Large-N Quantum Field Theories and Nonlinear Random Processes

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Motivation

Problems for modern Lattice QCD simulations(based on standard Monte-Carlo):

- Sign problem (finite chemical potential, fermions etc.)
- Finite-volume effects
- Difficult analysis of excited states
- Critical slowing-down
- Large-N extrapolation (AdS/CFT, AdS/QCD)

➡ Look for alternative numerical algorithms

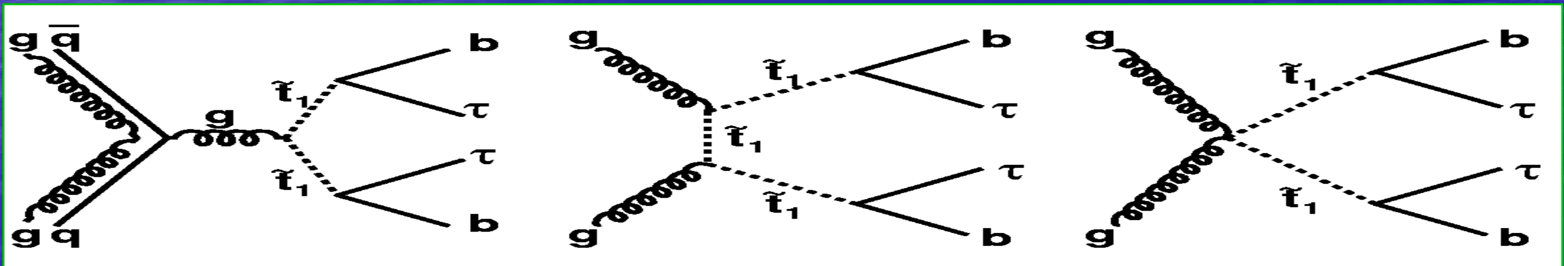
Motivation: Diagrammatic MC, Worm Algorithm, ...

- Standard Monte-Carlo: directly evaluate the path integral

$$\mathcal{Z} = \int \mathcal{D}\phi(x) \exp\left(-S[\phi(x)]\right)$$

- Diagrammatic Monte-Carlo: stochastically sum all the terms in the perturbative expansion

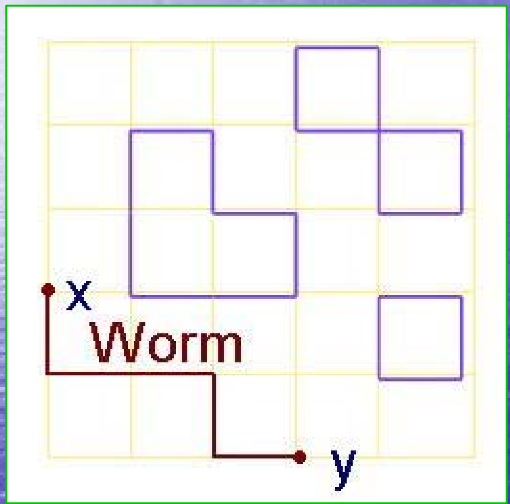
$$\mathcal{Z} = \int \mathcal{D}\phi(x) \exp\left(-S[\phi(x)]\right) = \int \mathcal{D}\phi(x) \exp\left(-S_0[\phi(x)]\right) \sum_k \left(\delta S[\phi(x)]\right)^k / k!$$



Motivation: Diagrammatic MC, Worm Algorithm, ...

- Worm Algorithm [Prokof'ev, Svistunov]:

Directly sample Green functions, Dedicated simulations!!!



Example:

Ising model

$$\langle \sigma_x \sigma_y \rangle \sim p(x,y)$$

x,y – head and tail of the worm

Applications:

- Discrete symmetry groups a-la Ising [Prokof'ev, Svistunov]
- O(N)/CP(N) lattice theories [Wolff] – difficult and limited

Extension to QCD?

And other quantum field theories with continuous symmetry groups ...

Typical problems:

- Nonconvergence of perturbative expansion (non-compact variables)
- Compact variables (SU(N), O(N), CP(N-1) etc.): finite convergence radius for strong coupling
- Algorithm complexity grows with N
- Weak-coupling expansion (=lattice perturbation theory): complicated, volume-dependent...

Large-N Quantum Field Theories

Situation might be better at large N ...

- Sums over PLANAR DIAGRAMS typically converge at weak coupling

- Large-N theories are quite nontrivial – confinement, asymptotic freedom, ...

- Interesting for AdS/CFT, quantum gravity, AdS/condmat ...

Main results to be presented:

- Stochastic summation over planar graphs: a general "genetic" random processes

➢ Noncompact Variables
➢ Itzykson-Zuber integrals
➢ Weingarten model = Random surfaces

- Stochastic resummation of divergent series: random processes with memory

➢ O(N) sigma-model
➢ outlook: non-Abelian large-N theories

Schwinger-Dyson equations at large N

- Example: φ^4 theory, φ – hermitian NxN matrix
- Action:

$$S[\phi] = N \mathrm{Tr}\, \phi(x) \left(m^2 - \Delta\right) \phi(x) - \frac{N\lambda}{4} \mathrm{Tr}\, \phi^4(x)$$

- Observables = Green functions (Factorization!!!):

$$w(k_1, \ldots, k_n) = \mathcal{N}\, c^{n-1}\, G(k_1, \ldots, k_n) =$$

$$\int d^4x_1 \ldots d^4x_n \exp(ik_A \cdot x_A) \left\langle \frac{1}{N} \mathrm{Tr}\left(\phi(x_1) \ldots \phi(x_n)\right) \right\rangle$$

- N, c – “renormalization constants”

