Fuzzy Topology of Phase Space and Gauge Fields

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Motivations:

Study of Mathematics foundations can be important for the construction of quantum space-time

Axioms of Set theory and Topology are the basis of any geometry

Examples:

Discrete space-time (Snyder, 1947)

Noncommutative geometry (Connes, 1991)

Sets, Topology and Geometry

Example: 1-dimensional Euclidian geometry is constructed on ordered set of elements $X = \{ x_l \}$; x_l - ***points***

$$\forall x_i , x_j \qquad x_i \le x_j .or. x_j \le x_i$$

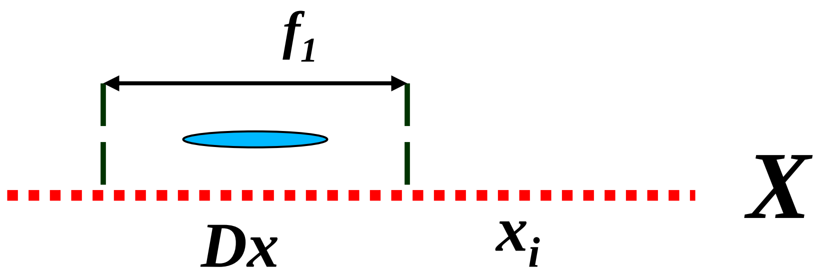
x_j $\qquad X$

Partial ordered set – Poset P^x **:**

Beside $x_i \le x_j$ it can be also $x_i \sim x_j$

x_i , x_j are incomparable elements

Example: $P = X \cup P^f; \quad P^f = \{ f_l \}$



$$f_1 \sim x_l \,, \; \forall x_l \in Dx$$

f_j **- fuzzy points,** ***(Zeeman, 1968*****)**

Fuzzy ordered set - foset F^x

$F^x \sim P^x$, but $\forall x_i, x_j \to w_{ij} \geq 0$

Example: $F^x = O^x \cup P'$

$x'_i \,||\, x_j \to w_{ij}$

$\sum_j w_{ij} = 1$

if O^x - is continuum ?

Fuzzy Geometry is consistent theory (Zeeman 1968, Dodson 1974)

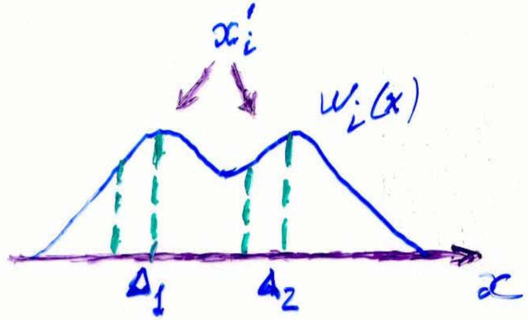
Fuzzy points x'_i - are particles with uncertain coordinate x in O^x

Foset $F^x = O^x \cup P'$

O^x - is continious $R^1 = \{x_a\}$

$P' = \{x_i'\}$, $i = 1, N$ discrete set

$\forall x_i' \sim w_i(x) \geq 0$; $\int_{O^x} w_i(x)dx = 1$



w_i support O_s^x

$O_s^x \in O^x$

$\forall x_a \in O_s^x$, $x_i' \parallel x_a$

$\forall \Delta_1, \Delta_2 \in O_s^x$; $x_i' \in \Delta_1$ and $x_i' \in \Delta_2$

Topological structure of F^x

F^x is nonprobabilistic structure!

Classical Mechanics

particle is ordered point $x(t)$ in O^*

its state: $|m\} = (x(t), \dot{x}(t))$

Fuzzy Mechanics (FM)

Particle is fuzzy point $m(t)$ in O^F

Fuzzy state $|m\} = (w(x,t), \ldots ?)$

$? = Q_1(x), \ldots Q_N(x); \; Q_1'(x, x'), \ldots; \; Q_1''(x, x', x'')$

Evolution:

$\hat{N}(t)|m_0\} = |m(t)\}$

minimal FM

Law of motion in Fuzzy Mechanics

a) Classical Mechanics: Minimal action

$$S = \int_{q_1}^{q_2} L(q, \dot{q})\,dt \; - \; \min$$

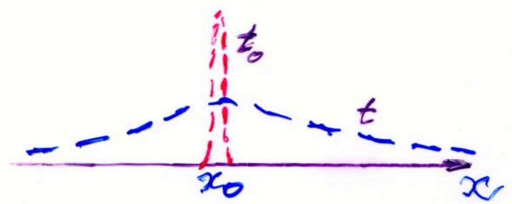
b) Quantum Mechanics – Path Integral

$$A(q_1, q_2) = e^{i\int_{q_1}^{q_2} S\,[dq]}$$

$$\Psi(t) = \hat{U}_t \Psi_0$$

c) Fuzzy Mechanics:

Fuzzy free state $g(x,t)$ tends to maximal space symmetry !?



