions [Ernst, Schmit, 1976], [Darbaidze et al, 1978].

$$\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz},$$
$$C_2 = 1 + \frac{1}{c} \tag{42}$$

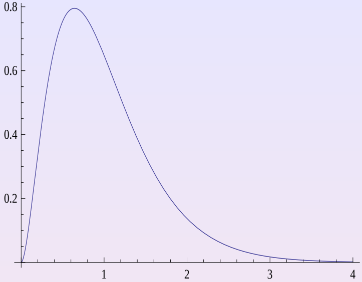

Figure: KNO distribution (42), $\Psi(z)$, with $c = 2.8$

The value of the parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= A < n >,$$
$$A = \frac{1}{\sqrt{c}} \simeq 0.6, \;\; c = 2.8;$$
$$(c = 3, \;\; A = 5.8) \tag{43}$$

which is in accordance with $n-$dimension counting.

We can calculate also $1/ < n >$ correction to the scaling function (see appendix)

$$< n > \frac{\sigma_n}{\sigma} = \Psi = \Psi_0(\frac{n}{< n >}) + \frac{1}{< n >}\Psi_1(\frac{n}{< n >}),$$
$$C_k = C_k^0 + \frac{1}{< n >}C_k^1,$$
$$C_k^0 = \int_0^\infty dx x^k \Psi_0(x),\ C_k^1 = \int_0^\infty dx x^k \Psi_1(x),$$
$$\Psi_1(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n^1 = \frac{C_2^1 c^2}{2}(z - 2 + \frac{c-1}{cz})\Psi_0 \quad (44)$$

The characteristic function we define as

$$\Phi(t) = \int_0^\infty dx e^{tx}\Psi(x) = (1 - t/c)^{-c},\ Re(t) < c \tag{45}$$

For the moments of the distribution, we have

$$\Phi^{(k)}(0) = C_k = (-c)(-c-1)...(-c-k+1)(-1/c)^k = \frac{\Gamma(c+k)}{\Gamma(c)c^k} \tag{46}$$

Note that it is an infinitely divisible characteristic function, i.e.

$$\Phi(t) = (\Phi_n(t))^n,\ \Phi_n(t) = (1 - t/c)^{-c/n} \tag{47}$$

If we calculate observable(mean) value of x, we find

$$\begin{aligned} &< x >= \Phi'(0) = n\Phi(0)_n{}' = n < x >_n, \\ &< x >_n = \frac{< x >}{n} \end{aligned} \tag{48}$$

For the second moment and dispersion, we have

$$\begin{aligned}
&< x^2 >= \Phi^{(2)}(0) = n < x^2 >_n + n(n-1) < x >_n^2,\\
&\mathrm{D}^2 = < x^2 > - < x >^2 = n(< x^2 >_n - < x >_n^2) = n\mathrm{D}_n^2\\
&\mathrm{D}_n^2 = \frac{\mathrm{D}^2}{n} = \frac{\mathrm{D}^2}{< x >} < x >_n
\end{aligned} \tag{49}$$

In a sense, any Hamiltonian quantum (and classical) system can be described by infinitely divisible distributions, because in the functional integral formulation, we use the following step

$$U(t) = e^{-itH} = (e^{-i\frac{t}{N}H})^N \tag{50}$$

In the case of our scalar field theory (1),

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n = g^{\frac{2}{2-n}}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{n}\phi^n), \quad (51)$$

so, to the constituent field ϕ_N corresponds higher value of the coupling constant,

$$g_N = gN^{\frac{n-2}{2}} \quad (52)$$

For weak nonlinearity, $n = 2 + 2\varepsilon, d = 2/\varepsilon + 2, g_N = g(1 + \varepsilon \ln N + O(\varepsilon^2))$

Closed equation of renormdynamics for the generating function of the observables

Let us consider a generating function of the topological crossections

$$F(h,g,m,\mu)=\Sigma_{n\geq 2}h^n\sigma_n,$$
$$\sigma_n=\frac{1}{n!}\frac{d^n}{dh^n}F|_{h=0},$$
$$\sigma=F|_{h=1},\quad <n>=\frac{d}{dh}\ln F|_{h=1},... \tag{53}$$

It is natural that for the generating function we have closed renormdynamic equation [Makhaldiani, 1980]

$$(D-\gamma(\frac{h\partial}{\partial h}+2))F=0,$$
$$F(h(\mu),g(\mu),m(\mu),\mu)=F(h(\bar\mu),g(\bar\mu),m(\bar\mu),\bar\mu)\exp(2\int_{\bar\mu}^{\mu}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar h=\bar h(\bar\mu)=h(\mu)\exp(\int_{\mu}^{\bar\mu}\frac{d\rho}{\rho}\gamma(g(\rho))),$$
$$\bar m=\bar m(\bar\mu)=m(\mu)\exp(\int_{\mu}^{\bar\mu}\frac{d\rho}{\rho}\eta(g(\rho))),\ \int_g^{\bar g}\frac{dg}{\beta(g)}=\ln\frac{\bar\mu}{\mu} \tag{54}$$

Negative binomial distribution (NBD) is defined as

$$P(n) = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n \geq 0} P(n) = 1, \tag{55}$$

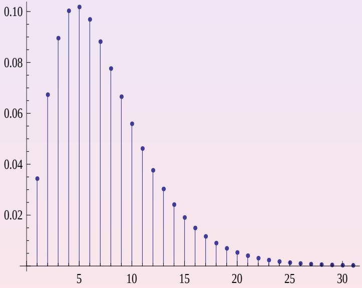

Figure: $P(n), (55), r = 2.8, p = 0.3, < n >= 6$

NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [ALICE,2010].

It is interesting to understand how NBD fits such a different reactions?

Let us consider NBD for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1+\frac{k}{<n>})^{-n}(1+\frac{<n>}{k})^{-k}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{<n>}{<n>+k})^n(\frac{k}{k+<n>})^k,$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}},$$
$$r = k > 0,\ p = \frac{<n>}{<n>+k}. \tag{56}$$

The generating function for NBD is

$$F(h) = (1+\frac{<n>}{k}(1-h))^{-k} = (1+\frac{<n>}{k})^{-k}(1-ah))^{-k},$$
$$a = p = \frac{<n>}{<n>+k}. \tag{57}$$

Indeed,

$$
\begin{aligned}
(1-ah))^{-k} &= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1}e^{-t(1-ah)} \\
&= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1}e^{-t}\sum_0^\infty \frac{(tah)^n}{n!} \\
&= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!}h^n, \\
P(n) &= (1+\frac{<n>}{k})^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}(\frac{<n>}{<n>+k})^n \\
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n>+k)^{-(n+k)}<n>^n \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)} \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)} \qquad (58)
\end{aligned}
$$

Note that KNO characteristic function (45) coincides with the NBD generating function (57) when $t =< n > (h-1),\ c = k.$
The Bose-Einstein distribution is a special case of NBD with $k = 1$.