Schwinger-Dyson equations at large N

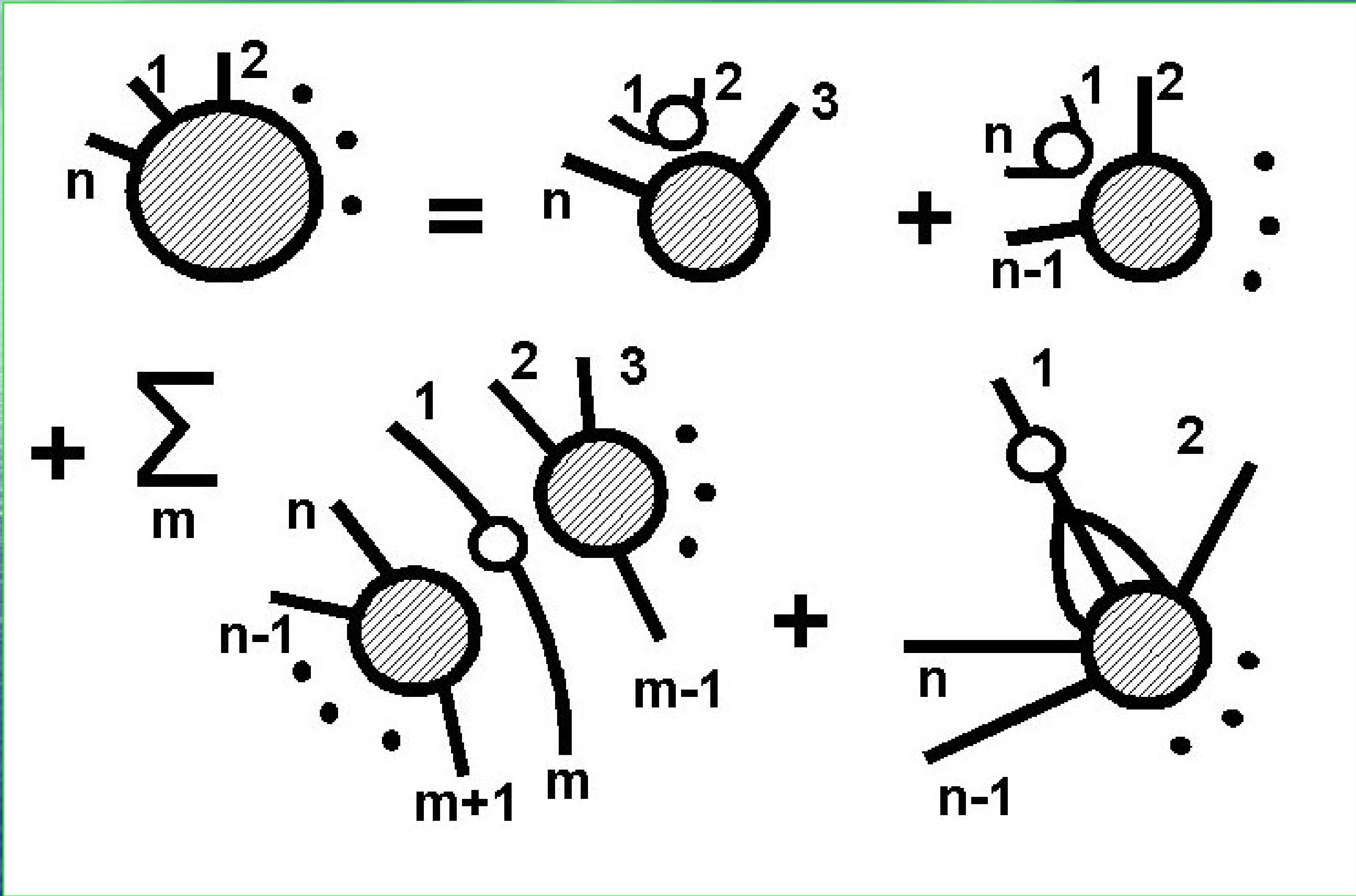
- Closed equations for $w(k_1, \ldots, k_n)$:

$$w(k_1, \ldots, k_n) =$$
$$G_0(k_1)\mathcal{N} c^{-4} \sum_{A=3}^{n-1} w(k_2, \ldots, k_{A-1})\, w(k_{A+1}, \ldots, k_n)\, \frac{\delta(k_1+k_A)}{V} +$$
$$G_0(k_1)\; c^{-2} \frac{\delta(k_1+k_2)}{V}\, w(k_3, \ldots, k_n) +$$
$$G_0(k_1)\; c^{-2} \frac{\delta(k_1+k_n)}{V}\, w(k_2, \ldots, k_{n-1}) +$$
$$G_0(k_1)\, \lambda c^2 \sum_{q_1, q_2, q_3} \delta(k_1 - q_1 - q_2 - q_3)\, w(q_1, q_2, q_3, k_2, \ldots, k_n)$$

- Always 2^{nd} order equations !
- Infinitely many unknowns, but simpler than at finite N

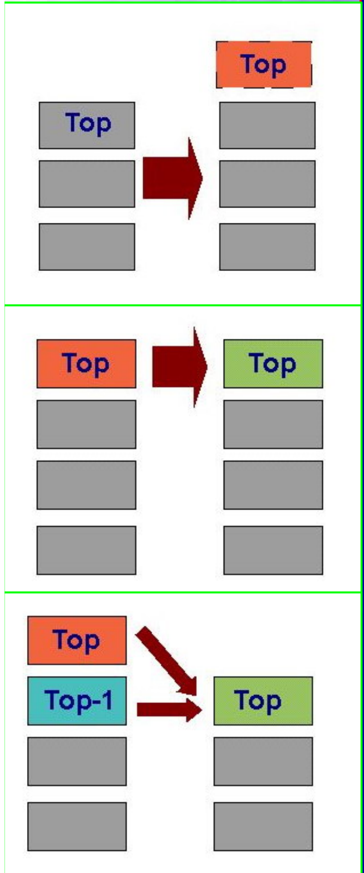
- Efficient numerical solution? Stochastic!
- Importance sampling: $w(k_1, \ldots, k_n)$ – probability

Schwinger-Dyson equations at large N



"Genetic" Random Process

Also: Recursive Markov Chain [Etessami, Yannakakis, 2005]



- Let X be some discrete set
- Consider stack of the elements of X
- At each process step:

Otherwise restart!!!

➢ **Create:** with probability $P_c(x)$ create new x and push it to stack

➢ **Evolve:** with probability $P_e(x|y)$ replace y on the top of the stack with x

➢ **Merge:** with probability $P_m(x|y_1,y_2)$ pop two elements y_1, y_2 from the stack and push x into the stack

"Genetic" Random Process: Steady State and Propagation of Chaos

- Probability to find n elements $x_1 \dots x_n$ in the stack:

  $$W(x_1, \dots, x_n)$$

- Propagation of chaos [McKean, 1966]
  (= factorization at large-N [tHooft, Witten, 197x]):

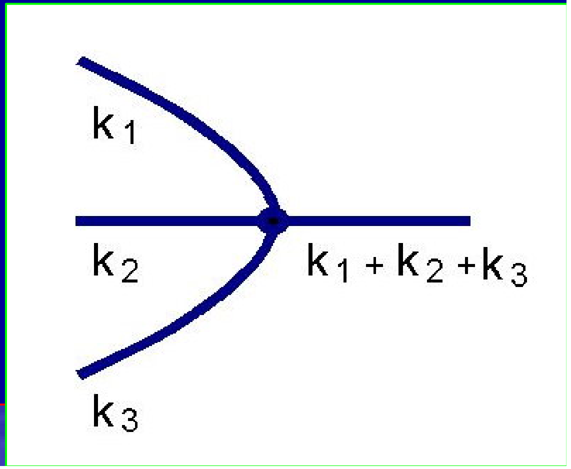
  $$W(x_1, \dots, x_n) = w_0(x_1)\, w(x_2) \dots w(x_n)$$