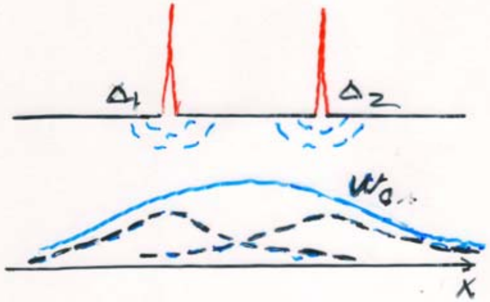
$$g(x,t) = \hat{N}_t\, g(x, t_0)$$

no parameters in $\hat{N}$

$$g(x,t) = \delta(x - x_0) \xrightarrow[\Delta t \to 0]{} \text{const}(x)$$

2 slits experiments

$w_0 \simeq \delta(x) \xrightarrow{t} \pm\infty$



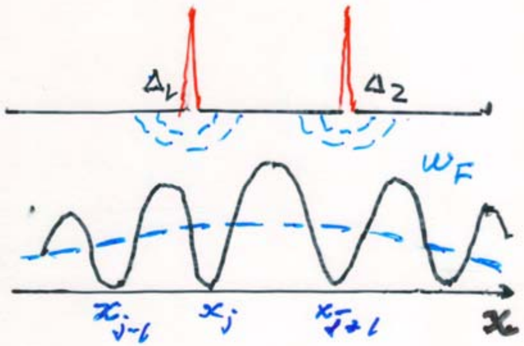
Stochastic mixture

$w_i(\vec{r}-\vec{r}_i, t)$

$L_c := m \in \Delta_1 .OR. m \in \Delta_2 \leftrightarrow g = g_1 .OR. g = g_2$

$w_c(x,t) = w_1(x,t) + w_2(x,t)$

$\forall x, \; w_c(x,t) > 0$



Topological structure of $\{m_0\}$

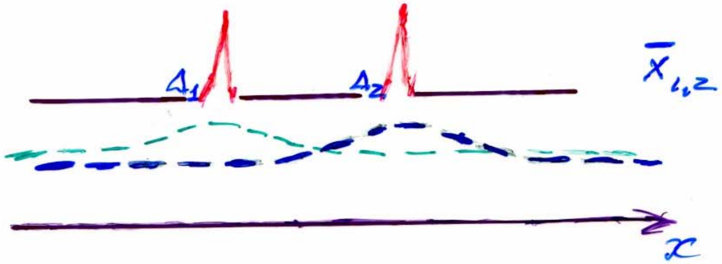
$L_F := m \in \Delta_1 .and. m \in \Delta_2 \rightarrow L_F \cap L_c = \emptyset$

$w_F(x,t) = w_n(x,t) + k(w_1(x,t) + w_2(x,t)) \Rightarrow k \simeq 0$

$\exists x_j, \; w_F(x,t) = 0$

$n=2$: $w_{1,2}(x,t)$ - Schwartz distributions

$w_{1,2}(x,t) \neq 0$
$x \to \pm\infty$

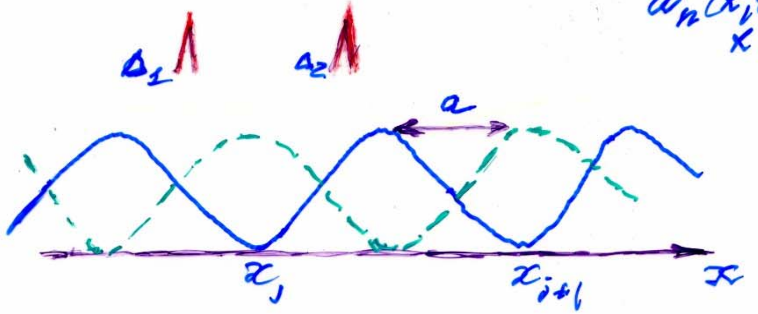


$\bar{x}_{1,2}$ - undefined

$m \in \Delta_1$. and . $m \in \Delta_2$;