If k is negative, the NBD becomes a positive binomial distribution, narrower than Poisson (corresponding to negative correlations).
For negative (integer) values of $k = -N$, we have Binomial GF

$$F_{bd} = (1 + \frac{< n >}{N}(h-1))^N = (a+bh)^N,\ a = 1 - \frac{< n >}{N}, b = \frac{< n >}{N},$$
$$P_{bd}(n) = C_N^n(\frac{< n >}{N})^n(1 - \frac{< n >}{N})^{N-n} \tag{59}$$

(In a sense) we have a (quantum) spectrum for the parameter k, which contains any (positive) real values and (with finite number of states) the negative integer values, $(0 \leq n \leq N)$

From the generating function we have

$$< n^2 >= (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} < n >^2 + < n >, \quad (60)$$

for dispersion we obtain

$$D = \sqrt{< n^2 > - < n >^2} = \frac{1}{\sqrt{k}} < n > (1 + \frac{k}{< n >})^{1/2}$$
$$= \frac{1}{\sqrt{k}} < n > + \frac{\sqrt{k}}{2} + O(1/ < n >), \quad (61)$$

so the dispersion low for KNO and NBD distributions are the same, with $c = k$, for high values of the mean multiplicity.
The factorial moments of NBD,

$$F_m = (\frac{d}{dh})^m F(h)|_{h=1} = \frac{< n(n-1)...(n-m+1) >}{< n >^m} = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \quad (62)$$

and usual normalized moments of KNO (46) coincides.

The KNO as asymptotic NBD

Let us show that NBD is a discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} <n> P_n|_{\frac{n}{<n>}=z} = \Psi(z) \qquad (63)$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1+\frac{1}{12x}+O(x^{-2})), \qquad (64)$$

we find

$$<n> P_n =<n> \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}}\frac{k^k}{n^k}<n> z^k e^{-k\frac{n+k}{<n>}}$$
$$= \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz} + O(1/<n>) \qquad (65)$$

We can calculate also $1/<n>$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz}(1+\frac{k^2}{2}(z-2+\frac{k-1}{kz})\frac{1}{<n>}) \qquad (66)$$

This form coincides with the corrected KNO (44) for $c=k$ and $C_2^1=1$.

We have seen that KNO characteristic function (45) and NBD GF (57) have almost same form. This relation become in coincidence if

$$c = k,\ t = (h-1)\frac{<n>}{k} \tag{67}$$

Now the definition of the characteristic function (45) can be read as

$$\int_0^\infty e^{-<n>z(1-h)}\Psi(z)dz = (1 + \frac{<n>}{k}(1-h))^{-k} \tag{68}$$

which means that Poisson GF weighted by KNO distribution gives NBD GF. Because of this, the NBD is the gamma-Poisson (mixture) distribution.

Fore high values of $x_2 = k$ the NBD distribution reduces to the Poisson distribution

$$F(x_1, x_2, h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \Rightarrow e^{-x_1(1-h)} = e^{-<n>} e^{h<n>}$$
$$= \sum P(n) h^n,$$
$$P(n) = e^{-<n>} \frac{< n >^n}{n!} \quad (69)$$

For the Poisson distribution

$$\frac{d^2 F(h)}{dh^2}|_{h=1} =< n(n-1) >=< n >^2,$$
$$D^2 =< n^2 > - < n >^2 =< n > . \quad (70)$$

In the case of NBD, we had the following dispersion low

$$D^2 = \frac{1}{k} < n >^2 + < n >, \quad (71)$$

which coincides withe previous expression for high values of k.
Poisson GF belongs to the class of the infinitely divisible distributions,

$$F(h, < n >) = (F(h, < n > /k))^k \quad (72)$$

For high values of $< n >$, the Poisson distribution reduces to the Gauss distribution

$$P(n) = e^{-<n>} \frac{< n >^n}{n!} = \frac{1}{\sqrt{2\pi < n >}} \exp(-\frac{(n- < n >)^2}{2 < n >}) \qquad (73)$$

For high values of k in the integral relation (68), in the KNO function dominates the value $z_c = 1$ and both sides of the relation reduce to the Poisson GF.

Multiplicative properties of KNO and NBD and corresponding motion equations

An useful property of the negative binomial distribution with parameters

$$< n >, k$$

is that it is (also) the distribution of a sum of k independent random variables drawn from a Bose-Einstein distribution[1] with mean $< n > /k$,

$$\begin{aligned}
&P_n = \frac{1}{< n > +1}(\frac{< n >}{< n > +1})^n \\
&= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \frac{\hbar\omega}{\ln\frac{< n >+1}{< n >}} \\
&\sum_{n\geq 0} P_n = 1,\ \sum nP_n =< n >= \frac{1}{e^{\beta\hbar\omega-1}},\ T \simeq \hbar\omega < n >,\ < n >\gg 1, \\
&P(x) = \sum_n x^n P_n = (1+ < n > (1-x))^{-1}.
\end{aligned} \tag{74}$$

[1] A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems.

This is easily seen from the generating function in (57), remembering that the generating function of a sum of independent random variables is the product of their generating functions.
Indeed, for

$$n = n_1 + n_2 + ... + n_k, \tag{75}$$

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1} ... p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \tag{76}$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, < n >$ produces a negative binomial distribution with parameters $Nk, N < n >$.

So, for the GF of NBD we have (N=2)

$$F(k,<n>)F(k,<n>) = F(2k, 2<n>) \quad (77)$$

And more general formula (N=m) is

$$F(k,<n>)^m = F(mk, m<n>) \quad (78)$$

We can put this equation in the closed nonlocal form

$$Q_q F = F^q, \quad (79)$$

where

$$Q_q = q^D, \quad D = \frac{kd}{dk} + \frac{<n>d}{d<n>} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2} \quad (80)$$

Note that temperature defined in (74) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega = 100MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 1.5$
We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

p-adic string amplitudes can be obtained as tree amplitudes of the field theory with the following lagrangian and motion equation (see e.g. [Brekke, Freund, 1993])

$$L = \frac{1}{2}\Phi Q_p \Phi - \frac{1}{p+1}\Phi^{p+1},$$

$$Q_p \Phi = \Phi^p,\ Q_p = p^D \tag{81}$$

$$D = -\frac{1}{2}\triangle,\ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}, \tag{82}$$

Φ - is real scalar field on D-dimensional space-time with coordinates $x = (x_0, x_1, ..., x_{D-1})$. We have trivial, $\Phi = 0$ and $\Phi = 1$, and following nontrivial solutions of the equation (81)

$$\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2\ln p}(x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)} \tag{83}$$

The equation (81) permits factorization of its solutions $\Phi(x) = \Phi(x_0)\Phi(x_1)...\Phi(x_{D-1})$, every factor of which fulfils one dimensional equation

$$p^{\varepsilon\partial_x^2}\Phi(x) = \Phi(x)^p,\ \varepsilon = \pm\frac{1}{2} \tag{84}$$

The trivial solution of the equations are $\Phi = 0$ and $\Phi = 1$. For nontrivial solution of (84), we have

$$p^{\varepsilon\partial_x^2}\Phi(x) = e^{a\partial^2}\Phi(x) = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{\infty} dy e^{-\frac{1}{4a}y^2+y\partial}\Phi(x)$$
$$= \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{\infty} dy e^{-\frac{1}{4a}y^2}\Phi(x+y) = \Phi(x)^p,\ a = \varepsilon\ln p \tag{85}$$

