Logarithmic Conformal Field Theory

AM Semikhatov

Lebedev Physics Institute

Dubna Workshop on LCFTetc, June 2007

Logarithmic Conformal Field Theory: How far can we go with representation theory?

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Plan of the Talk

1. Motivation
2. Representation theory and CFT
3. Quantum groups

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2 Representation theory and CFT

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nondiagonalizable action of a number of operators of the type of a Hamiltonian

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Being logarithmic/nonsemisimple is a property of representations chosen
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2 Representation theory and CFT

3 Quantum groups

Rational models: basic representation-theory unput

- Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$
- Highest-weight modules $L_{n\geqslant 1}|\Delta\rangle = 0$, $L_0|\Delta\rangle = \Delta|\Delta\rangle$
  - Verma
  - irreducible
- *Rational* (p, p')-models at $c = 13 - \frac{p}{p'} - \frac{p'}{p}$:
  - Kac table of "good" modules: $\frac{1}{2}(p-1)\times(p'-1)$ nonisomorphic Virasoro irreps

  - *These irreps have no extensions among themselves* $\Longrightarrow$ semisimple (diagonalizable)
  - $\Longrightarrow$ chiral space of states $= \bigoplus$(irreps)
  - $\Longrightarrow$ numerous deep properties of RCFT…

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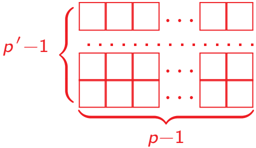

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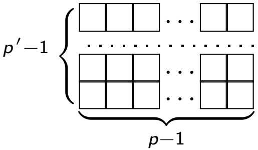

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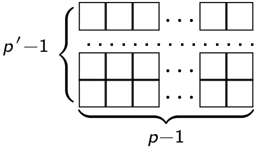

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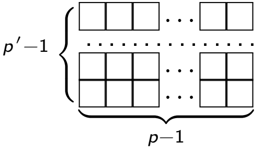

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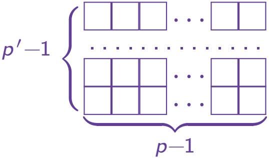

- Also, a new possibility: $(p,1)$ models with the extended Kac table

- Drastic consequences
  - Representations admit indecomposable extensions
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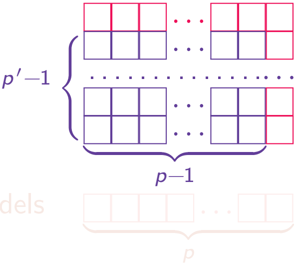

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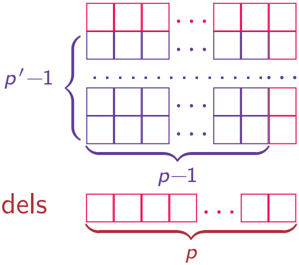

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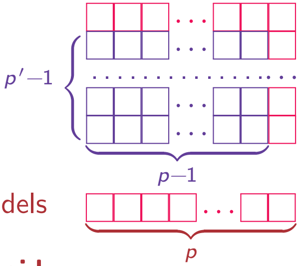

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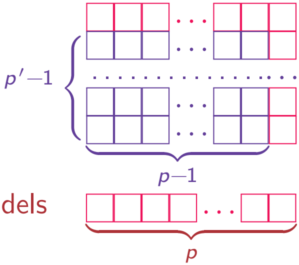

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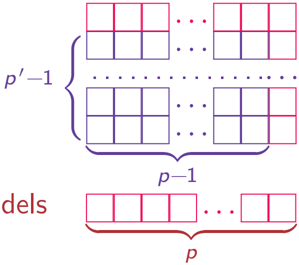

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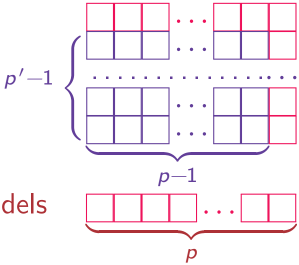

