Liouville field, Modular forms and Elliptic Genera

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How does one describe strings propagating on non-compact backgrounds with potential isolated singularities?

How to handle time-dependent string backgrounds?

Liouville-type theories are conformal field theories that describe Calabi-Yau manifolds which are non-compact or singular.

Characteristic of such theories: central charge `above threshold' and necessity to introduce continuous as well as discrete representations.

Such representations have radically different modular properties (compared to RCFT) and it is non-trivial to construct suitable modular invariants describing the geometry of non-compact Calabi-Yau manifolds.

Here we attempt to construct elliptic genera for non-compact CY manifolds modelled by ALE spaces.

1. Context

Supersymmetric 5-brane soliton solution to heterotic string

(Callan-Harvey-Strominger 91)

$$NS_5 \quad \times \quad S^3 \quad \times \quad R$$

$$\downarrow$$

$$\frac{SU(2)}{U(1)} \times \quad U(1) \quad \times \quad R$$

$\mathcal{N}=2$ minimal

$\mathcal{N}=2$ Liouville

$\downarrow$ T dual

$$\frac{SL(2,R)}{U(1)}$$

Kazama-Susuki supercoset
(2d Euclidean black hole)

Equivalence with the superconformal theory of A_{N-1} singularities of K3 (Ooguri-Vafa 95)

2. Testing algebraic techniques against conformal bootstrap

Bosonic Liouville Theory ($\mathcal{N}=0$)

Stress-energy tensor: $T(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{Q}{2}\partial^2\phi$

central charge $c = 1 + 3Q^2$, Q background charge.

If the background charge is parametrized as

$$Q = \sqrt{2}(b + 1/b)$$

the vertex operator $\exp(\sqrt{2}b\phi)$ has conformal dimension h=1. The Liouville theory is defined as a theory perturbed from free field by this marginal operator (Liouville potential)

The dynamics of bosonic Liouville theory has been clarified in the late 90's by the method of conformal bootstrap.
(Fateev, Zamolodchikov & Zamolodchikov 2000; Teschner 2000)

We reproduce the results of conformal bootstrap using the representation theory of the Virasoro algebra for central charge beyond threshold ($c = 1 + 3Q^2 \geq 1$)
and the modular properties of character formulas.

There exist two types of representations in bosonic Liouville:

continuous representations $p > 0$

$$\chi_p(\tau) = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{q^{\frac{p^2}{2}}}{\eta(\tau)}, h = \frac{p^2}{2} + \frac{Q^2}{8}$$

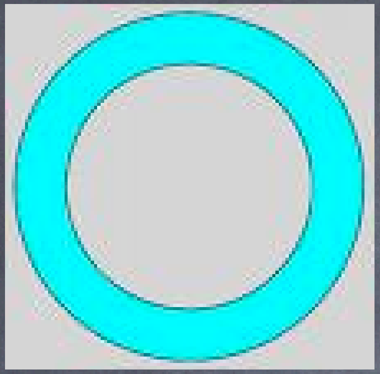
$$\chi_p(-\frac{1}{\tau}) = \int_0^{\infty} dp' \, \cos(2\pi p p') \, \chi_{p'}(\tau)$$

identity representation $h = 0$

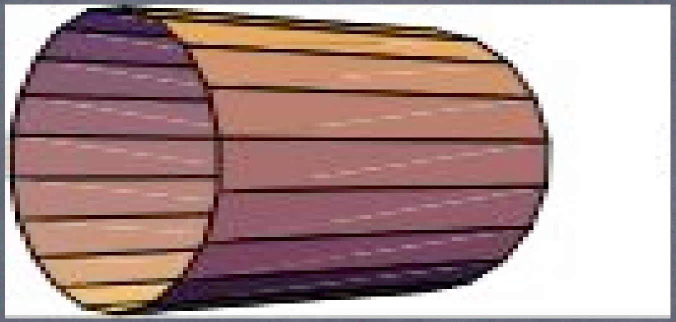
$$\chi_{h=0}(\tau) = \frac{q^{-\frac{Q^2}{8}}(1-q)}{\eta(\tau)}$$

$$\chi_{h=0}(-\frac{1}{\tau}) = \int_0^{\infty} dp \, \sinh(2\pi b p) \, \sinh(\frac{2\pi p}{b}) \, \chi_p(\tau)$$