If we (de quantize) put, $p = q$, and take (classical) limit, $q \to 1$, the motion equation reduce to

$$\varepsilon\partial_x^2\Phi = \Phi\ln\Phi, \tag{86}$$

with solution

$$\Phi(x) = e^{\frac{1}{2}}e^{\frac{x^2}{4\varepsilon}}. \tag{87}$$

It is obvious that the anzac

$$\Phi = Ae^{bx^2}, \tag{88}$$

can pass the equation (85). Indeed, the solution is

$$\begin{aligned} &\Phi(x) = p^{\frac{1}{2(p-1)}} e^{\frac{1-p^{-1}}{4\varepsilon \ln p} x^2}, \\ &\Phi(x_0, x_1, ..., x_{D-1}) = p^{\frac{D}{2(p-1)}} e^{\frac{1-p^{-1}}{2\ln p}(x_0^2 - x_1^2 - x_2^2 - ... - x_{D-1}^2)} \end{aligned} \tag{89}$$

Now, we can define the following class of motion equations

$$Q_q F = F^q, \tag{90}$$

where

$$Q_q = q^D, \quad D = D_1(x_1) + ... + D_l(x_l), \tag{91}$$

$D_k(x)$ is some (differential) operator depending on x. In the case of the NBD GF,

$$D_k(x) = \frac{xd}{dx}. \tag{92}$$

For this (Qlike) class of equations, we have factorization property

$$F = F(x_1, ..., x_l) = F_1(x_1)...F_l(x_l),$$
$$q^{D_k(x)} F_k(x) = c_k F_k(x)^q, 1 \leq k \leq l, \; c_1 c_2 ... c_l = 1. \tag{93}$$

For NBD distribution we have corresponding multiplication(convolution)formulas

$$(P \star P)_n \equiv \sum_{m=0}^{n} P_m(k, <n>) P_{n-m}(k, <n>)$$
$$= P_n(2k, 2 <n>) = Q_2 P_n(k, <n>), ... \quad (94)$$

So, we can say, that star-product on the distributions of NBD corresponds ordinary product for GF.
It will be nice to have similar things for string field theory(SFT) [Kaku,2000].
SFT motion equation is

$$Q\Phi = \Phi \star \Phi \quad (95)$$

For stringfield GF F we may have

$$QF = F^2. \quad (96)$$

By construction we know the solution of the nice equation (79) as GF of NBD, F. We obtain corresponding differential equations, if we consider $q = 1 + \varepsilon$, for small ε,

$$
\begin{aligned}
&(D(D-1)...(D-m+1) - (lnF)^m)\Psi = 0,\\
&(\frac{\Gamma(D+1)}{\Gamma(D+1-m)} - (\ln F)^m)\Psi = 0,\\
&(D_m - \Phi^m)\Psi = 0, m = 1, 2, 3, ...\\
&D_m = \frac{\Gamma(D+1)}{\Gamma(D+1-m)}, \Phi = \ln F,
\end{aligned} \tag{97}
$$

with the solution $\Psi = F = \exp(\Phi)$. In the case of the NBD and p-adic string, we have correspondingly

$$
\begin{aligned}
&D = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2};\\
&D = -\frac{1}{2}\triangle,\ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}.
\end{aligned} \tag{98}
$$

These equations have meaning not only for integer m.

For high mean multiplicities we have corresponding equations for KNO

$$Q_2\Psi(z) = \Psi \star \Psi \equiv \int_0^z \Psi(t)\Psi(z-t)dt \tag{99}$$

Due to the explicit form of the operator D, these equations and corresponding solutions have the symmetry under the change of the variables

$$k \to ak, \; <n> \to b <n> . \tag{100}$$

When

$$a = \frac{<n>}{k}, \; b = \frac{k}{<n>}, \tag{101}$$

we obtain the symmetry with respect to the transformations $k \leftrightarrow <n>, x_1 \leftrightarrow x_2$.

The Riemann zeta function $\zeta(s)$ is defined for complex $s = \sigma + it$ and $\sigma > 1$ by the expansion

$$\zeta(s) = \sum_{n \geq 1} n^{-s}, \ Res > 1. \tag{102}$$

All complex zeros, $s = \alpha + i\beta$, of $\zeta(\sigma + it)$ function lie in the critical stripe $0 < \sigma < 1$, symmetrically with respect to the real axe and critical line $\sigma = 1/2$. So it is enough to investigate zeros with $\alpha \leq 1/2$ and $\beta > 0$. These zeros are of three type, with small, intermediate and big ordinates.

The Riemann hypothesis [Titchmarsh,1986] states that the (non-trivial) complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$.
At the beginning of the XX century Polya and Hilbert made a conjecture that the imaginary part of the Riemann zeros could be the oscillation frequencies of a physical system (ζ - (mem)brane).
After the advent of Quantum Mechanics, the Polya-Hilbert conjecture was formulated as the existence of a self-adjoint operator whose spectrum contains the imaginary part of the Riemann zeros.
The Riemann hypothesis (RH) is a central problem in Pure Mathematics due to its connection with Number theory and other branches of Mathematics and Physics.

The functional equation is (see e.g. [Titchmarsh,1986])

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(s) \tag{103}$$

From this equation we see the real (trivial) zeros of zeta function:

$$\zeta(-2n) = 0,\ n = 1, 2, ... \tag{104}$$

Also, at s=1, zeta has pole with reside 1.
From Field theory and statistical physics point of view, the functional equation (103) is duality relation, with self dual (or critical) line in the complex plane, at $s = 1/2 + i\beta$,

$$\zeta(\frac{1}{2} - i\beta) = \frac{2\Gamma(s)}{(2\pi)^s}\cos(\frac{\pi s}{2})\zeta(\frac{1}{2} + i\beta), \tag{105}$$

we see that complex zeros lie symmetrically with respect to the real axe.
On the critical line, (nontrivial) zeros of zeta corresponds to the infinite value of the free energy,

$$F = -T\ln\zeta. \tag{106}$$

At the point with $\beta = 14.134725...$ is located the first zero. In the interval $10 < \beta < 100$, zeta has 29 zeros. The first few million zeros have been computed and all lie on the critical line. It has been proved that uncountably many zeros lie on critical line.
The first relation of zeta function with prime numbers is given by the following formula,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \ Res > 1. \tag{107}$$

Another formula, which can be used on critical line, is

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n \geq 1} (-1)^{n+1} n^{-s}, \ Res > 0. \tag{108}$$

Let us consider the values $q = n, n = 1, 2, 3, ...$ and take sum of the corresponding equations (90), we find

$$\zeta(-D)F = \frac{F}{1-F} \tag{109}$$

In the case of the NBD we know the solutions of this equation.
Now we invent a Hamiltonian H with spectrum corresponding to the set of nontrivial zeros of the zeta function, in correspondence with Riemann hypothesis,

$$\begin{aligned}
&-D_n = \frac{n}{2} + iH_n,\ H_n = i(\frac{n}{2} + D_n),\\
&D_n = x_1\partial_1 + x_2\partial_2 + ... + x_n\partial_n,\ H_n^+ = H_n = \sum_{m=1}^{n} H_1(x_m),\\
&H_1 = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\ \hat{p} = -i\partial_x
\end{aligned} \tag{110}$$

The Hamiltonian $H = H_n$ is hermitian, its spectrum is real. The case $n = 1$ corresponds to the Riemann hypothesis.