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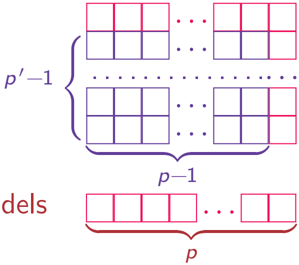

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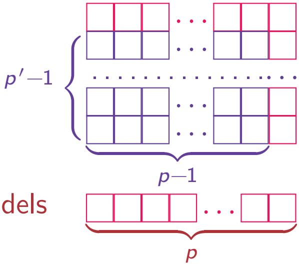

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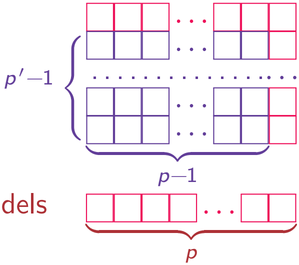

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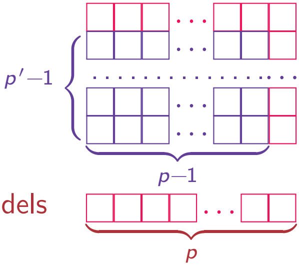

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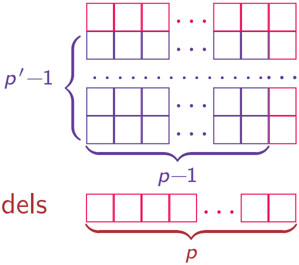

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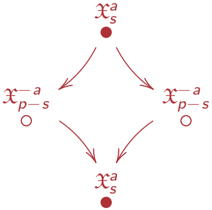

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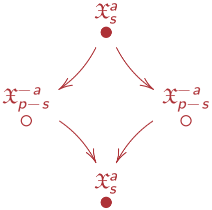

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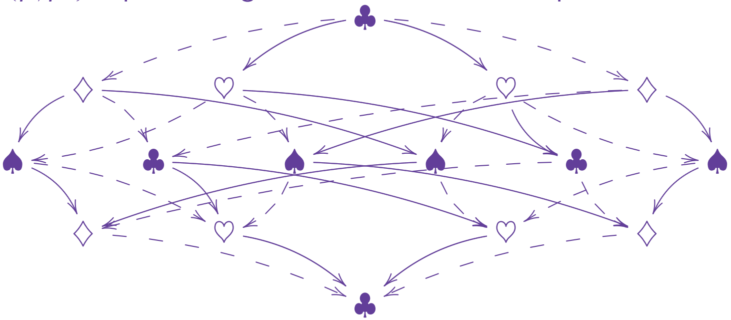

although involved in the true projective module, is insufficient.

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LCFTs in terms of free fields

1. Take screenings in a free-field realization: e.g., $S_{\pm} = \int e^{\alpha_{\pm}\varphi(z)}\,dz$
2. *rational* models are the *cohomology* of (the differential associated with) screenings
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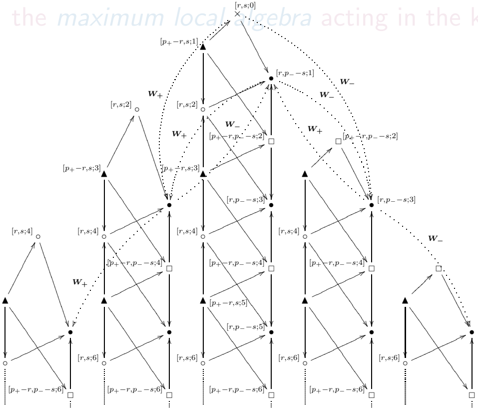

