THE LEPTON SECTOR AS AN AXIOMATIC-LIKE CONSTRUCTION

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INTRODUCTION

Necessity of conversion to axiomatics.

What means axiomatic-like construction?

Unique approach and unification of mathematical formalism.

STABLE LEPTONS

Suppositions, necessary and sufficient conditions.

Massive leptons

Massless leptons

Comparison with well known results.

UNSTABLE LEPTONS

Setting of the problem

Groups $\Delta_1, \Delta_2, \Delta_3$.

Discussion of the results.

PERSPECTIVES AND REPRESENTING PROBLEMS

Our suppositions are:

1. the equations must be invariant and covariant under homogeneous Lorentz transformations taken into account all four connected components;
2. the equations must be formulated on the base of irreducible representations of the groups determining every lepton equation;
3. conservation of four-vector of probability current must be fulfilled and fourth component of the current must be positively defined;
4. the lepton spin is supposed equal to 1/2;
5. every lepton equation must be reduced to Klein-Gordon equation;

We see here two kinds of symmetries: with respect to the homogeneous Lorentz group and relativistic quantum wave equations.
These requirements are necessary and sufficient for obtaining of lepton equations.

The unified form of four connected components

We will use contracted form for notation of connected components. It looks for d_γ-group as:

$$\{b_i, b_k\} = 2\delta_{ik}, \quad (i, k = 1, 2, 3). \tag{1}$$

Lie algebra of d_γ-group is:

$$\begin{array}{lll}
[a_1, a_2] = 2a_3, & [a_2, a_3] = 2a_1, & [a_3, a_1] = 2a_2, \\
[b_1, b_2] = -2a_3, & [b_2, b_3] = -2a_1, & [b_3, b_1] = -2a_2, \\
[a_1, b_1] = 0, & [a_2, b_2] = 0, & [a_3, b_3] = 0, \\
[a_1, b_2] = 2b_3, & [a_1, b_3] = -2b_2, & \\
[a_2, b_3] = 2b_1, & [a_2, b_1] = -2b_3, & \\
[a_3, b_1] = 2b_2, & [a_3, b_2] = -2b_1. &
\end{array}$$

The obtained commutation relations coincide with commutation relations of the infinitesimal matrices of the proper homogeneous Lorentz group. Due to construction of commutation relation, all six operators a_1, a_2, a_3 and b_1, b_2, b_3 have a definite physical meaning.

P-conjugate representation and duality of d_γ-group

The duality means, that d_γ contains apart from $Q_2[a_1, a_2]$ one more group of the eighth order $q_2[a_1, a'_2]$. Here $a'_2 = a_2 \cdot c, \quad c = \sigma_x\sigma_y\sigma_z$.
Lie algebra is

$$[a_1, a'_2] = 2a'_3, \qquad [a'_2, a'_3] = -2a_1, \qquad [a'_3, a_1] = 2a'_2, \tag{2}$$

where $a'_3 \equiv a_1 a'_2$. Let us coll this group quaternion group of the second kind $q_2[a_1, a_2]$. As corollary we have another Lie algebra. We will denote it as f_γ

$$\begin{array}{lll}
[a_1, a'_2] = 2a'_3, & [a'_2, a'_3] = -2a_1, & [a'_3, a_1] = 2a'_2, \\
[b'_1, b'_2] = -2a'_3, & [b'_2, b'_3] = 2a_1, & [b'_3, b'_1] = -2a'_2, \\
[a_1, b'_1] = 0, & [a'_2, b'_2] = 0, & [a'_3, b'_3] = 0, \\
[a_1, b'_2] = 2b'_3, & [a_1, b'_3] = -2b'_2, & \\
[a'_2, b'_3] = -2b'_1, & [a'_2, b'_1] = -2b'_3, & \\
[a'_3, b'_1] = 2b'_2, & [a'_3, b'_2] = 2b'_1. &
\end{array}$$

The contracted defining relations for f_γ-group take the form

$$\begin{array}{ll}
\{b_1, b_k\}_p = 2\delta_{1k}, & (k = 1, 2, 3), \\
\{b_i, b_k\}_p = -2\delta_{ik}, & (i, k = 2, 3).
\end{array} \tag{3}$$