The case $n = 2$, corresponds to NBD,

$$\zeta(1 + iH_2)F = \frac{F}{1 - F}, \ \zeta(1 + iH_2)|_F = \frac{1}{1 - F},$$
$$F(x_1, x_2; h) = (1 + \frac{x_1}{x_2}(1 - h))^{-x_2} \quad (111)$$

Let us scale $x_2 \to \lambda x_2$ and take $\lambda \to \infty$ in (111), we obtain

$$\zeta(\frac{1}{2} + iH_1(x))e^{-(1-h)x} = \frac{1}{e^{(1-h)x} - 1},$$
$$\frac{1}{\zeta(\frac{1}{2} + iH(x))} \frac{1}{e^{\varepsilon x} - 1} = e^{-\varepsilon x},$$
$$H(x) = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x), \ H^+ = H, \varepsilon = 1 - h. \quad (112)$$

Now we scale $x \to xy$, multiply the equation by y^{s-1} and integrate

$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \int_0^\infty dy e^{-\varepsilon xy}y^{s-1} = \frac{1}{(\varepsilon x)^s}\Gamma(s),$$

$$\frac{1}{\zeta(\frac{1}{2}+iH(x))}\int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \frac{1}{\zeta(\frac{1}{2}+iH(x))}x^{-s}\varepsilon^{-s}\Gamma(s)\zeta(s), \quad (113)$$

so

$$\zeta(\frac{1}{2}+iH(x))x^{-s} = \zeta(s)x^{-s} \Rightarrow H(x)\psi_E = E\psi_E,$$
$$\psi_E = cx^{-s},\ s = \frac{1}{2}+iE, \quad (114)$$

we have correct way and can return to the previous step (112) and take the following transformation

$$\frac{1}{e^{\varepsilon xy}-1} = \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} dE x^{-iE-1/2}\varphi(E,\varepsilon y),$$
$$\varphi(E,\varepsilon y) = \int_0^\infty dx \frac{x^{iE-\frac{1}{2}}}{e^{\varepsilon xy}-1} = \frac{\Gamma(\frac{1}{2}+iE)}{(\varepsilon y)^{iE+1/2}}\zeta(\frac{1}{2}+iE),$$
$$\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} dE x^{-iE-1/2}\varphi(E,\varepsilon y)\frac{1}{\zeta(1/2+iE)} = e^{-\varepsilon xy} \qquad (115)$$

If we take the following formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t - 1}, \qquad (116)$$

which says that ζ function is the Mellin transformation, we can find

$$\Gamma(1 + iH_2) \frac{F}{1 - F} = \int_0^\infty \frac{dt/t}{e^t - 1} F^{1/t}, \qquad (117)$$

or

$$\Gamma(1 + iH_2)\Phi = \int_0^\infty \frac{dt/t}{e^t - 1} (\frac{\Phi}{1 + \Phi})^{1/t},$$
$$\Phi = \frac{F}{1 - F} = \frac{1}{(1 + \frac{x_1}{x_2}(1 - h))^{x_2} - 1} \qquad (118)$$

We can obtain also the following equation with argument of ζ_N on critical axis

$$\zeta_N(\frac{1}{2}+iH_1(x_2))F(x_1,x_2,h)=\sum_{n=1}^{N}\frac{1}{(1+\frac{x_1}{nx_2}(1-h))^{nx_2}}$$
$$=\sum_{n=1}^{N}F(x_1,nx_2,h),$$
$$\zeta_N(\frac{1}{2}+iH_1(x_2))F(\lambda x_1,x_2,h)=\sum_{n=1}^{N}\frac{1}{(1+\frac{\lambda x_1}{nx_2}(1-h))^{nx_2}}$$
$$=\sum_{n=1}^{N}F(\lambda x_1,nx_2,h)\simeq Ne^{-\lambda(1-h)x_1},N\gg 1. \quad (119)$$

Let us calculate next therm in the $1/\lambda$ expansion in the (111)

$$F(x_1, \lambda x_2, h) = (1 + \frac{\varepsilon x_1}{\lambda x_2})^{-\lambda x_2} = e^{-\lambda x_2 \ln(1+\varepsilon \frac{x_1}{\lambda x_2})}$$
$$= e^{-\varepsilon x_1} e^{\frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})} = e^{-\varepsilon x_1}(1 + \frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})),$$
$$(F^{-1} - 1)^{-1} = (e^{\lambda x_2 \ln(1+\varepsilon \frac{x_1}{\lambda x_2})} - 1)^{-1}$$
$$= \frac{1}{e^{\varepsilon x_1} - 1}(1 + \frac{e^{\varepsilon x_1}}{e^{\varepsilon x_1} - 1} \frac{(\varepsilon x_1)^2}{2\lambda x_2} + O(\lambda^{-2})) \qquad (120)$$

The zero order term, λ^0 we already considered. The next, λ^{-1} order term gives the following relations

$$\zeta(-\delta_1 - \delta_2)\frac{x_1^2}{x_2} e^{-\varepsilon x_1} = \frac{1}{x_2}\zeta(1 - \delta_1) x_1^2 e^{-\varepsilon x_1} = \frac{x_1^2 e^{\varepsilon x_1}}{x_2 (e^{\varepsilon x_1} - 1)^2},$$
$$\zeta(1 - \delta) x^2 e^{-\varepsilon x} = \frac{x^2 e^{\varepsilon x}}{(e^{\varepsilon x} - 1)^2} = x^2 e^{-\varepsilon x} + O(e^{-2\varepsilon x})$$
$$\zeta(1 - \delta)\Psi = E\Psi + O(e^{-2\varepsilon x}), \Psi = x^2 e^{-\varepsilon x}, E = 1. \qquad (121)$$

There have been a number of approaches to understanding the Riemann hypothesis based on physics (for a comprehensive list see [Watkins]) According to the idea of Berry and Keating, [Berry,Keating,1997] the real solutions E_n of

$$\zeta(\frac{1}{2} + iE_n) = 0, \tag{122}$$

are energy levels, eigenvalues of a quantum Hermitian operator (the Riemann operator) associated with the one-dimensional classical hyperbolic Hamiltonian

$$H_c = xp, \tag{123}$$

where x and p are the conjugate coordinate and momentum.

They suggest a quantization condition generating Riemann zeros. This Hamiltonian breaks time-reversal invariance since $(x,p) \to (x,-p) \Rightarrow H \to -H$. The classical Hamiltonian $H = xp$ of linear dilation, i.e. multiplication in x and contraction in p, gives the Hamiltonian equations:

$$\begin{aligned} \dot{x} &= x, \\ \dot{p} &= -p, \end{aligned} \tag{124}$$

with the solution

$$\begin{aligned} x(t) &= x_0 e^{t}, \\ p(t) &= p_0 e^{-t} \end{aligned} \tag{125}$$

for any nonzero $E = x_0 p_0 = x(t)p(t)$ is hyperbola in phase space.