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W-algebra generators for $(3,2)$

$$W^+ = \Big(\tfrac{35}{27}(\partial^4\varphi)^2 + \tfrac{56}{27}\,\partial^5\varphi\,\partial^3\varphi + \tfrac{28}{27}\,\partial^6\varphi\,\partial^2\varphi + \tfrac{8}{27}\,\partial^7\varphi\,\partial\varphi - \tfrac{280}{9\sqrt{3}}(\partial^3\varphi)^2\,\partial^2\varphi$$
$$- \tfrac{70}{3\sqrt{3}}\,\partial^4\varphi\,(\partial^2\varphi)^2 - \tfrac{280}{9\sqrt{3}}\,\partial^4\varphi\,\partial^3\varphi\,\partial\varphi - \tfrac{56}{3\sqrt{3}}\,\partial^5\varphi\,\partial^2\varphi\,\partial\varphi - \tfrac{28}{9\sqrt{3}}\,\partial^6\varphi(\partial\varphi)^2$$
$$+ \tfrac{35}{3}(\partial^2\varphi)^4 + \tfrac{280}{3}\,\partial^3\varphi(\partial^2\varphi)^2\,\partial\varphi + \tfrac{280}{9}(\partial^3\varphi)^2(\partial\varphi)^2 + \tfrac{140}{3}\,\partial^4\varphi\,\partial^2\varphi(\partial\varphi)^2$$
$$+ \tfrac{56}{9}\,\partial^5\varphi(\partial\varphi)^3 - \tfrac{140}{\sqrt{3}}(\partial^2\varphi)^3(\partial\varphi)^2 - \tfrac{560}{3\sqrt{3}}\,\partial^3\varphi\,\partial^2\varphi\,(\partial\varphi)^2 - \tfrac{70}{3\sqrt{3}}\,\partial^4\varphi(\partial\varphi)^4$$
$$+ 70(\partial^2\varphi)^2(\partial\varphi)^4 + \tfrac{56}{3}\,\partial^3\varphi(\partial\varphi)^5 - \tfrac{28}{\sqrt{3}}\,\partial^2\varphi(\partial\varphi)^6 + (\partial\varphi)^8 - \tfrac{1}{27\sqrt{3}}\,\partial^8\varphi\Big)e^{2\sqrt{3}\varphi},$$

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W-algebra generators for $(3,2)$