- Steady-state equation (sum over y, z):

  $$w(x) = P_c(x) + P_e(x|y)\, w(y) + P_m(x|y,z)\, w(y)\, w(z)$$

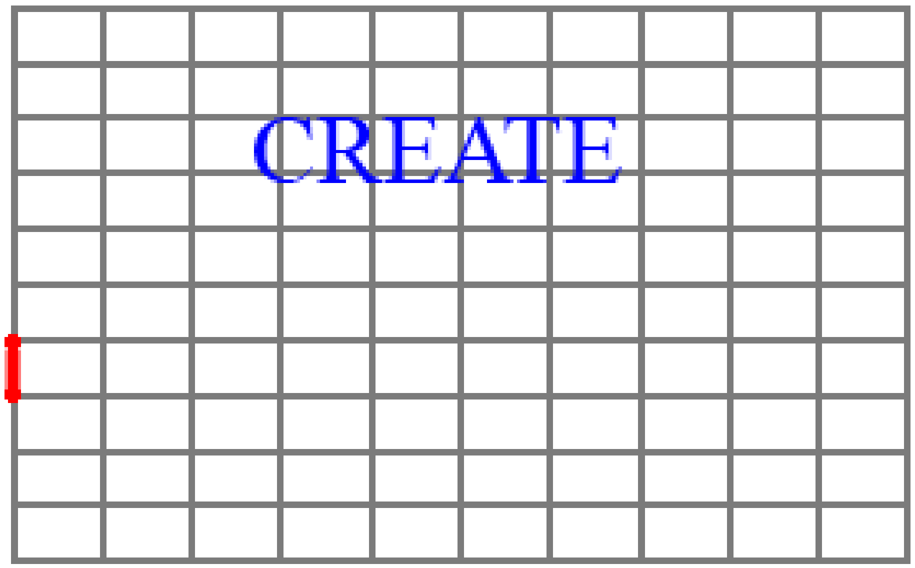
"Genetic" Random Process and Schwinger-Dyson equations

- Let X = set of all sequences $\{k_1, \ldots, k_n\}$, k – momenta
- Steady state equation for "Genetic" Random Process = Schwinger-Dyson equations, IF:
- **Create:** push a pair {k, -k}, $P \sim G_0(k)$
- **Merge:** pop two sequences and merge them
- **Evolve:**
  - add a pair {k, -k}, P ~ G0(k)
  - sum up three momenta on top of the stack, $P \sim \lambda\, G_0(k)$

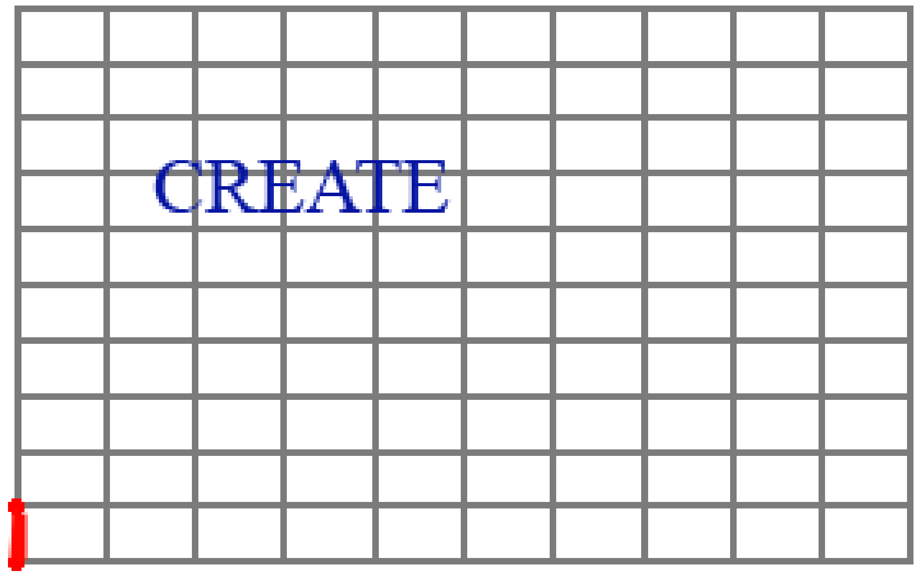


Examples: drawing diagrams

"Sunset" diagram



"Typical" diagram



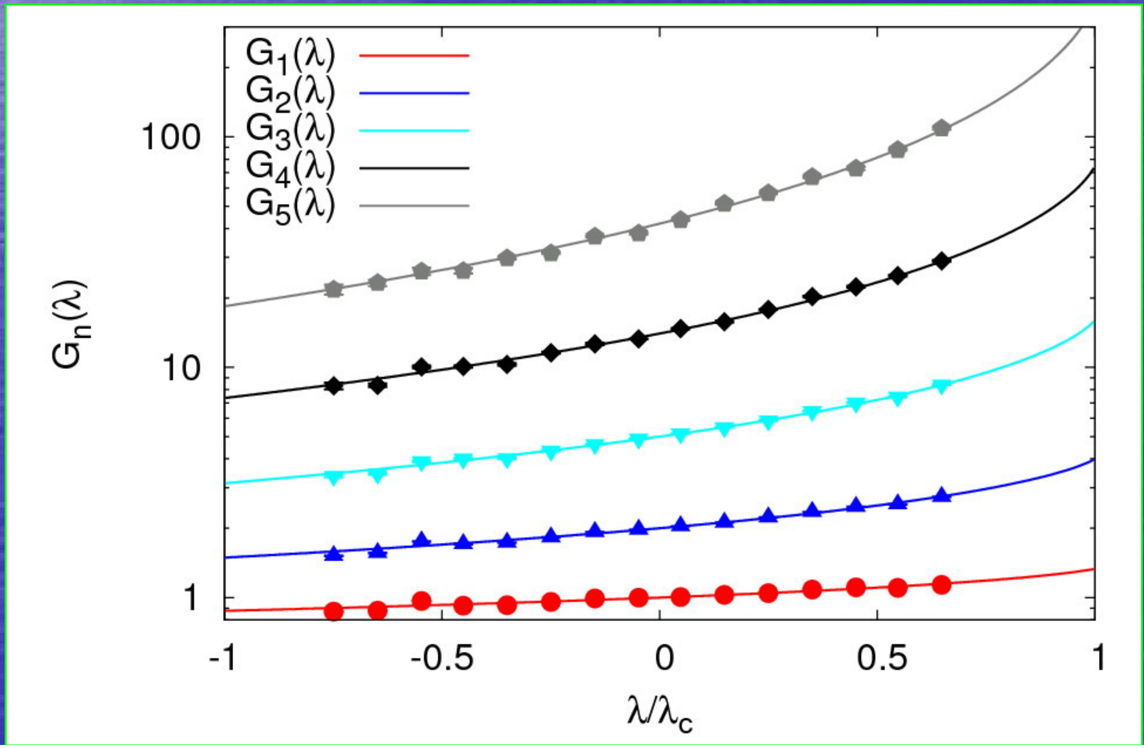
Only planar diagrams are drawn in this way!!!