T-conjugate representation

The contracted defining relations for b_γ-group take the form

$$\{b'_i, b'_k\} = -2\delta_{ik}, \quad (i, k = 1, 2, 3). \tag{4}$$

Lie algebra of b_γ-group is:

$$\begin{array}{lll}
[a_1, a_2] = 2a_3, & [a_2, a_3] = 2a_1, & [a_3, a_1] = 2a_2, \\
[b'_1, b'_2] = 2a_3, & [b'_2, b'_3] = 2a_1, & [b'_3, b'_1] = 2a_2, \\
[a_1, b'_1] = 0, & [a_2, b'_2] = 0, & [a_3, b'_3] = 0, \\
[a_1, b'_2] = 2b'_3 & [a_1, b'_3] = -2b'_2, & \\
[a_2, b'_3] = 2b'_1, & [a_2, b'_1] = -2b'_3, & \\
[a_3, b'_1] = 2b'_2, & [a_3, b'_2] = -2b'_1, &
\end{array}$$

(PT)-conjugate representation

The contracted defining relations for c_γ-group take the form

$$\begin{array}{ll} \{b_1^*, b_k^*\}_{pt} = -2\delta_{1k}, & (k = 1,2,3), \\ \{b_i^*, b_k^*\}_{pt} = 2\delta_{ik}, & (i,k = 2,3). \end{array} \qquad (5)$$

$$\begin{array}{lll} [a_1, a_2'] = 2a_3', & [a_2', a_3'] = -2a_1, & [a_3', a_1] = 2a_2', \\ [b_1^*, b_2^*] = 2a_3', & [b_2^*, b_3^*] = -2a_1, & [b_3^*, b_1^*] = 2a_2', \\ [a_1, b_1^*] = 0, & [a_2', b_2^*] = 0, & [a_3', b_3^*] = 0, \\ [a_1, b_2^*] = 2b_3^* & [a_1, b_3^*] = -2b_2^*, & \\ [a_2', b_3^*] = -2b_1^*, & [a_2', b_1^*] = -2b_3^*, & \\ [a_3', b_1^*] = 2b_2^*, & [a_3', b_2^*] = 2b_1^*. & \end{array}$$

Here (PT)=(P)(T)=(T)(P) means sequential action (P)- and (T)-conjugation.

Now we have complete system of constituents for constructing of lepton wave equations.

This set of the groups was become closed with respect to discrete transformations. It is expressed by following equalities:

$$\langle T\rangle d_\gamma = b_\gamma, \quad \langle P\rangle d_\gamma = f_\gamma, \quad \langle PT\rangle d_\gamma = c_\gamma,$$

$$\langle T^{-1}\rangle b_\gamma = d_\gamma, \quad \langle P\rangle b_\gamma = c_\gamma, \quad \langle T^{-1}P\rangle b_\gamma = f_\gamma,$$
$$\langle T^{-1}\rangle c_\gamma = f_\gamma, \quad \langle P^{-1}\rangle c_\gamma = b_\gamma, \quad \langle T^{-1}P^{-1}\rangle c_\gamma = d_\gamma,$$
$$\langle T\rangle f_\gamma = c_\gamma, \quad \langle P^{-1}\rangle f_\gamma = d_\gamma, \quad \langle P^{-1}T\rangle f_\gamma = b_\gamma.$$

Here:

$\langle T\rangle$ means transition $b_k \rightarrow b'_k = ib_k \quad (k = 1, 2, 3)$,
$\langle P\rangle$ means transition $a_2 \rightarrow a'_2 = ia_2$.

It is so-called analytic continuation by group parameters.

Structure of the stable lepton groups.

1. The Dirac equation — $D_\gamma(II)$: d_γ, b_γ, f_γ, structural invariant $In[D_\gamma(II)] = -1$.
2. The equation for a doublet of massive neutrinos — $D_\gamma(I)$: d_γ, c_γ, f_γ, structural invariant $In[D_\gamma(I)] = 1$.
3. The equation for a quartet of massless neutrinos — $D_\gamma(III)$: d_γ, b_γ, c_γ, f_γ, structural invariant $In[D_\gamma(III)] = 0$.
4. The equation for a massless T-singlet — $D_\gamma(IV)$: b_γ, structural invariant $In[D_\gamma(IV)] = -1$.
5. The equation for a massless P-singlet — $D_\gamma(V)$: c_γ, structural invariant $In[D_\gamma(V)] = 1$.

Structure of the unstable lepton groups.