so $w_s(x,t) = w_n(x,t)$; $\to$ $\exists x_j$; $w_s(x_j,t) = 0$

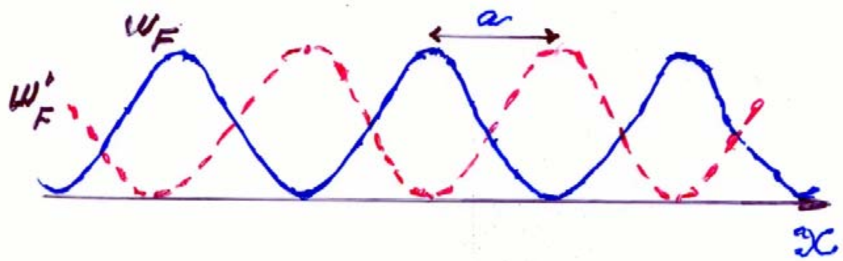
$w_n(x,t) \neq 0$
$x \to \pm\infty$



if $w_s(x,t)$ is solution , $\bar{x}$ - undefined
then $w_s'(x,t) = w_s(x+a,t)$ is also
solution $\forall a$, $-\infty \leq a \leq \infty$

$|g_0\} \sim w_0(x) = \frac{1}{2}\delta(x - x_1) + \frac{1}{2}\delta(x - x_2)$

Does $|g_0\}$ include other parameters q_i?



for $|g_F\}$ $\langle x \rangle$ is undefined

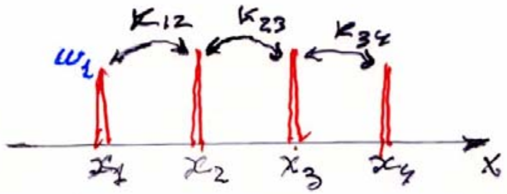
hence if $w_F(x,t)$ is solution

then $w_F'(x,t) = w_F(x+a,t)$ is solution

$\forall a$; $-\infty \leq a \leq \infty$

so, $|g_0\}$ has a-parameter
in addition to $w_0(x)$

a is $|g\}$ free parameter



$n = 4$

$$w_0(x) = \Sigma w_i \, \delta(x - x_i)$$

$n=2 \to a \sim k_{12}$ - correlation w_1 and w_2

if $k_{13} = k_{12} + k_{23}$

Then k_{ij} - phase $\alpha(x_i, x_j)$

$n \to \infty \quad |g\} = \{w(x), \alpha(x)\}$, $\cancel{x_0}$

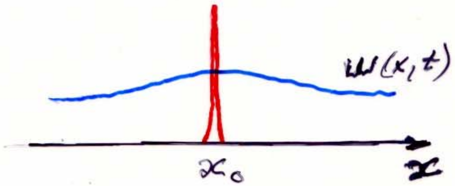
$$|g\} \to g(x,t) = \sqrt{w(x)}\, e^{i\alpha(x)} = q_1(x) + i q_2(x)$$

$$g(x) \to 0 \Longleftrightarrow w(x) \to 0$$

$|g\} = \{w(x), \alpha(x, x')$ - Bilocal state!

$$g(x) = \{w(x), a(x)\} \rightarrow g_1(x) + i\, g_2(x)$$

$$g(x,t_0) = c\,\delta(x - x_0)$$



$$g(x) \sim \sum c_i\, \delta(x - x_i)!$$

$g(t) = \hat{U}_t\, g(t_0)$; $\hat{U}_t$ - unitary

$U_{t_1+t_2} = U_{t_1} \cdot U_{t_2} \rightarrow U_t = e^{-i\hat{H}t}$, $\forall t$

$\hat{H}$ - time independent

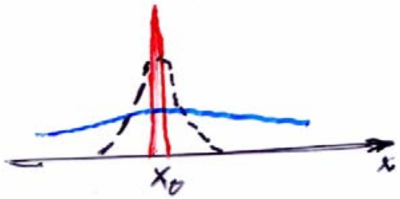
$\hat{V}_a$ - space shift ; $\hat{V}_a = e^{a\frac{\partial}{\partial x}}$