Examples: tr φ^4 Matrix Model

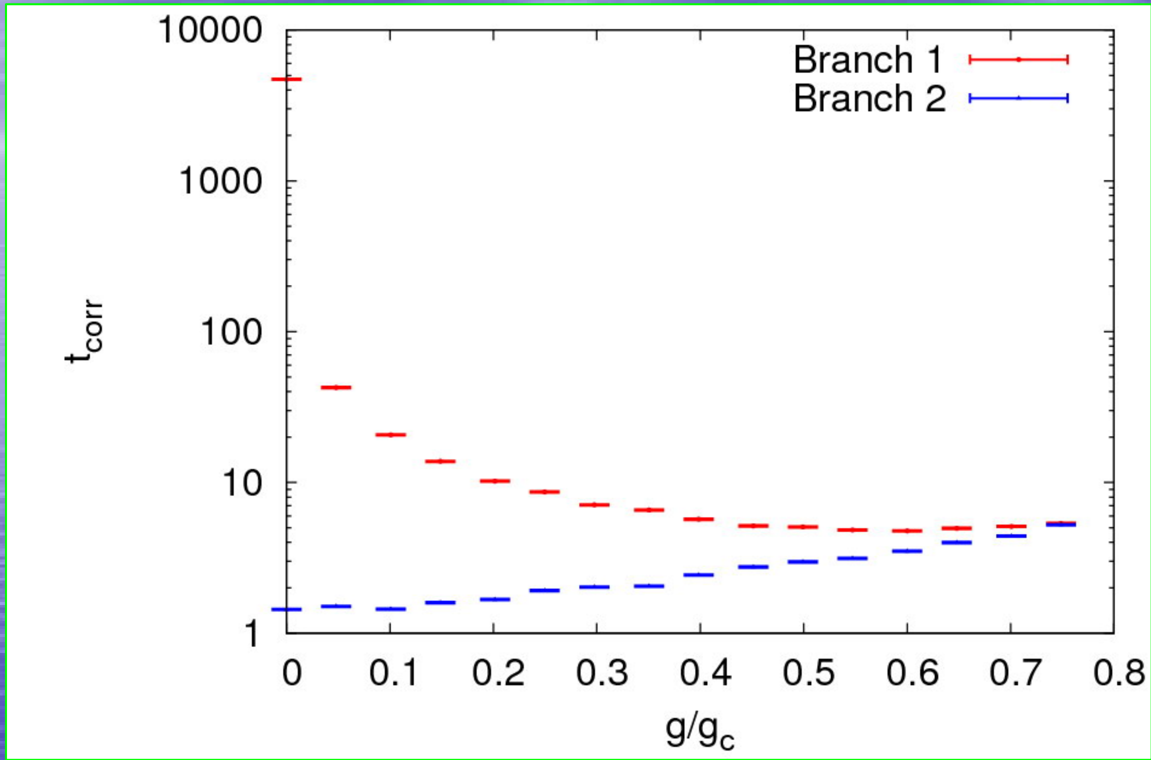
$$I_k = \mathcal{Z}^{-1} \int \mathcal{D}X \, \frac{1}{N} \mathrm{Tr}\, X^{2k} \, \exp\left(-N S[X]\right),$$

$$S[X] = \mathrm{Tr}\, X^2 - g \mathrm{Tr}\, X^4$$

Exact answer known [Brezin, Itzykson, Zuber]

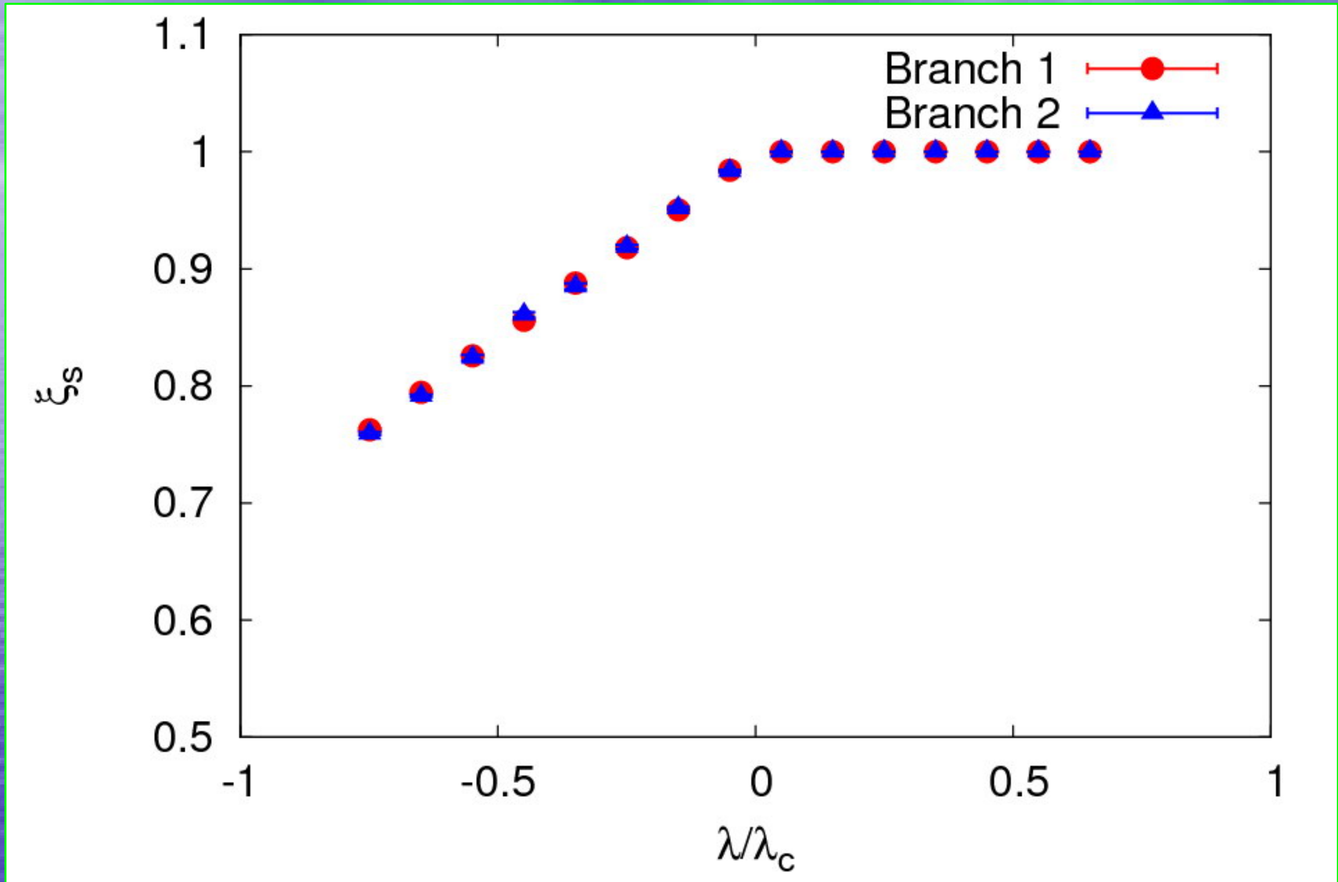


Examples: tr φ^4 Matrix Model



- Autocorrelation time vs. coupling:
  No critical slowing–down
- Peculiar: only $g < ¾\ g_c$ can be reached!!!
  Not a dynamical, but an algorithmic limitation...

Examples: tr ϕ^4 Matrix Model



Sign problem vs. coupling: No severe sign problem!!!