$$
\begin{aligned}
W^{-} = \Bigl(& \tfrac{217}{192}(\partial^5\varphi)^2 - \tfrac{2653}{3456}\,\partial^6\varphi\,\partial^4\varphi - \tfrac{23}{384}\,\partial^7\varphi\,\partial^3\varphi - \tfrac{11}{1152}\,\partial^8\varphi\,\partial^2\varphi - \tfrac{1}{768}\,\partial^9\varphi\,\partial\varphi - \tfrac{1225}{64\sqrt{3}}\,\partial^4\varphi(\partial^3\varphi)^2 \\
& - \tfrac{13475}{576\sqrt{3}}(\partial^4\varphi)^2\,\partial^2\varphi + \tfrac{2695}{64\sqrt{3}}\,\partial^5\varphi\,\partial^3\varphi\,\partial^2\varphi + \tfrac{2555}{192\sqrt{3}}\,\partial^5\varphi\,\partial^4\varphi\,\partial\varphi - \tfrac{2891}{576\sqrt{3}}\,\partial^6\varphi(\partial^2\varphi)^2 - \tfrac{1351}{192\sqrt{3}}\,\partial^6\varphi\,\partial^3\varphi\,\partial\varphi \\
& - \tfrac{103}{192\sqrt{3}}\,\partial^7\varphi\,\partial^2\varphi\,\partial\varphi - \tfrac{13}{384\sqrt{3}}\,\partial^8\varphi(\partial\varphi)^2 + \tfrac{3535}{32}(\partial^3\varphi)^2(\partial^2\varphi)^2 - \tfrac{735}{16}(\partial^3\varphi)^3\,\partial\varphi - \tfrac{3395}{54}\,\partial^4\varphi(\partial^2\varphi)^3 \\
& + \tfrac{245}{24}\,\partial^4\varphi\,\partial^3\varphi\,\partial^2\varphi\,\partial\varphi + \tfrac{12635}{576}(\partial^4\varphi)^2(\partial\varphi)^2 + \tfrac{245}{12}\,\partial^5\varphi(\partial^2\varphi)^2\,\partial\varphi + \tfrac{105}{32}\,\partial^5\varphi\,\partial^3\varphi(\partial\varphi)^2 \\
& - \tfrac{2443}{288}\,\partial^6\varphi\,\partial^2\varphi(\partial\varphi)^2 - \tfrac{19}{96}\,\partial^7\varphi(\partial\varphi)^3 - \tfrac{13405}{144\sqrt{3}}(\partial^2\varphi)^5 + \tfrac{8225}{24\sqrt{3}}\,\partial^3\varphi(\partial^2\varphi)^3\,\partial\varphi - \tfrac{105\sqrt{3}}{4}(\partial^3\varphi)^2\,\partial^2\varphi(\partial\varphi)^2 \\
& + \tfrac{665}{24\sqrt{3}}\,\partial^4\varphi(\partial^2\varphi)^2(\partial\varphi)^2 + \tfrac{245}{2\sqrt{3}}\,\partial^4\varphi\,\partial^3\varphi(\partial\varphi)^3 - \tfrac{245}{8\sqrt{3}}\,\partial^5\varphi\,\partial^2\varphi(\partial\varphi)^3 - \tfrac{91}{24\sqrt{3}}\,\partial^6\varphi(\partial\varphi)^4 + \tfrac{16205}{144}(\partial^2\varphi)^4(\partial\varphi)^2 \\
& + \tfrac{385}{4}\,\partial^3\varphi(\partial^2\varphi)^2(\partial\varphi)^3 + \tfrac{525}{8}(\partial^3\varphi)^2(\partial\varphi)^4 + \tfrac{35}{3}\,\partial^4\varphi\,\partial^2\varphi(\partial\varphi)^4 - 7\,\partial^5\varphi(\partial\varphi)^5 + \tfrac{665}{3\sqrt{3}}(\partial^2\varphi)^3(\partial\varphi)^4 \\
& + \tfrac{105\sqrt{3}}{2}\,\partial^3\varphi\,\partial^2\varphi(\partial\varphi)^5 - \tfrac{35}{3\sqrt{3}}\,\partial^4\varphi(\partial\varphi)^6 + \tfrac{455}{6}(\partial^2\varphi)^2(\partial\varphi)^6 + 5\,\partial^3\varphi(\partial\varphi)^7 + \tfrac{25}{\sqrt{3}}\,\partial^2\varphi(\partial\varphi)^8 \\
& + (\partial\varphi)^{10} - \tfrac{1}{13824\sqrt{3}}\,\partial^{10}\varphi\Bigr)\,e^{-2\sqrt{3}\varphi},
\end{aligned}
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Key differences from the rational case

- The LCFT model may be dependent on the free-field representation taken, on the screenings chosen, etc.
- The *symmetry algebra* of a LCFT model is larger than the "naive" algebra (e.g., Virasoro).
- The *space of states* in a LCFT is *not* the direct sum of irreducible representations but the sum of all (finitely many) projective modules

$$\mathbb{P} = \bigoplus_{\iota} \mathfrak{P}_\iota.$$

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$\mathcal{W}$-algebras and their representations

- $(p,1)$ MODELS: the *triplet* W-algebra $\mathcal{W}_p$
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$$\chi_r^+(q) = \frac{1}{\eta(q)}\Big(\frac{r}{p}\,\theta_{p-r,p}(q) + \frac{2}{p}\,\theta'_{p-r,p}(q)\Big),$$
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$\mathcal{W}$-Characters $\Longrightarrow$ Modular Group Representation

The need for *generalized characters*:

In LCFT, characters alone are not closed under $SL(2,\mathbb{Z})$ action

- $(p,1)$ MODELS: The $2p$ characters give rise to a $(3p-1)$-dimensional $SL(2,\mathbb{Z})$-representation.
- (p,p') MODELS: The $2pp' + \frac{1}{2}(p-1)(p'-1)$ characters give rise to a $\frac{1}{2}(3p-1)(3p'-1)$-dimensional $SL(2,\mathbb{Z})$-representation.