Group Δ_1 has the following defining relations:

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu\nu}, \quad (\mu, \nu = 1, 2, 3, 4, 5) \tag{6}$$

As a result we obtain the following composition:

$$\Delta_1\{D_\gamma(II), \quad D_\gamma(III), \quad D_\gamma(IV)\}, \quad In[\Delta_1] = -1. \tag{7}$$

Group Δ_3 has the following defining relations:

$$\begin{array}{ll} \Gamma_s\Gamma_t + \Gamma_t\Gamma_s = 2\delta_{st}, & (s, t = 1, 2, 3, 4), \\ \Gamma_s\Gamma_5 + \Gamma_5\Gamma_s = 0, & (s = 1, 2, 3, 4), \\ \Gamma_5^2 = -I. & \end{array}$$

It follows from here:

$$\Delta_3\{D_\gamma(II), \quad D_\gamma(I), \quad D_\gamma(III)\}, \quad In[\Delta_3] = 0. \tag{8}$$

Structure of the unstable lepton groups.

Group Δ_2 has the following defining relations:

$$\begin{array}{ll}\Gamma_s\Gamma_t+\Gamma_t\Gamma_s=2\delta_{st}, & (s,t=1,2,3),\\ \Gamma_s\Gamma_4+\Gamma_4\Gamma_s=0, & (s=1,2,3),\\ \Gamma_4^2=-I. & \\ \Gamma_u\Gamma_5+\Gamma_5\Gamma_u=0, & (u=1,2,3,4),\\ \Gamma_5^2=-I. & \end{array}$$

We obtain in this case:

$$\Delta_2\{D_\gamma(I),\quad D_\gamma(III),\quad D_\gamma(V)\},\quad In[\Delta_2]=1. \tag{9}$$

All three groups have its own structures.

conclusion

Nearest representing problems are determined by perspectives, which was opened on the base of obtained results.
Obviously that immediate task among the perspective problems is extension of obtained results on hadron sector.
It should be noted that this goal raises a set of problems. Some of them are well known, but some of them appear for the first time.

APPENDICES

A new (for physical applications) and effective tool for analysis and constructing lepton equations was found, i.e. numerical characteristic of irreducible matrix group.

Theorem. If $D = \{\gamma_1, ..., \gamma_\rho\}$ is an irreducible matrix group, then

$$In[D] = \frac{1}{\rho} \sum_{i=1}^{\rho} \chi(\gamma_i^2) = \left\{ \begin{array}{c} 1 \\ -1 \\ 0 \end{array} \right. \tag{10}$$

Here ρ - is order of the group, $\chi(\gamma_i^2)$ - is a trace of i-matrix squared. $In[D]$ — will be called **structural invariant** of D-group.

The defining relations for the groups of stable leptons

Dirac

$$\begin{array}{l} D_\gamma(II) : d_\gamma, b_\gamma, f_\gamma. \\ \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \\ \mu, \nu = 1, 2, 3, 4. \end{array} \tag{11}$$

Majorana

$$\begin{array}{l} D_\gamma(I) : d_\gamma, c_\gamma, f_\gamma. \\ \gamma_s\gamma_t + \gamma_t\gamma_s = 2\delta_{st}, \\ \gamma_4\gamma_s + \gamma_s\gamma_4 = 0 \\ \gamma_4^2 = -1, s, t = 1, 2, 3 \end{array} \tag{12}$$

Pauli

$$\begin{array}{l} D_\gamma(III) : d_\gamma, b_\gamma, c_\gamma, f_\gamma. \\ \gamma_s\gamma_t + \gamma_t\gamma_s = 2\delta_{st}, \\ \gamma_4\gamma_s - \gamma_s\gamma_4 = 0 \\ \gamma_4^2 = 1, s, t = 1, 2, 3 \end{array} \tag{13}$$

The defining relations for the groups of stable leptons

T-singlet

$$\begin{array}{l} D_\gamma(IV) : b_\gamma. \\ \gamma_s\gamma_t + \gamma_t\gamma_s = -2\delta_{st}, s, t = 1, 2, 3 \\ \gamma_4\gamma_s - \gamma_s\gamma_4 = 0, s = 1, 2, 3, \gamma_4^2 = 1 \end{array} \tag{14}$$

P-singlet

$$\begin{array}{l} D_\gamma(V) : c_\gamma. \\ \gamma_s\gamma_t + \gamma_t\gamma_s = 0, s \neq t, s, t = 1, 2, 3 \\ \gamma_1^2 = \gamma_2^2 = 1, \gamma_3^2 = -1, \\ \gamma_4\gamma_s - \gamma_s\gamma_4 = 0, \gamma_4^2 = 1, s = 1, 2, 3. \end{array} \tag{15}$$