for free motion $[\hat{V}_a, \hat{U}_t] = 0$

fourier transform : $\varphi(p,t) = \int g(x,t)\, e^{ipx} dx$

$[\hat{V}_a, \hat{H}] = 0 \rightarrow \hat{H}_p = F(p)$

for $g(x, t=0) = \delta(x - x_0) \rightarrow \varphi(p,t) = e^{-iF(p)t + ipx_0}$



$$g(x,t) \underset{t \to t_0 +}{\longrightarrow} \delta(x-x_0)$$

$$\int g(x,t_j)\,dx = 1$$

$g(x,t_j)$ - δ-sequence $\{t_j\} \to t_0$

$$z = \frac{x}{f(t)} \;;\; \underset{t \to t_0}{f(t) \to 0} \;;\; g(x,t) = \frac{e^{i\gamma(z)}}{f(t)}$$

if $\int e^{i\gamma(z)} dz = 1 + O(t)$

$$\eta(p,t) = \int g(x,t) e^{ipx} dx = e^{-i\Gamma(pf(t))}$$

$$\varphi(p,t) = \eta(p,t) \to e^{-iF(p)t} = e^{-i\Gamma(pf(t))}$$

$$\hat{H}_p = F(p) = \frac{p^s}{2m_0} \;;\; m_0 > 0$$

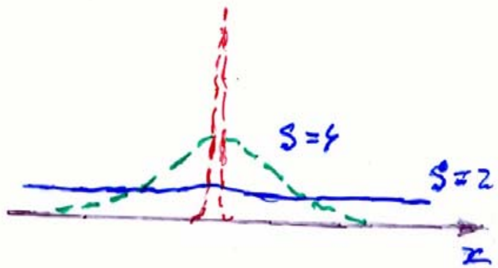
$$s = 2, 4, \ldots, 2n, \ldots$$

only $s=2$ gives $\underset{x \to \infty}{w(x,t) \neq 0}$!

$$-i\frac{\partial g}{\partial t} = \hat{H}_0 g = \frac{\hat{p}^2}{2m_0} g$$ - Schrödinger Equation

$$\hat{H}_0 = \frac{p^s}{2m_0} \qquad s = 2, 4, 6, \ldots 2n$$

$$\varphi(p,t) = e^{-i\frac{p^s}{2m}t} \to g(x,t)$$



$$w(x,t) \underset{x\to\infty}{\neq} 0$$

$$s = 4$$

$$w \sim \frac{1}{(tx^2)^{\frac{1}{6}}}$$

$$s = 2 \to g_0 = \delta(x - x_0) \to g(x,t) = \frac{e^{-\frac{ix^2m}{t}}}{c\sqrt{t}}$$

$$\psi(x) = \sum a_i g_i(x,t) \; ; \quad -i\frac{\partial\psi}{\partial t} = \frac{1}{2m}\frac{\partial^2}{\partial x^2}\psi$$

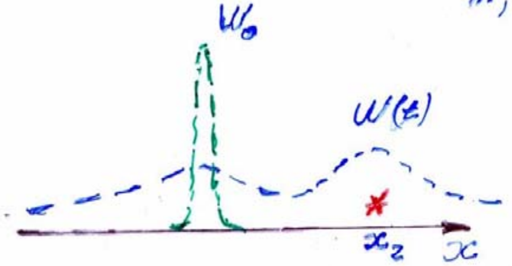
RF : $m \to \infty$ Galilean Transform.

$c \neq \infty$, From ψ linearity

$$-i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\hat{p} + \beta m)\psi :$$

ψ – 4-spinor

Interactions on Fuzzy Manifold



m, m_2 interactions

$m_2 \to \infty$

$\sigma_2(x) \to 0$

$\hat{H}_{int} = Q_1 \cdot Q_2 \cdot \hat{B}_{int}$

$\hat{H}_{int}$ perturbs $g(x,t)$ restoration of R^3 symmetry by m free evolution

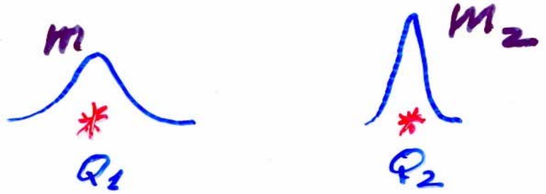
$g(x) \sim (w(x), \alpha(x)\}$ components

$$-i \frac{\partial g}{\partial t} = -i \frac{\partial}{\partial t}\left(\sqrt{w}\, e^{i\alpha(x)}\right) = \left(\hat{H}_0 + \hat{H}_{int}\right) g$$

$$\begin{cases} \dfrac{\partial \sqrt{w}}{\partial t} = \dfrac{\sqrt{w}}{2m} \dfrac{\partial^2 \alpha}{\partial x^2} + \dfrac{\partial \sqrt{w}}{\partial x} \cdot \dfrac{\partial \alpha}{\partial x} \\ \dfrac{\partial \alpha}{\partial t} = \dfrac{1}{2m} \dfrac{\partial^2 w^{\frac{1}{2}}}{\partial x^2} - \dfrac{\sqrt{w}}{2m}\left(\dfrac{\partial \alpha}{\partial x}\right)^2 + \hat{H}_{int} \end{cases}$$