Examples: Weingarten model

Weak-coupling expansion = sum over bosonic random surfaces [Weingarten, 1980]

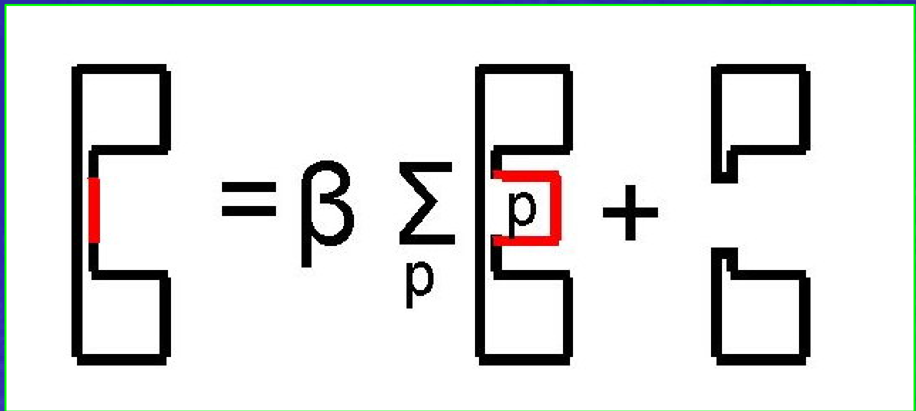
Complex NxN matrices on lattice links:

$$\mathcal{Z} = \int \mathcal{D}U(x,\mu)$$
$$\exp\left(-N\sum_{x,\mu}\mathrm{Tr}\,U(x,\mu)\,U^{\dagger}(x,\mu) + N\beta\sum_{p}U(p)\right)$$

"Genetic" random process:

- Stack of loops!
- Basic steps:
  - Join loops
  - Remove plaquette

Loop equations:

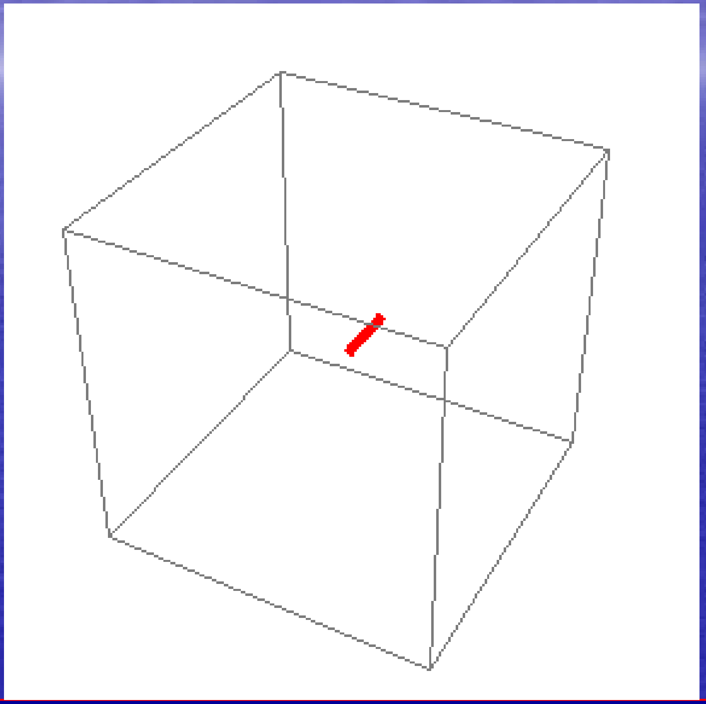


Examples: Weingarten model

Randomly evolving loops sweep out all possible surfaces with spherical topology

"Genetic" random process:

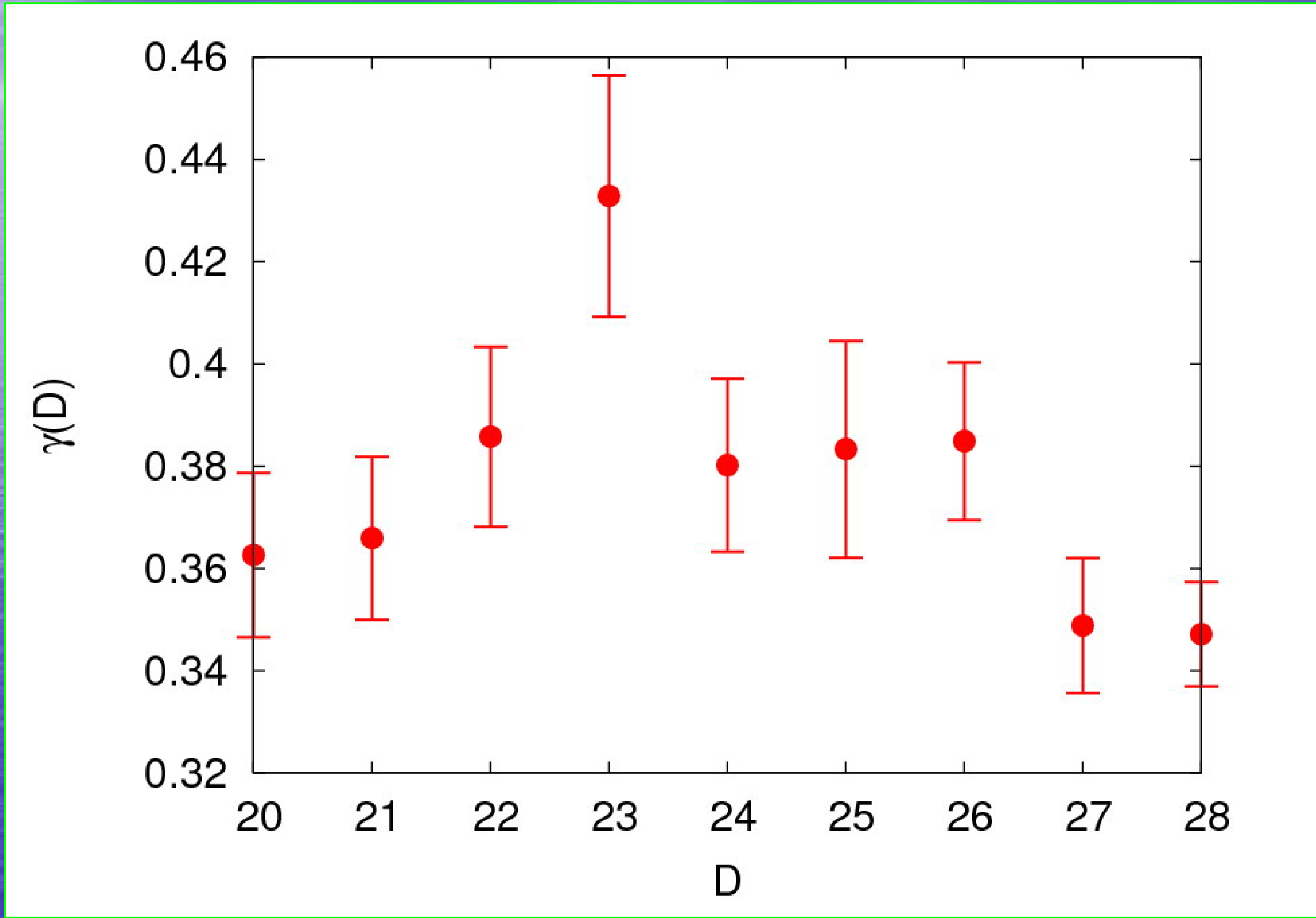
- Stack of loops!
- Basic steps:
  - Join loops
  - Remove plaquette