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It is highly probable that these dimensions $3p-1$ and $\frac{1}{2}(3p-1)(3p'-1)$ are the dimensions of the spaces of torus amplitudes.

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The $(3p-1)$-dimension $SL(2,\mathbb{Z})$-representation $\mathfrak{Z}_{\mathrm{cft}}$ has the structure

$$\mathfrak{Z}_{\mathrm{cft}} = \mathcal{R}_{p+1} \oplus \mathbb{C}^2 \otimes \mathcal{R}_{p-1},$$

$\mathcal{R}_{p-1}$ is the ["$\sin\frac{\pi rs}{p}$"] $SL(2,\mathbb{Z})$-representation realized in the $\widehat{s\ell}(2)_{p-2}$ minimal model, $\mathcal{R}_{p+1}$ is a ["$\cos\frac{\pi rs}{p}$"] $SL(2,\mathbb{Z})$-representations of dimension $p+1$, and $\mathbb{C}^2$ is the defining two-dimensional representation of $SL(2,\mathbb{Z})$.

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- $(p, 1)$ MODELS: The $2p$ characters give rise to a $(3p-1)$-dimensional $SL(2,\mathbb{Z})$-representation.
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Theorem

The $\frac{1}{2}(3p-1)(3p'-1)$-dimensional $SL(2,\mathbb{Z})$-representation $\mathfrak{Z}_{\mathrm{cft}}$ has the structure

$$\mathfrak{Z}_{\mathrm{cft}} = R_{\mathrm{min}} \oplus R_{\mathrm{proj}} \oplus \mathbb{C}^2 \otimes \left(R_{\boxslash} \oplus R_{\boxbslash}\right) \oplus \mathbb{C}^3 \otimes R_{\mathrm{min}},$$

R_{min} is the $\frac{1}{2}(p-1)(p'-1)$-dimensional $SL(2,\mathbb{Z})$-representation on the characters of the (p, p') Virasoro minimal model, $\mathbb{C}^3 \simeq \mathrm{S}^2(\mathbb{C}^2)$, and R_{proj}, $R_{\boxslash}$, and $R_{\boxbslash}$ are $SL(2,\mathbb{Z})$-representations of the respective dimensions $\frac{1}{2}(p+1)(p'+1)$, $\frac{1}{2}(p-1)(p'+1)$, and $\frac{1}{2}(p+1)(p'-1)$.

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It is highly probable that these dimensions $3p-1$ **and** $\frac{1}{2}(3p-1)(3p'-1)$ **are the dimensions of the spaces of torus amplitudes**.

Some details for (p, p')

Generalized characters:

subrep.	dimension	basis
R_{min}	$\frac{1}{2}(p-1)(p'-1)$	$\chi_{r,r'}$, $(r,r') \in \mathcal{J}_1$
R_{proj}	$\frac{1}{2}(p+1)(p'+1)$	$\varkappa_{r,r'}$, $(r,r') \in \mathcal{J}_0$
$\mathbb{C}^2 \otimes R_{\boxslash}$	$2 \cdot \frac{1}{2}(p-1)(p'+1)$	$\rho^{\boxslash}_{r,r'}$, $\varphi^{\boxslash}_{r,r'}$, $(r,r') \in \mathcal{J}_{\boxslash}$
$\mathbb{C}^2 \otimes R_{\boxbslash}$	$2 \cdot \frac{1}{2}(p+1)(p'-1)$	$\rho^{\boxbslash}_{r,r'}$, $\varphi^{\boxbslash}_{r,r'}$, $(r,r') \in \mathcal{J}_{\boxbslash}$
$\mathbb{C}^3 \otimes R_{\mathrm{min}}$	$3 \cdot \frac{1}{2}(p-1)(p'-1)$	$\rho_{r,r'}$, $\psi_{r,r'}$, $\varphi_{r,r'}$, $(r,r') \in \mathcal{J}_1$