$\hat{H}_{int} = Q_1 \cdot Q_2 \cdot F(r_{12}, t)$

$$\frac{\partial \alpha}{\partial t} = \left(\frac{\partial \alpha}{\partial t}\right)_{free} + Q_1 Q_2 F(r_{12}, t)$$

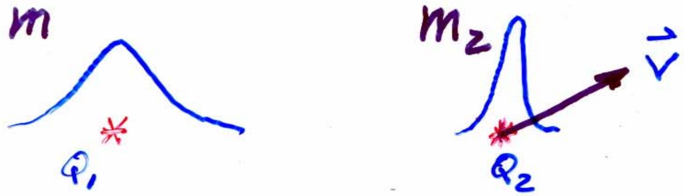


$$H_{int} = Q_1 Q_2 F(x_{12})$$

$$\frac{\partial \mathcal{L}(x)}{\partial t} = \left[\frac{\partial \mathcal{L}}{\partial t}(x)\right]_{free} + Q_1 \cdot Q_2 F(z_{12})$$

deformation of $\mathcal{H}$

Relativistic case



$$Q_2 \rightarrow J_\mu \sim \left\{ \frac{Q_2}{\sqrt{1-v^2}}, \ \frac{Q_2 \vec{V}}{\sqrt{1-v^2}} \right\}$$

$$H_{int} \sim Q_2 F(z_{12}) \longrightarrow A_\mu \simeq \{A_0(x), \vec{A}(x)\}$$

$$A_0(x),\ \vec{A}(x) \sim \frac{Q_2}{\sqrt{1-V^2}},\ \frac{Q_2\vec{V}}{\sqrt{1-V^2}}$$

Then $\exists\ RF'(\vec{v}')$ $\quad \vec{A}(x) \to \vec{A}'(x) = 0$

for m : $\vec{p} \to \vec{p}\,' = \dfrac{\vec{p} - m\vec{v}'}{\sqrt{1-v'^2}}$

hence : $\vec{p} \to \vec{p} + Q_1\vec{A}(x)$

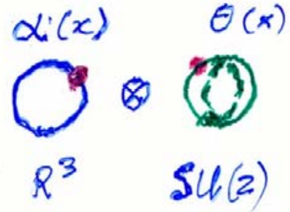
$$\frac{\partial\alpha}{\partial t} \to \frac{\partial\alpha}{\partial t} + Q_1 A_0(x)\ ;\ \frac{\partial\alpha}{\partial \vec{x}} \to \frac{\partial\alpha}{\partial \vec{x}} + Q_1\vec{A}(x)$$

$$-i\,\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\left(\vec{p} + Q_1\vec{A}(x)\right) + \beta m + Q_1 A_0(x)\right)\psi$$

Local $U(1)$ gauge invariance - QED

SU(2) Gauge Invariance

$$\Psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \sqrt{w(x)}\, e^{i\alpha(x) + i\vec{\theta}(x)\vec{\sigma}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



3 phase parameters

$$-i\frac{\partial \psi}{\partial t} = -i\frac{\partial}{\partial t}\left(\sqrt{w}\, e^{i\alpha + \vec{\theta}\vec{\sigma}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left(\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \hat{H}_{int}\right)\psi$$

$$\begin{cases} \dfrac{\partial \sqrt{w}}{\partial t} = \left(\dfrac{\partial w}{\partial t}\right)_{free} \\ \dfrac{\partial}{\partial t}\left(\alpha + \vec{\theta}\vec{\sigma}\right) = \dfrac{\partial}{\partial t}\left(\alpha + \vec{\theta}\vec{\sigma}\right)_{free} + \hat{H}_{int} \end{cases}$$