Two interpretations of the same physical reality



$\tau \leftrightarrow \sigma$



Annulus - Loop of open string

$\chi_p(-\frac{1}{\tau})$, $\chi_{h=0}(-\frac{1}{\tau})$

Continuous, identity (discrete)

Cylinder - Tree of closed string

$\chi_p(\tau)$

Continuous

There is no identity representation in the closed string channel. This is consistent with the presence of a mass gap and the decoupling of gravity

$$h(\mathrm{e}^{\alpha\phi}) = -\frac{\alpha^2}{2} + \frac{\alpha Q}{2} = -\frac{(\alpha - \frac{1}{2}Q)^2}{2} + \frac{Q^2}{8}$$
$$= \frac{p^2}{2} + \frac{Q^2}{8} \geq \frac{Q^2}{8} \text{ for } \alpha = ip + \frac{1}{2}Q$$

Brane Interpretation in bosonic Liouville

Introduce ZZ and FZZT brane boundary states $|ZZ>, |FZZT>$ and identify the character functions as the inner products

$$\chi_{h=0}(-\frac{1}{\tau}) =< ZZ|\, e^{i\pi\tau H^{(c)}} |ZZ>$$

$$\chi_p(-\frac{1}{\tau}) =< FZZT;p|\, e^{i\pi\tau H^{(c)}} |ZZ>$$

where $H^{(c)} = (L_0 + \bar{L}_0)/2$ is the closed string Hamiltonian. The boundary states may be expanded in terms of Ishibashi states $|p>>$:

$$|ZZ>= \int_0^\infty dp\Psi_0(p)\,|p>> \quad \text{and} \quad |FZZT;p>= \int_0^\infty dp'\,\Psi_p(p')\,|p'>>$$

where

$$<< p|\, e^{i\pi\tau H^{(c)}}\, |p'>>= \delta(p-p')\chi_p(\tau)$$

One then has

$$|\Psi_0(p)^2| = \sinh(2\pi p b)\ \sinh(\frac{2\pi p}{b})$$

$$\Psi_p(p')^*\Psi_0(p') = \cos 2\pi p p'$$

Solving these relations one finds the boundary wave functions

$$\Psi_0(p) = \frac{2i\pi p}{\Gamma(1+2ibp)\Gamma(1+\frac{2ip}{b})}$$

$$\Psi_p(p') = -\frac{1}{2i\pi p'}\Gamma(1-2ibp')\Gamma(1-\frac{2ip'}{b})\ \cos(2\pi p p')$$

Agrees with results from conformal bootstrap (up to phase factors)

Conclusion: the data from representation theory is closely related to the properties of D-branes

String applications: Supersymmetric Liouville ($\mathcal{N}=2$)

$NS_5 \quad \times \quad S^3 \quad \times \quad R \qquad c=9 \; + \; c=6$

$\downarrow$

$$\frac{SU(2)}{U(1)} \times U(1) \times R$$

$\mathcal{N}=2$ minimal
compact

$\hat{c}_M = 1 - \frac{2}{N}$

$\downarrow$

Landau-Ginsburg

$\mathcal{N}=2$ Liouville
non compact

$$\hat{c}_L = 1 + \frac{2}{N}\,, \qquad Q = \sqrt{\frac{2}{N}}$$

$\downarrow$ T dual

$$\frac{SL(2,R)_{N+2}}{U(1)}$$

Kazama-Susuki supercoset
(2d Euclidean black hole with
asymptotic radius of cigar
$\sqrt{2N}$)

Equivalence with the superconformal theory of A_{N-1} singularities of K3 (Ooguri-Vafa 95)

Unitary representations of $\mathcal{N}=2$ algebra with $\hat{c}_L = 1 + \frac{2}{N}$