The system is quantized by considering the dilation operator in the x space

$$H = \frac{1}{2}(xp + px) = -i\hbar(\frac{1}{2} + x\partial_x), \tag{126}$$

which is the simplest formally Hermitian operator corresponding to the classical Hamiltonian. The eigenvalue equation

$$H\psi_E = E\psi_E, \tag{127}$$

is satisfied by the eigenfunctions

$$\psi_E(x) = cx^{-\frac{1}{2}+\frac{i}{\hbar}E}, \tag{128}$$

where the complex constant c is arbitrary, since the solutions are not square-integrable. To the normalization

$$\int_0^\infty dx\psi_E(x)^*\psi_{E'}(x) = \delta(E - E'), \tag{129}$$

corresponds $c = 1/\sqrt{2\pi}$.

We have seen that

$$\zeta(\frac{1}{2}+iH)e^{-\varepsilon x} = \frac{1}{e^{\varepsilon x}-1},$$
$$H = -i(\frac{1}{2}+x\partial_x) = x^{1/2}px^{1/2}, p = -i\partial_x, \tag{130}$$

than

$$e^{-\varepsilon x} = \int dEx^{-1/2+iE}\varphi(E,\varepsilon), \varphi(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dxx^{-1/2-iE}e^{-\varepsilon x}$$
$$= \frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE);$$
$$\zeta(\frac{1}{2}+iE)\varphi(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dx\frac{x^{-1/2-iE}}{e^{\varepsilon x}-1}$$
$$= \frac{\varepsilon^{-1/2+iE}}{2\pi}\Gamma(1/2+iE)\zeta(\frac{1}{2}+iE). \tag{131}$$

From the equation (112) we have

$$\zeta(\frac{1}{2}+iH_1(x))e^{-\varepsilon x}=\frac{1}{e^{\varepsilon x}-1},\ H_1=i(\frac{1}{2}+x\partial_x),$$
$$\zeta(-x\partial_x)(1-\varepsilon x+\frac{(\varepsilon x)^2}{2}+...)=\frac{1}{\varepsilon x}(1-(\frac{\varepsilon x}{2}+\frac{(\varepsilon x)^2}{6}+...)+$$
$$+(\frac{\varepsilon x}{2}+...)^2+...), \tag{132}$$

so

$$\zeta(0)=-\frac{1}{2},\ \zeta(-1)=-\frac{1}{12},... \tag{133}$$

Note that, a little calculation shows that, the $(\varepsilon x)^2$ terms cancels on the r.h.s, in accordance with $\zeta(-2)=0$.

More curious question concerns with the term $1/\varepsilon x$ on the r.h.s. To it corresponds the term with actual infinitesimal coefficient on the l.h.s.

$$\frac{1}{\zeta(1)}\frac{1}{\varepsilon x}, \tag{134}$$

in the spirit of the nonstandard analysis (see, e.g. [Davis,1977]), we can imagine that such a terms always present but on the r.h.s we may not note them.

For other values of zeta function we will use the following expansion

$$\frac{1}{e^x-1} = \frac{1}{x+\frac{x^2}{2}+\frac{x^3}{3!}+...} = \frac{1}{x} - \frac{1}{2} + \sum_{k\geq 1}\frac{B_{2k}x^{2k-1}}{(2k)!},$$
$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, ... \tag{135}$$

and obtain

$$\zeta(1-2n) = -\frac{B_{2n}}{2n},\ n\geq 1. \tag{136}$$

Let us imagine space-time development of the the multiparticle process and try to describe it by some (phenomenological) dynamical equation. We start to find the equation for the Poisson distribution and than naturally extend them for the NBD case.
Let us define an integer valued variable $n(t)$ as a number of events (produced particles) at the time $t,\ n(0) = 0$. The probability of event $n(t),\ P(t,n)$, is defined from the following motion equation

$$\begin{aligned}
&P_t \equiv \frac{\partial P(t,n)}{\partial t} = r(P(t,n-1) - P(t,n)),\ n \geq 1 \\
&P_t(t,0)) = -rP(t,0), \\
&P(t,n) = 0,\ n < 0,
\end{aligned} \tag{137}$$

so

$$\begin{aligned}
&P(t,0) \equiv P_0(t) = e^{-rt}, \\
&P(t,n) = Q(t,n)P_0(t), \\
&Q_t(t,n) = rQ(t,n-1),\ Q(t,0) = 1.
\end{aligned} \tag{138}$$

To solve the equation for Q, we invent its generating function

$$F(t,h) = \sum_{n\geq 0} h^n Q(t,n), \tag{139}$$

and solve corresponding equation

$$F_t = rhF,\ F(t,h) = e^{rth} = \sum h^n \frac{(rt)^n}{n!},\ Q(t,n) = \frac{(rt)^n}{n!}, \tag{140}$$

so

$$P(t,n) = e^{-rt}\frac{(rt)^n}{n!} \tag{141}$$

is the Poisson distribution.
If we compare this distribution with (73), we identify $< n >= rt$, as if we have a free particle motion with velocity r and the distance is the mean multiplicity. This way we have a connection between n-dimension of the multiplicity and the usual dimension of trajectory.

As the equation gives right solution, its generalization may give more general distribution, so we will generalize the equation (137). For this, we put the equation in the closed form

$$P_t(t,n) = r(e^{-\partial_n} - 1)P(t,n) = \sum_{k\geq 1} D_k \partial^k P(t,n),\ D_k = (-1)^k \frac{r}{k!}, \tag{142}$$

where the $D_k,\ k \geq 1$, are generalized diffusion coefficients.
For other values of the coefficients, we will have other distributions.

For mean square deviation of the trajectory we have

$$< (x-\bar{x})^2 >=< x^2 > - < x >^2 \equiv D(x)^2 \sim t^{2/d_f}, \tag{143}$$

where d_f is fractal dimension. For smooth classical trajectory of particles we have $d_f = 1$; for free stochastic, Brownian, trajectory, all diffusion coefficients are zero but D_2, we have $d_f = 2$. In the case of Poisson process we have,

$$D(n)^2 =< n^2 > - < n >^2 \sim t,\ d_f = 2. \tag{144}$$

In the case of the NBD and KNO distributions

$$D(n)^2 \sim t^2,\ d_f = 1. \tag{145}$$

As we have seen, rasing k, KNO reduce to the Poisson, so we have a dimensional (phase) transition from the phase with dimension 1 to the phase with dimension 2. It is interesting, if somehow this phase transition is connected to the other phase transitions in strong interaction processes. For the Poisson distribution GF is solution of the following equation,

$$\dot{F} = -r(1-h)F, \tag{146}$$

For the NBD corresponding equation is

$$\dot{F} = \frac{-r(1-h)}{1+\frac{rt}{k}(1-h)}F = -R(t)F,\ \ R(t) = \frac{r(1-h)}{1+\frac{rt}{k}(1-h)}. \tag{147}$$