The process mostly produces "spiky" loops = random walks

Noncritical string theory degenerates into scalar particle [Polyakov 1980]

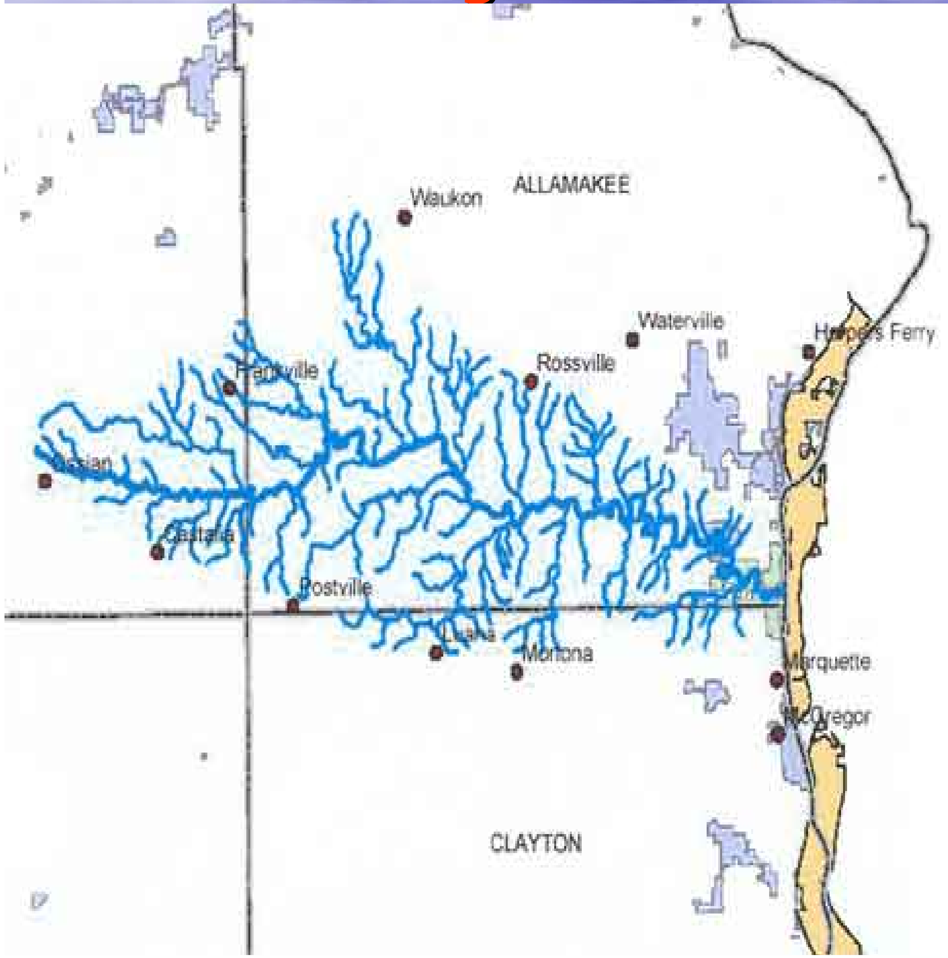
Examples: Weingarten model Critical index vs. dimension

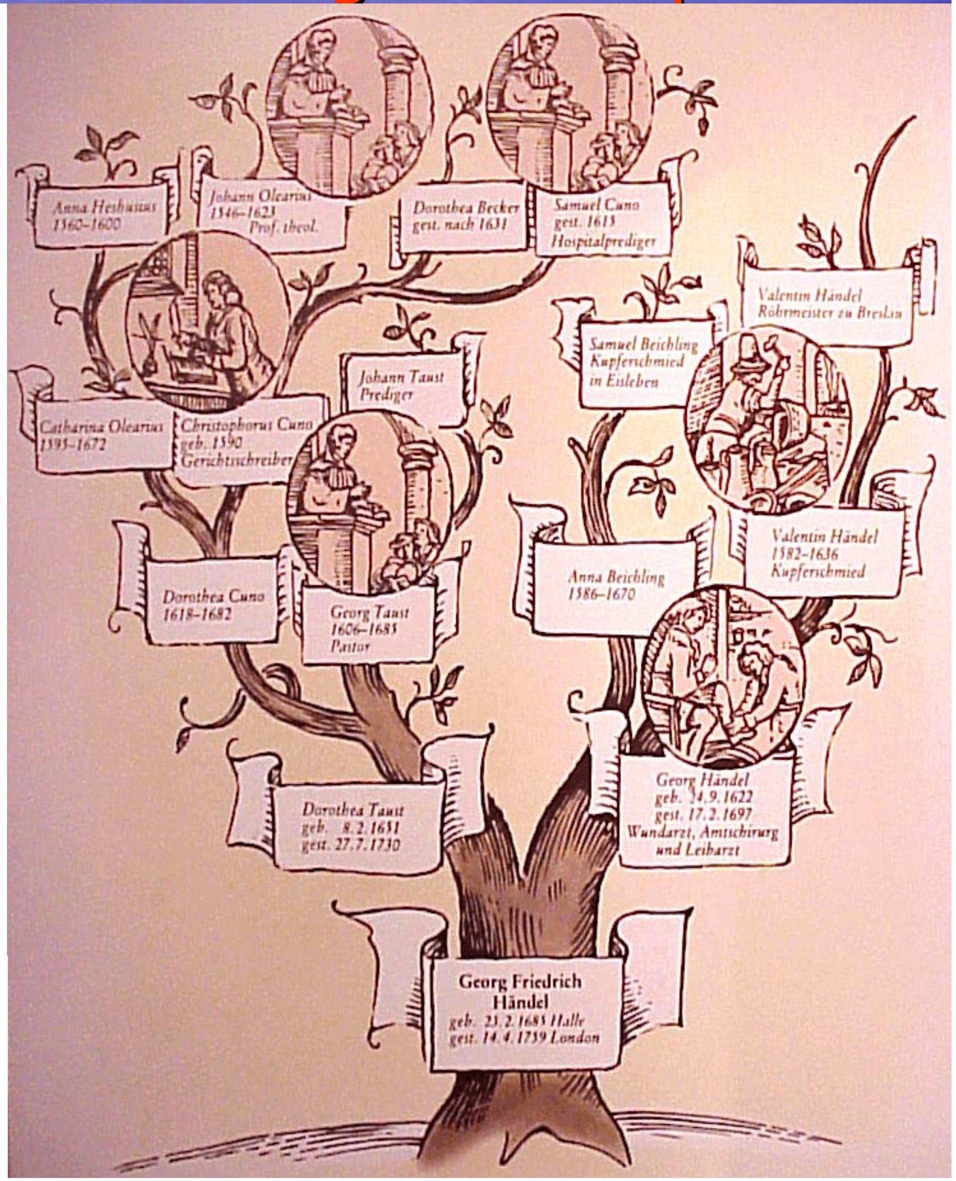


Peak around D=23, close to D_c=26 !!!