Identity rep. $j = 0$ vacuum

continuous reps. $j = \frac{1}{2} + i\frac{p}{Q}$ non-BPS states

discrete reps. $j = \frac{s}{2}, 1 \leq s \leq N$ BPS states, chiral primaries

Application to string theory requires the sum over spectral flows of each $\mathcal{N}=2$ representation:

$$\chi_*(r;\tau,z) = \sum_{n \in r+NZ} q^{\frac{\hat{c}_L}{2}n^2} e^{2i\pi \hat{c}_L z n}\, ch_*(\tau, z+n\tau)$$

continuous reps.:

$$\chi_{cont}(p,m;\tau) : p \geq 0, m \in Z_{2N}, h = \frac{p^2}{2} + \frac{m^2+1}{4N}, Q = \frac{m}{N}$$

identity rep. $\chi_{id}(r;\tau);\ r \in Z_N$

discrete reps. $\chi_{dis}(s,r;\tau);\ r \in Z_N, 1 \leq s \leq N$

The S transformation of these characters has the pattern

(continuous rep) $\xrightarrow{S}$ (continuous rep)

(identity rep) $\xrightarrow{S}$ (discrete rep) + (continuous rep)

(discrete rep) $\xrightarrow{S}$ (discrete rep) + (continuous rep)

Such a pattern was first observed in $\mathcal{N}=4$ representation theory.

There are 3 types of boundary states in the $\mathcal{N}=2$ theory, whose boundary wave functions are given by the elements of the modular S-matrix. One can reproduce the wave functions of the D0, D1 and D2 branes of the 2d black hole, first calculated by Ribault and Schomerus using semi-classical methods.

3. The geometry of $\mathcal{N}=2$ Liouville fields

Consider $\mathcal{N}=2$ minimal $\otimes$ $\mathcal{N}=2$ Liouville

$$\hat{c}_M + \hat{c}_L = 1 - \frac{2}{N} + 1 + \frac{2}{N} = 2, \; i.e. \; c_{tot} = 6$$

This theory describes (complex) 2d Calabi-Yau manifolds (ALE spaces)

Elliptic Genus (invariant under smooth variations of parameters and useful in counting the number of BPS states):

$$Z(\tau; z) = \mathrm{Tr}_{R\otimes R}(-1)^{F_L+F_R} e^{2i\pi z J_0^L} \, q^{L_0} \bar{q}^{\bar{L}_0}$$

$Z(\tau; z=0) = \chi$ Euler Number

$Z(\tau; z=\frac{1}{2}) = \sigma + \dots$ Hirzebruch signature

$Z(\tau; z=\frac{\tau+1}{2}) = \hat{A}\, q^{-1/4} + \dots$ $\hat{A}$ genus

AIM: compute the elliptic genus of non-compact CY manifolds by pairing $\mathcal{N}=2$ minimal models with the $\mathcal{N}=2$ Liouville theory.

The contribution to the elliptic genus from the minimal theory comes from Ramond ground states:

$$Z_{\mathrm{minimal}}(\tau,z)=\sum_{\ell=0}^{N-2}\mathrm{ch}^{\tilde{R}}_{\ell,\ell+1}(\tau,z)$$

On the other hand, the Landau-Ginsburg theory with superpotential

$$W=g\,(X^N+Y^2+Z^2)$$

acquires scale invariance in the infrared and reproduces the $\mathcal{N}=2$ minimal theory with $\hat{c}_M=1-2/N$.

As $g\to 0$, LG becomes free (chiral field with U(1) charge 1/N)

$$Z_{\mathrm{minimal}}(\tau,z)=Z_{LG}(\tau,z)=\frac{\theta_1(\tau,(1-\frac{1}{N})z)}{\theta_1(\tau,\frac{1}{N}z)}$$

free fermion charge 1-1/N

free boson charge 1/N

↓ Witten 94

Can we do something similar in the Liouville sector?

$$Z_{\mathrm{Liouville}}(\tau,z)=\sum_{s=1}^{N}\chi_{\mathrm{dis}}^{\tilde{R}}(s,s-1;\tau,z) \quad =\mathcal{K}_{2N}(\tau,\frac{z}{N})\,\frac{\theta_1(\tau,z)}{\eta^3(\tau)}$$