If we change the time variable as $t = T^{d_f}$, we reduce the dispersion low from general fractal to the NBD like case. Corresponding transformation for the evolution equation is

$$F_T = -d_f T^{d_f-1} R(T^{d_F})F, \tag{148}$$

we ask that this equation coincides with NBD motion equation, and define rate function $R(T)$

$$d_f T^{d_f-1} R(T^{d_F}) = \frac{r(1-h)}{1+\frac{rT}{k}(1-h)}, \tag{149}$$

now the following equation defines a production processes with fractal dimension d_F

$$F_t = -R(t)F,\ \ R(t) = \frac{r(1-h)}{d_F t^{\frac{d_F-1}{d_F}}(1+\frac{rt^{1/d_F}}{k}(1-h))} \tag{150}$$

Now we would like to consider a model of multiparticle production based on the d-dimensional sphere, and (try to) motivate the values of the NBD parameter k. The volum of the d-dimensional sphere with radius r, in units of hadron size r_h is

$$v(d,r) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}(\frac{r}{r_h})^d \tag{151}$$

Note that,

$$v(0,r) = 1,\ v(1,r) = 2\frac{r}{r_h},$$
$$v(-1,r) = \frac{1}{\pi}\frac{r_h}{r} \tag{152}$$

If we identify this dimensionless quantity with corresponding coulomb energy formula,

$$\frac{1}{\pi} = \frac{e^2}{4\pi}, \tag{153}$$

we find $e = \pm 2$.

For less then -1 even integer values of d, and $r \neq 0,\ v = 0$. For negative odd integer $d = -2n + 1$

$$v(-2n+1, r) = \frac{\pi^{-n+1/2}}{\Gamma(-n+3/2)}(\frac{r_h}{r})^{2n-1},\ n \geq 1, \tag{154}$$

$$v(-3, r) = -\frac{1}{2\pi^2}(\frac{r_h}{r})^3,\ v(-5, r) = \frac{3}{4\pi^3}(\frac{r_h}{r})^5 \tag{155}$$

Note that,

$$v(2, r)v(3, r)v(-5, r) = \frac{1}{\pi},\ v(1, r)v(2, r)v(-3, r) = -\frac{1}{\pi} \tag{156}$$

We postulate that after collision,it appear intermediate state with almost spherical form and constant energy density. Than the radius of the sphere rise dimension decrease, volume remains constant. At the last moment of the expansion, when the crossection of the one dimensional sphere - string become of order of hadron size, hadronic string divide in k independent sectors which start to radiate hadrons with geometric (Boze-Einstein) distribution, so all of the string final state radiate according to the NBD distribution.

So, from the volume of the hadronic string,

$$v = \pi(\frac{r}{r_h})^2 \frac{l}{r_h} = \pi k, \tag{157}$$

we find the NBD parameter k,

$$k = \frac{\pi^{d/2-1}}{\Gamma(d/2+1)}(\frac{r}{r_h})^d \tag{158}$$

Knowing, from experimental date, the parameter k, we can restrict the region of the values of the parameters d and r of the primordial sphere (PS),

$$r(d) = (\frac{\Gamma(d/2+1)}{\pi^{d/2-1}}k)^{1/d} r_h,$$
$$r(3) = (\frac{3}{4}k)^{1/3} r_h,\ r(2) = k^{1/2} r_h,\ r(1) = \frac{\pi}{2} k r_h \tag{159}$$

If the value of $r(d)$ will be a few r_h, the matter in the PS will be in the hadronic phase. If the value of r will be of order $10r_h$, we can speak about deconfined, quark-gluon, Glukvar, phase. From the formula (160), we see, that to have for the r, the value of order $10r_h$, in $d = 3$ dimension, we need the value for k of order 1000, which is not realistic.

So in our model, we need to consider the lower than one, fractal, dimensions. It is consistent with the following intuitive picture. Confined matter have point-like geometry, withe dimension zero. Primordial sphere of Glukvar have nonzero fractal dimension, which is less than one,

$$k = 3,\ r(0.7395)/r_h = 10.00,$$
$$k = 4,\ r(0.8384)/r_h = 10.00 \tag{160}$$

From the experimental data we find the parameter k of the NBD as a function of energy, $k = k(s)$. Then, by our spherical model, we construct fractal dimension of the Glukvar as a function of $k(s)$.
If we suppose that radius of the primordial sphere r is of order (or less) of r_h. Than we will have higher dimensional PS, e.g.

d	r/r_h	k
3	1.3104	3.0002
4	1.1756	3.0003
6	1.1053	2.9994
8	1.1517	3.9990

With extra dimensions gravitation interactions may become strong at the LHC energies,

$$V(r) = \frac{m_1 m_2}{m^{2+d}} \frac{1}{r^{1+d}} \tag{161}$$

If the extra dimensions are compactified with(in) size R, at $r >> R$,

$$V(r) \simeq \frac{m_1 m_2}{m^2 (mR)^d} \frac{1}{r} = \frac{m_1 m_2}{M_{Pl}^2} \frac{1}{r}, \tag{162}$$

where (4-dimensional) Planck mass is given by

$$M_{Pl}^2 = m^{2+d} R^d, \tag{163}$$

so the scale of extra dimensions is given as

$$R = \frac{1}{m} (\frac{M_{Pl}}{m})^{\frac{2}{d}} \tag{164}$$

If we take $m = 1TeV, (GeV^{-1} = 0.2fm)$

$$R(d) = 2 \cdot 10^{-17} \cdot (\frac{M_{Pl}}{1TeV})^{\frac{2}{d}} \cdot cm,$$
$$R(1) = 2 \cdot 10^{15} cm,$$
$$R(2) = 0.2\ cm\ !$$
$$R(3) = 10^{-7} cm\ !$$
$$R(4) = 2 \cdot 10^{-9} cm,$$
$$R(6) \sim 10^{-11} cm \qquad (165)$$

Note that lab measurements of $G_N (= 1/M_{Pl}^2, M_{Pl} = 1.2 \cdot 10^{19} GeV)$ have been made only on scales of about 1 cm to 1 m; 1 astronomical unit(AU) (mean distance between Sun and Earth) is $1.5 \cdot 10^{13} cm$; the scale of the periodic structure of the Universe, $L = 128 Mps \simeq 4 \cdot 10^{26} cm$. It is curious which (small) value of the extra dimension corresponds to L?

$$\begin{aligned} d &= 2\frac{\ln \frac{M_{Pl}}{m}}{\ln(mL)} = 0.74,\ m = 1TeV, \\ &= 0.81,\ m = 100GeV, \\ &= 0.07,\ m = 10^{17} GeV. \end{aligned} \qquad (166)$$

Motion equations of physics (applied mathematics in general) connect different observable quantities and reduce the number of independently measurable quantities. More fundamental equation contains less number of independent quantities. When (before) we solve the equations, we invent dimensionless invariant variables, than one solution can describe all of the class of phenomena.
In the z - Scaling (zS) approach to the inclusive multiparticle distributions (MPD) (see, e.g. [Tokarev, Zborovsry, 2007a]), different inclusive distributions depending on the variables $x_1, ...x_n$, are described by universal function $\Psi(z)$ of fractal variable z,

$$z = x_1^{-\alpha_1}...x_n^{-\alpha_n}. \tag{167}$$

It is interesting to find a dynamical system which generates this distributions and describes corresponding MPD.