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$\mathbb{C}^2 \otimes R_{\boxslash}$	$2 \cdot \frac{1}{2}(p-1)(p'+1)$	$\rho^{\boxslash}_{r,r'}$, $\varphi^{\boxslash}_{r,r'}$, $(r,r') \in \mathcal{I}_{\boxslash}$
$\mathbb{C}^2 \otimes R_{\boxbslash}$	$2 \cdot \frac{1}{2}(p+1)(p'-1)$	$\rho^{\boxbslash}_{r,r'}$, $\varphi^{\boxbslash}_{r,r'}$, $(r,r') \in \mathcal{I}_{\boxbslash}$
$\mathbb{C}^3 \otimes R_{\min}$	$3 \cdot \frac{1}{2}(p-1)(p'-1)$	$\rho_{r,r'}$, $\psi_{r,r'}$, $\varphi_{r,r'}$, $(r,r') \in \mathcal{I}_1$

$$\begin{aligned}
\varkappa_{r,r'} &= \chi_{r,r'} + 2\chi^{+}_{r,r'} + 2\chi^{-}_{r,p'-r'} + 2\chi^{-}_{p-r,r'} + 2\chi^{+}_{p-r,p'-r'}, && (r,r') \in \mathcal{I}_1,\\
\varkappa_{0,r'} &= 2\chi^{+}_{p,p'-r'} + 2\chi^{-}_{p,r'}, && 1 \leqslant r' \leqslant p'-1,\\
\varkappa_{r,0} &= 2\chi^{+}_{p-r,p'} + 2\chi^{-}_{r,p'}, && 1 \leqslant r \leqslant p-1,\\
\varkappa_{0,0} &= 2\chi^{+}_{p,p'},\\
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$\mathbb{C}^3 \otimes R_{\min}$	$3 \cdot \frac{1}{2}(p-1)(p'-1)$	$\rho_{r,r'}$, $\psi_{r,r'}$, $\varphi_{r,r'}$, $(r,r') \in \mathcal{I}_1$

$$\rho^{\boxslash}_{r,r'}(\tau) = \frac{p'r-pr'}{2}\chi_{r,r'}(\tau) + p'(r-p)(\chi^{+}_{r,r'}(\tau) + \chi^{-}_{r,p'-r'}(\tau)) + p'r(\chi^{+}_{p-r,p'-r'}(\tau) + \chi^{-}_{p-r,r'}(\tau)), \qquad (r,r') \in \mathcal{I}_1,$$

$$\rho^{\boxslash}_{r,0}(\tau) = p'(r\chi^{+}_{p-r,p'}(\tau) - (p-r)\chi^{-}_{r,p'}(\tau)), \qquad 1 \leqslant r \leqslant p-1,$$

$$\varphi^{\boxslash}_{r,r'}(\tau) = \tau\rho^{\boxslash}_{r,r'}(\tau), \qquad (r,r') \in \mathcal{I}_{\boxslash},$$

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$\mathbb{C}^3 \otimes R_{\min}$	$3 \cdot \frac{1}{2}(p-1)(p'-1)$	$\rho_{r,r'}$, $\psi_{r,r'}$, $\varphi_{r,r'}$, $(r,r') \in \mathcal{I}_1$