$$\hat{H}_{int} = Q_1 \cdot Q_2\, \vec{A}(x,t)\,\vec{\sigma}$$

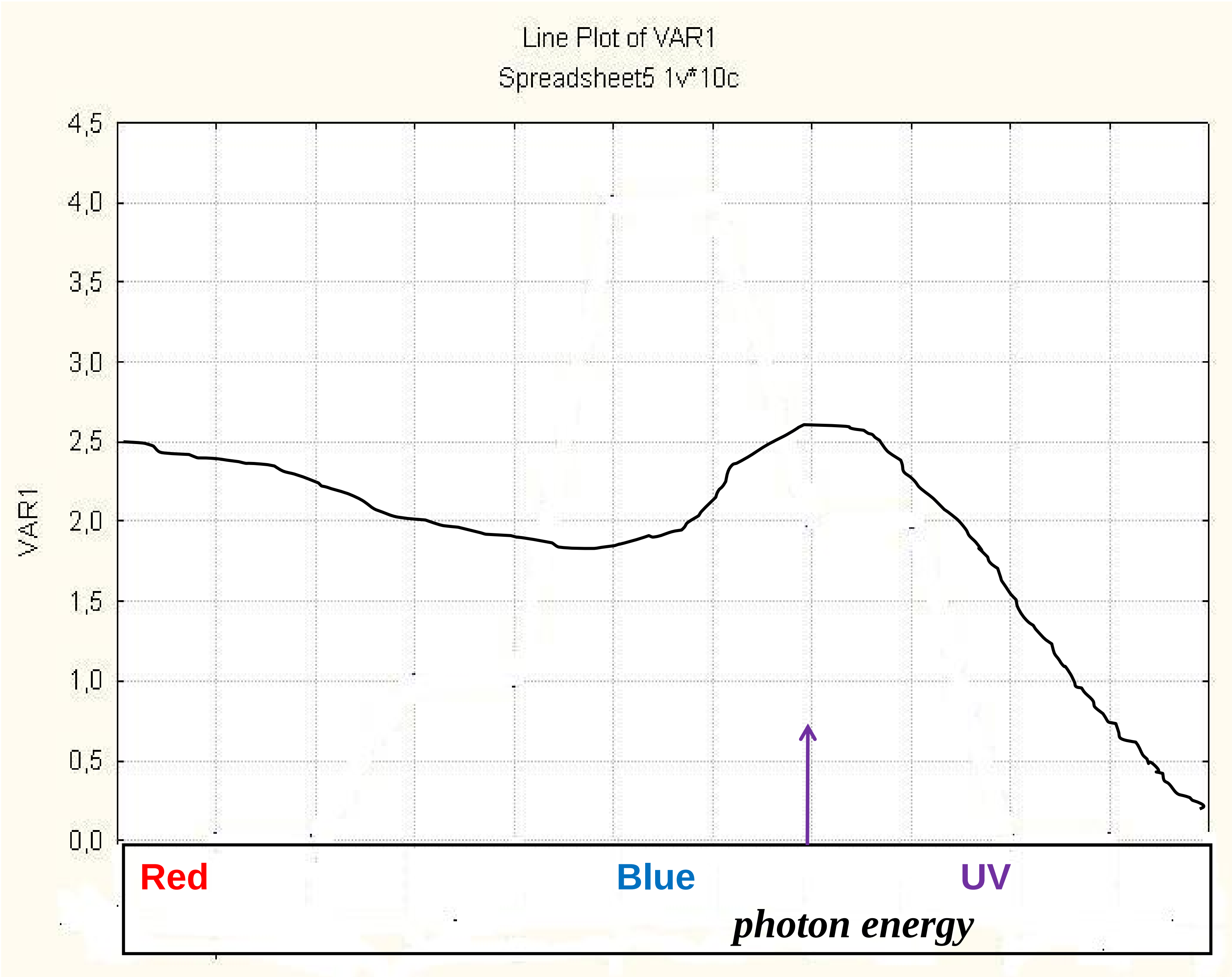
$\vec{A} \parallel \vec{S}_2$

if $Q_2, |\vec{S}_2| \to \infty$

$$\langle \vec{S}_{\perp}(t) \rangle \to 0$$

Conclusions

1. Fuzzy topology is the most simple and natural formalism for introduction of quantization into physical theory
2. Shroedinger equation is obtained from simple assumptions
3. Gauge invariance of fields corresponds to dynamics on fuzzy manifold


Line Plot of VAR1
Spreadsheet5 1v*10c
4,5
4,0
3,5
3,0
2,5
2,0
1,5
1,0
0,5
0,0
VAR1
Red
Blue
UV
photon energy