We can find a good function if we know its derivative. Let us consider the following RD like equation

$$z\frac{d}{dz}\Psi = V(\Psi),$$
$$\int_{\Psi(z_0)}^{\Psi(z)} \frac{dx}{V(x)} = \ln\frac{z}{z_0} \tag{168}$$

In $x-$representation,

$$\ln z = -\sum_{k=1}^{n} \alpha_k \ln x_k,\ \ \delta_z = z\frac{d}{dz} = -\sum_k \frac{\delta_k}{n_h \alpha_k},$$
$$\sum_{k=1}^{n} \frac{x_k}{n_h \alpha_k}\frac{\partial}{\partial x_k}\Psi(x_1, ..., x_n) + V(\Psi) = 0, \tag{169}$$

e.g.

$$z = \delta_z z = -\sum_{k=1}^{n} \frac{x_k}{n_h \alpha_k}\frac{\partial}{\partial x_k} x_1^{-\alpha_1} ... x_n^{-\alpha_n} = z,\ \ n_h = n. \tag{170}$$

In the case of NBD GF (79), we have

$$\begin{aligned} &n=2,\ x_1=k,\ x_2=<n>,\ \alpha_1=\alpha_2=1,\ n_h=1,\\ &\Psi=F,\ V(\Psi)=-\Psi\ln\Psi. \end{aligned} \tag{171}$$

In the case of the $z-$scaling, [Tokarev, Zborovsry, 2007a],

$$\begin{aligned} &n=4,\ x_3=y_a,\ x_4=y_b,\\ &\alpha_1=\delta_1,\ \alpha_2=\delta_2,\ \alpha_3=\varepsilon_a,\ \alpha_4=\varepsilon_b, n_h=4, \end{aligned} \tag{172}$$

for infinite resolution, $\alpha_n=1,\ n=1,2,3,4$. In z variable the equation for Ψ has universal form. In the case of $n=2, \alpha_1=\alpha_2=1,\ n_h=1$, we find that $V(\Psi)=-\Psi\ln\Psi$, so if this form is applicable also in the case of n=4,

$$\begin{aligned} &z\frac{d}{dz}\Psi(z)=-\Psi\ln\Psi,\\ &\Psi(z)=e^{c/z}=(\Psi(z_0)^{z_0})^{\frac{1}{z}}=\Psi(z_0)^{\frac{z_0}{z}},\\ &c=z_0\ln\Psi(z_0)<0,\ z\in(0,\infty),\ \Psi(z)\in(0,1). \end{aligned} \tag{173}$$

Note that the fundamental equation is invariant with respect to the scale transformation $z\to\lambda z$, but the solution is not, the scale transformation transforms one solution into another solution. This is an example of the spontaneous breaking of the (scale) symmetry by the states of the system.

Formal motivation (foundation) of the RD motion equation for Ψ

As a dimensionless physical quantity $\Psi(z)$ may depend only on the running coupling constant $g(\tau), \ \tau = \ln z/z_0$

$$z\frac{d}{dz}\Psi = \dot{\Psi} = \frac{d\Psi}{dg}\beta(g) = U(g) = U(f^{-1}(\Psi)) = V(\Psi),$$
$$\Psi(\tau) = f(g(\tau)), \ g = f^{-1}(\Psi(\tau)) \quad (174)$$

According to the paper [Tokarev, Zborovsry, 2007a], for high values of z, $\Psi(z) \sim z^{-\beta}$; for small z, $\Psi(z) \sim$ const.
So, for high z,

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = -\beta\Psi(z); \tag{175}$$

for smaller values of z, $\Psi(z)$ rise and we expect nonlinear terms in $V(\Psi)$,

$$V(\Psi) = -\beta\Psi + \gamma\Psi^2. \tag{176}$$

With this function, we can solve the equation for Ψ(see appendix) and find

$$\Psi(z) = \frac{1}{\frac{\gamma}{\beta} + cz^{\beta}}. \tag{177}$$

RD equation of the z-Scaling,

$$z\frac{d}{dz}\Psi(z) = V(\Psi),\ V(\Psi) = V_1\Psi + V_2\Psi^2 + ... + V_n\Psi^n + ... \tag{178}$$

can be reparametrized,

$$\begin{aligned}
&\Psi(z) = f(\psi(z)) = \psi(z) + f_2\psi^2 + ... + f_n\psi^n + ...\\
&z\frac{d}{dz}\psi(z) = v(z) = v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...\\
&(v_1\psi(z) + v_2\psi^2 + ... + v_n\psi^n + ...)(1 + 2f_2\psi + ... + nf_n\psi^{n-1} + ...)\\
&= V_1(\psi + f_2\psi^2 + ... + f_n\psi^n + ...)\\
&+V_2(\psi^2 + 2f_2\psi^3 + ...) + ... + V_n(\psi^n + nf_2\psi^{n+1} + ...) + ...\\
&= V_1\psi + (V_2 + V_1f_2)\psi^2 + (V_3 + 2V_2f_2 + V_1f_3)\psi^3 +\\
&... + (V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n)\psi^n + ...\\
&v_1 = V_1,\\
&v_2 = V_2 - f_2V_1,\\
&v_3 = V_3 + 2V_2f_2 + V_1f_3 - 2f_2v_2 - 3f_3v_1 = V_3 + 2(f_2^2 - f_3)V_1, ...\\
&v_n = V_n + (n-1)V_{n-1}f_2 + ... + V_1f_n - 2f_2v_{n-1} - ... - nf_nv_1,
\end{aligned} \tag{179}$$

so, by reparametrization, we can change any coefficient of potential V but V_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{V_2}{V_1},\ f_3 = \frac{V_3}{2V_1} + f_2^2 = \frac{V_3}{2V_1} + (\frac{V_2}{V_1}^2),\ \ldots$$
$$f_n = \frac{V_n + (n-1)V_{n-1}f_2 + \ldots + 2V_2 f_{n-1}}{(n-1)V_1}, \ldots \qquad (180)$$

We will consider the case when only one of higher coefficient is nonzero and give explicit form of the solution Ψ.

Let us consider more general potential V

$$z\frac{d}{dz}\Psi = V(\Psi) = -\beta\Psi(z) + \gamma\Psi(z)^{1+n} \tag{181}$$

Corresponding solution for Ψ is

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz^{n\beta})^{\frac{1}{n}}} \tag{182}$$

More general solution contains three parameters and may better describe the data of inclusive distributions.

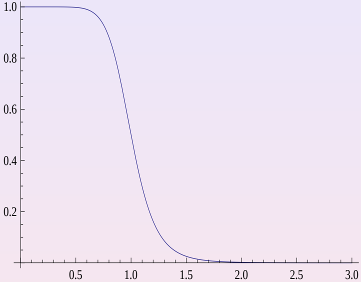

Figure: z-scaling distribution (182), $\Psi(z, 9, 9, 1, 1)$

In the case of $n = 1$ we reproduce the previous solution.