$$\begin{aligned}
\rho^{\boxbslash}_{r,r'}(\tau) &= \tfrac{pr'-p'r}{2}\chi_{r,r'}(\tau) - p(p'-r')(\chi^{+}_{r,r'}(\tau) + \chi^{-}_{p-r,r'}(\tau)) \\
&\quad + pr'(\chi^{+}_{p-r,p'-r'}(\tau) + \chi^{-}_{r,p'-r'}(\tau)), && (r,r') \in \mathcal{I}_1, \\
\rho^{\boxbslash}_{0,r'}(\tau) &= p(r'\chi^{+}_{p,p'-r'}(\tau) - (p'-r')\chi^{-}_{p,r'}(\tau)), && 1 \leqslant r' \leqslant p'-1, \\
\varphi^{\boxbslash}_{r,r'}(\tau) &= \tau\rho^{\boxbslash}_{r,r'}(\tau), && (r,r') \in \mathcal{I}_{\boxbslash},
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$$\rho_{r,r'}(\tau) = pp'\Big((p-r)(p'-r')\chi^{+}_{r,r'}(\tau) + rr'\chi^{+}_{p-r,p'-r'}(\tau) - \frac{(pr'-p'r)^2}{4pp'}\chi_{r,r'}(\tau) - (p-r)r'\chi^{-}_{r,p'-r'}(\tau) - r(p'-r')\chi^{-}_{p-r,r'}(\tau)\Big), \qquad (r,r') \in \mathcal{I}_1$$

$$\psi_{r,r'}(\tau) = 2\tau\rho_{r,r'}(\tau) + i\pi pp'\chi_{r,r'}(\tau), \qquad (r,r') \in \mathcal{I}_1,$$

$$\varphi_{r,r'}(\tau) = \tau^2\rho_{r,r'}(\tau) + i\pi pp'\tau\chi_{r,r'}(\tau), \qquad (r,r') \in \mathcal{I}_1.$$

Corollary: Several modular invariants

- involving τ explicitly:

$$\rho^{\boxslash}(\tau,\bar{\tau}) = \sum_{r=1}^{p-1} \operatorname{im}\tau\, |\rho^{\boxslash}_{r,0}(\tau)|^2 + 2 \sum_{(r,r')\in\mathcal{I}_1} \operatorname{im}\tau\, |\rho^{\boxslash}_{r,r'}(\tau)|^2,$$

$$\begin{aligned}\rho(\tau,\bar{\tau}) = \sum_{(r,r')\in\mathcal{I}_1} & \bar{\rho}_{r,r'}(\bar{\tau})\big(8(\operatorname{im}\tau)^2\rho_{r,r'}(\tau) + 4pp'\pi \operatorname{im}\tau\, \chi_{r,r'}(\tau)\big) \\ & + \bar{\chi}_{r,r'}(\bar{\tau})\big(4pp'\pi \operatorname{im}\tau\, \rho_{r,r'}(\tau) + (\pi pp')^2 \chi_{r,r'}(\tau)\big).\end{aligned}$$

Corollary: Several modular invariants

- *A*-series:

$$\varkappa_{[A]}(\tau,\overline{\tau}) = $$

$$= |\varkappa_{0,0}(\tau)|^2 + |\varkappa_{p,0}(\tau)|^2 + 2\sum_{r=1}^{p-1} |\varkappa_{r,0}(\tau)|^2 + 2\sum_{r'=1}^{p'-1} |\varkappa_{0,r'}(\tau)|^2 + 4\sum_{(r,r')\in\mathcal{I}_1} |\varkappa_{r,r'}(\tau)|^2$$

- *D*-series (in the case $p' \equiv 0 \bmod 4$):

$$\varkappa_{[D]}(\tau,\overline{\tau}) = |\varkappa_{0,0}(\tau) + \varkappa_{p,0}(\tau)|^2 + \sum_{r=1}^{p-1} |\varkappa_{r,0}(\tau) + \varkappa_{p-r,0}(\tau)|^2$$

$$+ \sum_{\substack{2\leqslant r'\leqslant p'-1\\ r' \text{ even}}} |\varkappa_{0,r'}(\tau) + \varkappa_{0,p'-r'}(\tau)|^2 + \sum_{\substack{(r,r')\in\mathcal{I}_1\\ r' \text{ even}}} 2|\varkappa_{r,r'}(\tau) + \varkappa_{r,p'-r'}(\tau)|^2.$$