↓ Appell function

$$\mathcal{K}_\ell(\tau,z)=\sum_m \frac{e^{i\pi m^2\ell\tau+2i\pi m\ell\nu}}{1-e^{2i\pi(\nu+m\tau)}}$$

no `good' modular tfs (Semikhatov, A.T.,Tipunin 03)
unlike in minimal sector

First attempt at constructing an elliptic genus for ALE A_{N-1} spaces: orbifoldization when coupling minimal and Liouville

$$Z_{ALE(A_{N-1})}(\tau,z)=\tfrac{1}{N}\textstyle\sum_{a,b\in Z_N}\ q^{a^2}e^{4i\pi az}Z_{\mathrm{minimal}}(\tau,z+a\tau+b)$$

$$\times\ Z_{\mathrm{Liouville}}(\tau,z+a\tau+b)$$

So one gets

$$Z_{ALE(A_{N-1})}(\tau,z) = \frac{1}{N}\sum_{a,b\in Z_N} q^{a^2} e^{4i\pi az}(-1)^{a+b}\frac{\theta_1(\tau,\dfrac{N-1}{N}(z+a\tau+b))}{\theta_1(\tau,\dfrac{1}{N}(z+a\tau+b))}$$

$$\times\mathcal{K}_{2N}(\tau,\tfrac{1}{N}(z+a\tau+b))\frac{\theta_1(\tau,z)}{\eta^3(\tau)}$$

The elliptic genus associated with a CFT defined on a torus must be invariant under SL(2,Z) or one of its subgroups. Since we deal with a SCFT, it seems natural to demand invariance under the subgroup

$$\Gamma(2) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,Z), a=d=1, b=c=0 \bmod 2\right\}$$

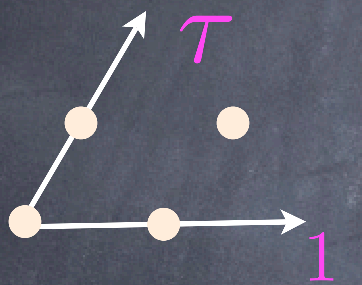
which leaves the spin structures fixed.

The formula proposed above must be modified to qualify as an elliptic genus.

4. Interlude: N=4 characters - Some basic facts

$$Ch^{NS}_{h,\ell}(q,z) = Tr_H(\, q^{L_0 - c/24} z^{T^3_0}\,), \qquad q = e^{2i\pi\tau},\ z = e^{2i\pi\mu}$$

$$\tau \in \mathcal{H}^+, \quad \mu \in \mathbf{C}$$



2d torus

● : points of order two

Spectral flow

$$\mu \to \mu + \frac{1}{2}.\tau\,(\,z \to zq^{1/2}\,)$$ NS to R

$$\mu \to \mu + \frac{1}{2}.1\ (\,z \to -z\,)$$ NS to NS'

$$\mu \to \mu + \frac{1}{2}.\tau + \frac{1}{2}.1 \qquad (\,z \to -zq^{1/2}\,)$$ NS to R'

Massive N=4 characters have the structure of

$$su(2)_{k-1} \times (4\,\text{free fermions}) \times 1\,\text{boson}$$

$$c = \frac{3(k-1)}{k+1} + (4 \times \frac{1}{2}) + 1 + 3Q^2 = 6k \ \text{for}\ Q^2 = \frac{2k^2}{k+1}$$

For k=1, the NS massive characters are

$$Ch^{NS}_{[h]}(q,z) = q^{h-1/4} \prod_{n=1}^{\infty} \frac{(1+zq^{n-1/2})^2(1+z^{-1}q^{n-1/2})^2}{1-q^n}$$

$$= q^{h-1/8} \frac{\vartheta_3(q,z)^2}{\eta(q)^3}, \quad h > 0$$

continuum

hence deformable

The structure of massless characters is more complicated due to the presence of fermionic null vectors

For k=1, the 2 NS massless characters are:

$$Ch_\ell^{NS}(q,z) = q^{\ell-1/8}\frac{\vartheta_3(q,z)^2}{\eta(q)^3}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-z^2q^n)(1-z^{-2}q^{n-1})}$$

$$\times\sum_{m\in Z} q^{2m^2+(2\ell+1)m}\left\{\frac{z^{4m+2\ell}}{(1+zq^{m+1/2})^2}-\frac{z^{-4m-2\ell-2}}{(1+z^{-1}q^{m+1/2})^2}\right\}$$

with $\ell = 0$ and $\ell = 1/2$

discrete
hence topological

S transform of N=4 characters :
Massless to Massless + continuum of Massive

$$Ch^{NS}_{k,\ell}(-\frac{1}{\tau},\frac{\mu}{\tau}) = (-1)^{2\ell}(k-2\ell+1)e^{\frac{2i\pi\mu^2 k}{\tau}}\, Ch^{NS}_{k,\frac{k}{2}}(\tau,\mu) \;+\mathcal{M}$$