Another "natural" case is $n = 1/\beta$,

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz)^{\beta}} \tag{183}$$

In this case, we can absorb (interpret) the combined parameter by shift and scaling

$$z \to \frac{1}{c}(z - \frac{\gamma}{\beta}) \tag{184}$$

Another interesting point of view is to predict the value of β

$$\beta = \frac{1}{n} = 0.5; 0.33; 0.25; 0.2; ..., \;\; n = 2, 3, 4, 5, ... \tag{185}$$

For experimentally suggested value $\beta \simeq 9, n = 0.11$

In the case of $n = -\varepsilon,\ \beta = \gamma = 1/\varepsilon,\ c = \varepsilon k$, we will have

$$V(\Psi) = -\Psi \ln \Psi,\ \Psi(z) = e^{\frac{k}{z}} \tag{186}$$

This form of $\Psi-$function interpolates between asymptotic values of Ψ and predicts its behavior in the intermediate region. These three parameter function is restricted by the normalization condition

$$\int_0^\infty \Psi(z)dz = 1,$$
$$B(\frac{\beta-1}{\beta n}, \frac{1}{\beta n}) = (\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}} \frac{\beta n}{c^{\beta n}}, \tag{187}$$

so remains only two free parameter. When $\beta n = 1$, we have

$$c = (\beta - 1)(\frac{\beta}{\gamma})^{\beta-1} \tag{188}$$

If $\beta n = 1$ and $\beta = \gamma$, than $c = \beta - 1$.
In general

$$c^{\beta n} = (\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}} \frac{\beta n}{B(\frac{\beta-1}{\beta n}, \frac{1}{\beta n})} \tag{189}$$

RD equation of the z-scaling (181), after substitution,

$$\Psi(z) = (\varphi(z))^{\frac{1}{n}}, \tag{190}$$

reduce to the $n = 1$ case with scaled parameters,

$$\dot{\varphi} = -\beta n \varphi + \gamma n \varphi^2, \tag{191}$$

this substitution could be motivated also by the structure of the solution (182),

$$\Psi(z, \beta, \gamma, n, c) = \Psi(z, \beta n, \gamma n, 1, c)^{\frac{1}{n}} = \Psi(z, \beta, \gamma, \beta n, c)^{\beta}. \tag{192}$$

General RD equation takes form

$$\dot{\varphi} = nv_1\varphi + nv_2\varphi^{1+\frac{1}{n}} + nv_3\varphi^{1+\frac{2}{n}} + ... + nv_n\varphi^2 + nv_{n+1}\varphi^{2+\frac{1}{n}} + ... \tag{193}$$

The dimension of the space(-time) is the model dependent concept. E.g. for the fundamental bosonic string model (in flat space-time) the dimension is 26; for superstring model the dimension is 10 [Kaku,2000].
Let us imagine that we have some action-functional formulation with the fundamental motion equation

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = V(\Psi) = -\beta\Psi + \gamma\Psi^{1+n}. \tag{194}$$

Than, the corresponding Lagrangian contains the following mass and interaction parts

$$-\frac{\beta}{2}\Psi^2 + \frac{\gamma}{2+n}\Psi^{2+n} \tag{195}$$

The action gives renormalizable (effective quantum field theory) model when

$$d+2 = \frac{2N}{N-2} = \frac{2(2+n)}{n} = 2 + \frac{4}{n} = 2 + 4\beta, \tag{196}$$

so, measuring the parameter β inside hadronic and nuclear matters, we find corresponding (fractal) dimension.

From fundamental equation we obtain

$$(z\frac{d}{dz})^2\Psi \equiv \ddot{\Psi} = V'(\Psi)V(\Psi) = \frac{1}{2}(V^2)'$$
$$= \beta^2\Psi - \beta\gamma(n+2)\Psi^{n+1} + \gamma^2(n+1)\Psi^{2n+1} \quad (197)$$

Corresponding action Lagrangian is

$$L = \frac{1}{2}\dot{\Psi}^2 + U(\Psi),\ U = \frac{1}{2}V^2 = \frac{1}{2}\Psi^2(\beta - \gamma\Psi^n)^2$$
$$= \frac{\beta^2}{2}\Psi^2 - \beta\gamma\Psi^{2+n} + \frac{\gamma^2}{2}\Psi^{2+2n} \quad (198)$$

This potential, $-U$, has two maximums, when $V = 0$, and minimum, when $V' = 0$, at $\Psi = 0$ and $\Psi = (\beta/\gamma)^{1/n}$, and $\Psi = (\beta/(n+1)\gamma)^{1/n}$, correspondingly.

We define time-space-scale field $\Psi(t,x,\eta)$, where $\eta = \ln z-$ is scale coordinate variable, with corresponding action functional

$$A = \int dtd^d x d\eta (\frac{1}{2} g^{ab} \partial_a \Psi \partial_b \Psi + U(\Psi)) \quad (199)$$

The renormalization constraint for this action is

$$N = 2 + 2n = \frac{2(2+d)}{2+d-2} = 2 + \frac{4}{d},\ dn = 2,\ d = 2/n = 2\beta. \quad (200)$$

So we have two models for spase-time dimension, (196) and (200),

$$d_1 = 4\beta;\ d_2 = 2\beta \quad (201)$$

The coordinate η characterise (multiparticle production) physical process at the (external) space-time point (x,t). The dimension of the space-time inside hadrons and nuclei, where multiparticle production takes place is

$$d + 1 = 1 + 2\beta \quad (202)$$

Note that this formula reminds the dimension of the spin s state, $d_s = 2s + 1$. If we take $\beta(= s) = 0; 1/2; 1; 3/2; 2; ...$ We will have $d + 1 = 1; 2; 3; 4; 5; ...$

Note that as we invent Ψ as a real field, we ought to take another normalization

$$\int d^dx|\Psi|^2 = 1 \tag{203}$$

for the solutions of the motion equation. This case extra values of the parameter β is possible, $\beta > d/2$.

We can take a renormdynamic scheme were $\Psi(g)$ is running coupling constant. The variable z is a formation length and has dimension -1, RD equation for Ψ in φ_D^3 model is

$$z\frac{d}{dz}\Psi = \frac{6-D}{2}\Psi + \gamma\Psi^2 \tag{204}$$

$$\beta = \frac{D-6}{2} \tag{205}$$

For high values of z, $\beta = 9$, so $D = 24$. This value of D corresponds to the physical (transverse) degrees of freedom of the relativistic string, to the dimension of the external space in which this relativistic string lives. This is also the number of the quark - lepton matter degrees of freedom, $3 \cdot 6 + 6$. So, in these high energy reactions we measured the dimension of the space-time and matter and find the values predicted by relativistic string and SM. For lower energies, in this model, D monotonically decrees until $D = 6$, than the model (may) change form on the $\varphi_D^4, \beta = D - 4$. So we have two scenarios of behavior. In one of them the dimension of the space-time inside hadrons has value 6 and higher. In another the dimension is 4 and higher.

Perturbative QCD indicates that we have a fixed point of RD in dimension slightly higher than 4, and ordinary to hadron phase transition corresponds to the dimensional phase transition from slightly lower than 4, in QED, to slightly higher than 4 dimension in QCD. In general scalar field model φ^n_D,

$$\beta = -d_g = \frac{nD}{2} - n - D. \tag{206}$$

For φ^3 model, $\beta = 9$ corresponds to $D = 24$. In tha case of the $O(N)-$sigma model

$$\beta = D - 2, \tag{207}$$

For the experimental value of $\beta = 9$, we have the dimension of the $M-$theory, $D = 11$!

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