Some historical remarks

"Genetic" algorithm vs. branching random process





Some historical remarks

"Genetic" algorithm vs. branching random process

Probability to find some configuration of branches obeys nonlinear equation

Steady state due to creation and merging

Recursive Markov Chains
[Etessami, Yannakakis, 2005]

Also some modification of McKean-Vlasov-Kac models
[McKean, Vlasov, Kac, 196x]

"Extinction probability" obeys nonlinear equation
[Galton, Watson, 1974]
"Extinction of peerage"

Attempts to solve QCD loop equations
[Migdal, Marchesini, 1981]

"Loop extinction":
No importance sampling

Compact variables? QCD, CP(N),...

- Schwinger-Dyson equations: still quadratic
- Problem: alternating signs!!!
- Convergence only at strong coupling
- Weak coupling is most interesting...

Example: O(N) sigma model on the lattice

$$\mathcal{Z} = \int_{|n(x)|=1} \mathcal{D}n(x) \exp\left(\frac{N}{\lambda} \sum_{<xy>} n(x) \cdot n(y) \right)$$

Observables:

$$\xi(x, y) = \langle n(x) \cdot n(y) \rangle, \quad \xi(x) \equiv \xi(x, 0)$$

O(N) σ-model: Schwinger-Dyson

Schwinger-Dyson equations:

$$\xi(x) = \lambda^{-1} \sum_{\mu} \xi(x \pm e_\mu) - \lambda^{-1} \sum_{\mu} \xi(x)\, \xi(\pm e_\mu) + \delta(x)$$

Strong-coupling expansion does NOT converge !!!

Rewrite as:

$$\xi(x) = \frac{1}{\lambda + \sum_{\mu} \xi(\pm e_\mu)} \sum_{\mu} \xi(x \pm e_\mu) + \frac{\lambda}{\lambda + \sum_{\mu} \xi(\pm e_\mu)} \delta(x)$$

Now define a "probability" w(x): $\xi(x) = c\, w(x),\ \sum_{x} w(x) = 1$

O(N) σ-model: Random walk

Introduce the "hopping parameter":

$$\kappa = \frac{1}{\lambda + \sum_{\mu} \xi(\pm e_{\mu})} = \frac{1}{2D + \lambda w(0)}$$

Schwinger-Dyson equations
= Steady-state equation for Bosonic Random Walk:

$$\xi(x) = \frac{1}{\lambda + \sum_{\mu} \xi(\pm e_{\mu})} \sum_{\mu} \xi(x \pm e_{\mu}) + \frac{\lambda}{\lambda + \sum_{\mu} \xi(\pm e_{\mu})} \delta(x)$$

$$w(x) = \kappa \sum_{\mu} w(x \pm e_{\mu}) + (1 - 2D\kappa)\, \delta(x)$$

Random walks with memory

"hopping parameter" depends on the return probability w(0):

$$\kappa = \frac{1}{\lambda + \sum_{\mu} \xi\left(\pm e_{\mu}\right)} = \frac{1}{2D + \lambda w\left(0\right)}$$

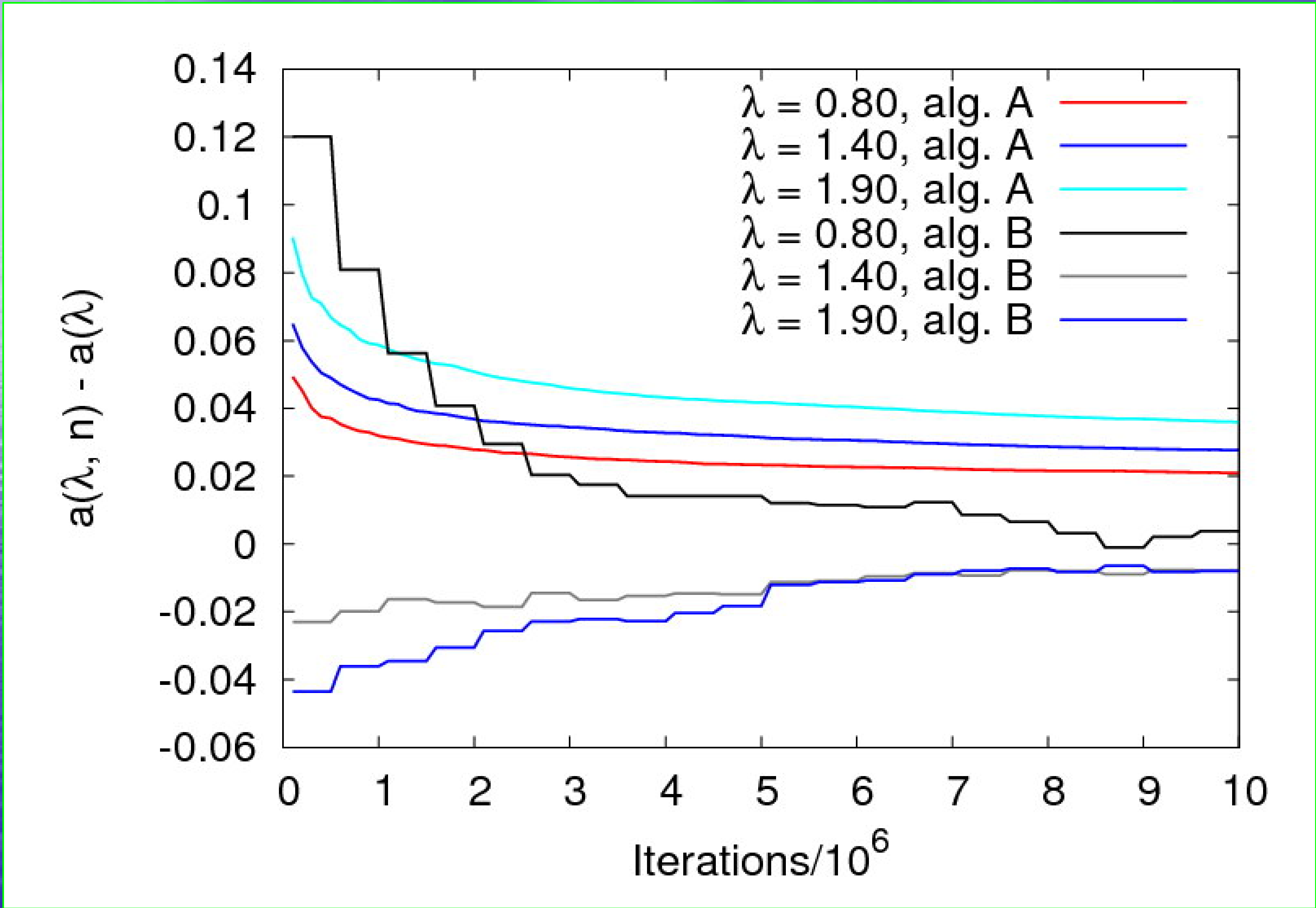
Iterative solution:

- Start with some initial hopping parameter
- Estimate w(0) from previous history ➡ memory

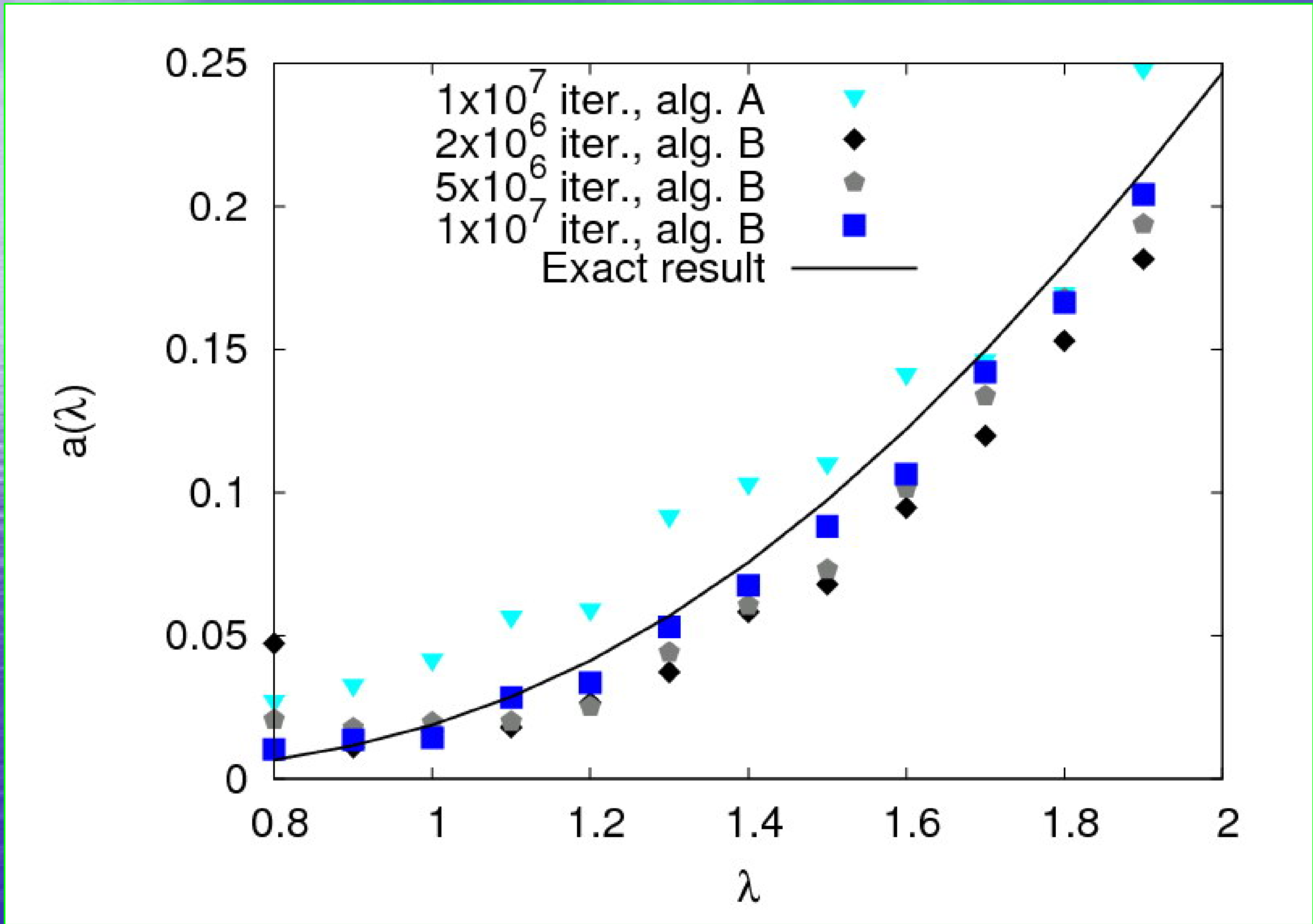
- Algorithm A: continuously update hopping parameter and w(0)
- Algorithm B: iterations

$$\kappa_{i+1} = \frac{1}{2D + \lambda w\left(0; \kappa_i\right)}$$

Random walks with memory: convergence



Random walks with memory: asymptotic freedom in 2D



Random walks with memory: condensates and renormalons

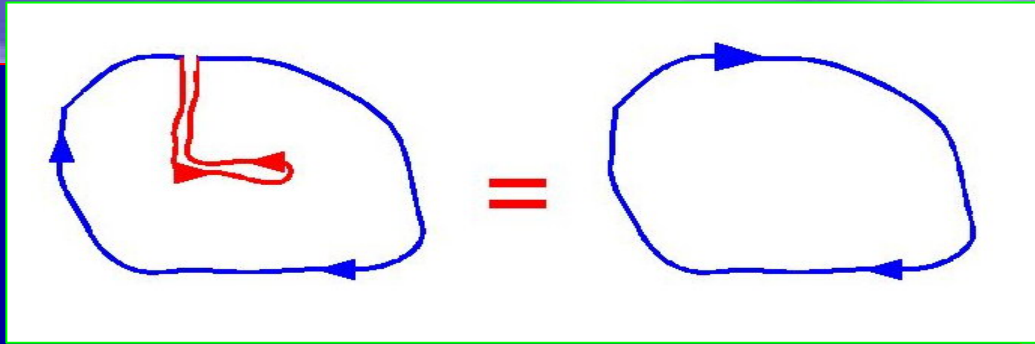
- O(N) σ-model at large N: divergent strong coupling expansion
- Absorb divergence into a redefined expansion parameter
- Similar to renormalons [Parisi, Zakharov, ...]
- ⇒ Nice convergent expansion

$$\kappa = \frac{1}{\lambda + \sum_{\mu} \langle n(0) \cdot n(\pm e_{\mu}) \rangle}$$

- $\langle n(0) \cdot n(+/- e_{\mu}) \rangle$ – "Condensate"
- Non-analytic dependence on λ
- O(N) σ-model = Random Walk in its own "condensate"

Outlook: large-N gauge theory

- |n(x)|=1 = "Zigzag symmetry"



- Self-consistent condensates = Lagrange multipliers for "Zigzag symmetry" [Kazakov 93]: "String project in multicolor QCD", ArXiv:hep-th/9308135

→ "QCD String" in its own condensate???

- AdS/QCD: String in its own gravitation field

- AdS: "Zigzag symmetry" at the boundary [Gubser, Klebanov, Polyakov 98], ArXiv:hep-th/9802109

Summary

- Stochastic summation of planar diagrams at large N is possible

➡ Random process of "Genetic" type

- Useful also for Random Surfaces
- Divergent expansions: absorb divergences into redefined self-consistent expansion parameters
- Solving for self-consistency

➡ Random process with memory