$$\mathcal{M} \;=\frac{1}{2}(-1)^{k-2\ell+1}e^{2i\pi\frac{\mu^2 k}{\tau}}\sum_{a=1}^{k} q^{\frac{k+1}{4}(1-\frac{a}{k+1})^2}\,\widetilde{Ch}^{NS}_{k,\frac{a-1}{2}}(\tau,\mu)$$
$$\times\int d\alpha\, q^{\frac{k+1}{4}\alpha^2}\left\{\sum_{n=0}^{k-2\ell} e^{\alpha n\pi+i\pi n}\frac{e^{\alpha\pi}\sin\frac{na}{k+1}\pi+\sin\frac{(n+1)a}{k+1}\pi}{\cosh\pi\alpha+\cos\frac{a}{k+1}\pi}\right\}$$

$$\widetilde{Ch}^{NS}_{k,\ell}(q,z) = Ch^{NS}_{k,\ell}(q,z)+(z+z^{-1})Ch^{NS}_{k,\ell+1/2}(q,z)+Ch^{NS}_{k,\ell+1}(q,z)$$
$$= Ch^{NS}_{[h=\ell]}(q,z)$$

massive at threshold

Special case: Mordell Integral (k=1)

The S transform of the massless NS character with $\ell = \frac{1}{2}$ has

$$\mathcal{M} = \int d\alpha \frac{q^{\frac{\alpha^2}{2}+\frac{1}{8}}}{2\cosh \pi\alpha} \widetilde{Ch}^{NS}_{k=1,\ell'=0}(q,z) \quad e^{\frac{2i\pi\mu^2}{\tau}}$$

$$= \int d\alpha \frac{1}{2\cosh \pi\alpha} Ch^{NS}_{[h=\frac{\alpha^2}{2}+\frac{1}{8}]}(q,z) \; e^{\frac{2i\pi\mu^2}{\tau}}$$

$$= \frac{1}{\eta(q)} \int d\alpha \frac{q^{\frac{\alpha^2}{2}}}{2\cosh \pi\alpha} \quad \frac{\vartheta_3(q,z)^2}{\eta(q)^2} \, e^{\frac{2i\pi\mu^2}{\tau}}$$

$\downarrow$

Mordell integral

The Mordell integral is S invariant:

$$f(\tau) = \frac{1}{\eta(q)} \int d\alpha \frac{q^{\frac{\alpha^2}{2}}}{2\cosh \pi\alpha} = h_3(\tau) + h_3(-\frac{1}{\tau})$$

$$h_3(q) = \frac{1}{\eta(q)\vartheta_3(q)} \sum_{m \in Z} \frac{q^{\frac{m^2}{2}-\frac{1}{8}}}{1+q^{m-1/2}}$$

cf. Lerch sums

The function $h_3(q)$ plays a central role in any theory with N=4 superconformal symmetry at c=6 .
It is a building block of unitary hws massless characters.

5. The K3 elliptic genus and its decompactification

How can we modify the `failed' formula for elliptic genus in manifolds with A_{N-1} singularities? Start with the K3 elliptic genus:

$$Z_{K3}(\tau,z) = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2 + \left(\frac{\theta_2(\tau,z)}{\theta_2(\tau)}\right)^2\right]$$

$$\begin{aligned}
&Z_{K3}(\tau, z=0) = 24\,(\text{Euler Characteristic});\\
&Z_{K3}(\tau, z=\tfrac{1}{2}) = 16 + \ldots\,(\text{Signature});\\
&Z_{K3}(\tau, z=\tfrac{\tau}{2}) = -2q^{-1/4} + \ldots\,(\hat{A}\,\text{genus})
\end{aligned}$$

Rewrite it using $\mathcal{N}=4$ representation theory. At $\hat{c}=2(c=6)$, the theory contains SU(2) symmetry at level 1. The unitary reps in the NS sector are:

massive reps : $\quad \mathrm{ch}^{NS}(h;\tau,z) = q^{h-\frac{1}{8}}\frac{\theta_3(\tau,z)^2}{\eta^3(\tau)}$

massless reps : $\quad \mathrm{ch}_0^{NS}(\ell=0;\tau,z);\mathrm{ch}_0^{NS}(\ell=\tfrac{1}{2};\tau,z)$

relation : $\quad \mathrm{ch}_0^{NS}(\ell=0) + 2\,\mathrm{ch}_0^{NS}(\ell=\tfrac{1}{2}) = q^{-\frac{1}{8}}\frac{\theta_3^2}{\eta^3}$

(Eguchi-Taormina 88)

The characters may be rewritten in terms of 3 functions

$$h_2(\tau) = \frac{1}{\eta(\tau)\theta_2(\tau)} \sum_{m\in Z} \frac{q^{m^2/2+m/2}}{1+q^m}$$
$$h_3(\tau) = \frac{1}{\eta(\tau)\theta_3(\tau)} \sum_{m\in Z} \frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}$$
$$h_4(\tau) = \frac{1}{\eta(\tau)\theta_4(\tau)} \sum_{m\in Z} \frac{(-1)^m q^{m^2/2-1/8}}{1-q^{m-1/2}}$$

as

$$\begin{aligned} \mathrm{ch}_0^{NS}(\ell = \tfrac{1}{2}; \tau, z) &= -\left(\frac{\theta_1(z)}{\theta_3(0)}\right)^2 + h_3(\tau)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2 \\ &= \left(\frac{\theta_2(z)}{\theta_4(0)}\right)^2 + h_4(\tau)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2 \\ &= \left(\frac{\theta_4(z)}{\theta_2(0)}\right)^2 + h_2(\tau)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2 \end{aligned}$$

Since

$$q^{1/4}e^{-2i\pi z}\times Z_{K3}(z - \frac{1}{2}(\tau+1)) = 8\left[-\left(\frac{\theta_1(z)}{\theta_3(0)}\right)^2 + \left(\frac{\theta_2(z)}{\theta_4(0)}\right)^2 + \left(\frac{\theta_4(z)}{\theta_2(0)}\right)^2\right],$$ we get

$$q^{1/4}e^{-2i\pi z}\times Z_{K3}(z - \frac{1}{2}(\tau+1)) = 24\mathrm{ch}_0^{NS}(\ell = \frac{1}{2}; z) - 8\sum_{i=2,3,4} h_i(\tau)\left(\frac{\theta_3(z)}{\eta(\tau)}\right)^2$$

Note that $$8\eta(\tau) \sum_{i=2,3,4} h_i(\tau) = q^{-1/8}[2 - \sum_{n=1} a_n q^n]$$

where the integer coefficients a_n are all positive (Wendland,2000) so that we may write

$$Z_{K3}(z - \frac{1}{2}(\tau+1)) = 20\mathrm{ch}_0^{NS}(\ell = \frac{1}{2}; z) - 2\mathrm{ch}_0^{NS}(\ell = 0; z) + \sum_{n=1} a_n q^{n-1/8} \frac{\theta_3(z)^2}{\eta(\tau)^3}$$

The theory contains

- 1 $\ell = 0$ representation (gravity)
- 20 $\ell = \frac{1}{2}$ representations (matter)

IIA vector; IIB tensor

(Seiberg 88)

- ∞ of massive reps (h=1,2,...)

K3 may be decomposed into a sum of 16 A_1 ALE spaces (Page 78). The decompactification is achieved by decoupling gravity, i.e. by dropping the $\ell = 0$ massless rep. This suggests

$$Z_{K3,\mathrm{decompactified}}(\tau, z) = 8\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]$$

6. Elliptic genus for ALE spaces

Proposal 1: The elliptic genus for the A_1ALE space is

$$Z_{A_1}(\tau, z) = \frac{1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]$$

Proposal 2: The elliptic genus for the A_{N-1} ALE space is

$$Z_{A_{N-1}}(\tau, z) = \frac{N-1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]$$

Note that

$$Z_{A_1}(z - \tfrac{1}{2}(\tau+1)) = \mathrm{ch}_0^{NS}(\ell = \tfrac{1}{2}; z) - \tfrac{1}{2}\eta(\tau)\left[h_3(\tau) + h_4(\tau)\right]\frac{\theta_3(z)^2}{\eta(\tau)^3}$$

where the expansion

$$\tfrac{1}{2}\eta(\tau)\sum_{i=3,4} h_i(\tau) = -\sum_{n=1} b_n q^{n-1/8}$$

has all positive integer coefficients b_n.

The above construction suggests that instead of using the irreducible character ch_0^{NS} one should use the combination of massless/ive reps

$$\mathrm{ch}_0^{NS}(\ell=\tfrac{1}{2};z)-\tfrac{1}{2}\eta(\tau)\left[h_3(\tau)+h_4(\tau)\right]\frac{\theta_3(z)^2}{\eta(\tau)^3}=\frac{1}{2}\left[\left(\frac{\theta_2(z)}{\theta_4(\tau)}\right)^2-\left(\frac{\theta_1(z)}{\theta_3(\tau)}\right)^2\right]$$

which is invariant under the congruence subgroup $\Gamma(2)$.

We call this combination the $\Gamma(2)$-invariant completion of the massless rep. and consider it as a conformal block in non-compact CFT.

Can one, for a given representation of a superconformal algebra, always define the $\Gamma(2)$- invariant completion uniquely by adding a suitable amount of non-BPS reps?

Yes, if massive contributions have only integer q-powers in the R sector and their conformal dimensions are above the gap (h=1,2,...)

7. Merging the two approaches on elliptic genus

K3 decompactification approach:

$$Z_{ALE(A_{N-1})}(\tau, z) = \frac{N-1}{2}\left[\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\frac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right]$$

$\mathcal{N} = 2\,\text{minimal} \quad \times \mathcal{N} = 2\,\text{Liouville}$ approach: orbifoldization

$$Z_{ALE(A_{N-1})}(\tau, z) = \frac{1}{N}\sum_{a,b\in Z_N} q^{a^2} e^{4i\pi a z}(-1)^{a+b}\frac{\theta_1(\tau, \frac{N-1}{N}(z+a\tau+b))}{\theta_1(\tau, \frac{1}{N}(z+a\tau+b))}$$

$$\times\mathcal{K}_{2N}(\tau, \tfrac{1}{N}(z+a\tau+b))\frac{\theta_1(\tau,z)}{\eta^3(\tau)}$$

↓

use the $\Gamma(2)$ invariant completion of the Appell function

$$Z_{ALE(A_{N-1})}(\tau,z) = \tfrac{N-1}{2}\left[\left(\tfrac{\theta_3(\tau,z)}{\theta_3(\tau)}\right)^2 + \left(\tfrac{\theta_4(\tau,z)}{\theta_4(\tau)}\right)^2\right] =$$

remarkable identity

$$\tfrac{1}{4N}\sum_{a,b=1}^{N} q^{\frac{a^2}{2}} e^{2i\pi a z}(-1)^{a+b}\frac{\theta_1(\tau,\frac{N-1}{N}z_{a,b})\,\theta_1(\tau,\frac{2}{N}z_{a,b})\,\theta_1(\tau,z)}{\theta_1(\tau,\frac{1}{N}z_{a,b})^3}$$

$$\times\left[\left(\frac{\theta_3(\tau,\frac{1}{N}z_{a,b})}{\theta_3(\tau)}\right)^{2(N-1)} + \left(\frac{\theta_4(\tau,\frac{1}{N}z_{a,b})}{\theta_4(\tau)}\right)^{2(N-1)}\right]$$

with $z_{a,b} = z + a\tau + b$.

Don Zagier has given a very elegant proof of this identity (see appendix of our paper). This certainly reinforces the plausibility of our conjectured definition of elliptic genus for non-compact CYs.

8. Summary

When a CY manifold is non-compact, string theory is described by a CFT possessing continuous as well as discrete representations. Their characters transform under the S transform of the modular group in a way reminiscent of the behaviour of characters in LCFT:

$$\text{discrete} \ \stackrel{S}{\longrightarrow} \sum \text{discrete} \ + \int \ \text{continuous}$$

$$\text{continuous} \ \stackrel{S}{\longrightarrow} \int \text{continuous}$$

The deep meaning of such transformations is currently not well understood. We found an empirical rule to construct conformal blocks which behave `nicely' under the modular group and proposed a formula for the elliptic genera of some non-compact CY manifolds.