- E_6-type invariant for $(p,p') = (5,12)$:

$$\varkappa_{[E_6]}(\tau,\overline{\tau}) = |\varkappa_{0,1}(\tau) - \varkappa_{0,7}(\tau)|^2 + |\varkappa_{0,2}(\tau) - \varkappa_{0,10}(\tau)|^2 + |\varkappa_{0,5}(\tau) - \varkappa_{0,11}(\tau)|^2$$

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1 Motivation

2 Representation theory and CFT

3 Quantum groups

Quantum groups: Kazhdan–Lusztig correspondence

THE SAME $SL(2,\mathbb{Z})$ REPRESENTATIONS ARE REALIZED ON CENTERS OF THE CORRESPONDING QUANTUM GROUPS

Quantum groups: Kazhdan–Lusztig correspondence

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- Screenings $\Longrightarrow$ quantum group $\mathfrak{g}$ ("Kazhdan–Lusztig-dual")
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- Center $\mathfrak{Z}$
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- Quantum group $\mathfrak{g}$ is ribbon and factorizable $\Longrightarrow$ *its center carries an $SL(2,\mathbb{Z})$ representation*

Quantum groups: Kazhdan–Lusztig correspondence

THE SAME $SL(2,\mathbb{Z})$ REPRESENTATIONS ARE REALIZED ON CENTERS OF THE CORRESPONDING QUANTUM GROUPS

- Screenings $\Longrightarrow$ quantum group $\mathfrak{g}$ ("Kazhdan–Lusztig-dual")
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Theorem

This $SL(2,\mathbb{Z})$-representation on $\mathfrak{Z}$ coincides with the $SL(2,\mathbb{Z})$-representation generated by the LCFT characters

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The quantum group knows surprisingly much about the LCFT

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Anything else?

More on KL: Grothendieck rings/Fusion

The "$(p,1)$" quantum group has $2p$ irreps $\mathcal{X}_r^{\pm}$, $1 \leqslant r \leqslant p$. Grothendieck ring:

$$\mathcal{X}_r^{\alpha}\,\mathcal{X}_s^{\alpha'} = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \widetilde{\mathcal{X}}_t^{\alpha\alpha'},$$

$$\widetilde{\mathcal{X}}_r^{\alpha} = \begin{cases} \mathcal{X}_r^{\alpha}, & 1 \leqslant r \leqslant p,\\ \mathcal{X}_{2p-r}^{\alpha} + 2\mathcal{X}_{r-p}^{-\alpha}, & p+1 \leqslant r \leqslant 2p-1. \end{cases}$$

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This nonsemisimple algebra $\mathfrak{G}_{2p}$ contains the ideal $\mathfrak{V}_{p+1}$ of projective modules; the quotient $\mathfrak{G}_{2p}/\mathfrak{V}_{p+1}$ is a *semisimple* fusion algebra — the fusion of the unitary $\widehat{s\ell}(2)$ representations of level $k = p-2$:

$$\overline{\mathcal{X}}_r\,\overline{\mathcal{X}}_s = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{p-1-|p-r-s|} \overline{\mathcal{X}}_t, \quad r,s = 1,\dots,p-1.$$

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CORROBORATED BY a derivation from characters [FHST (2003)]

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1. This algebra is generated by two elements $\mathcal{X}^{+}_{1,2}$ and $\mathcal{X}^{+}_{2,1}$;
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Related to fusion in (p, p') LCFT models?!

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CORROBORATED BY computer calculations [EF (2006)]

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Example: $(2,3)$ model

The $2pp' = 12$ representations:

$\mathcal{X}^+_{1,1}$ (2), $\mathcal{X}^-_{1,1}$ (7) $\mathcal{X}^+_{2,1}$ (1), $\mathcal{X}^-_{2,1}$ (5) $\mathcal{X}^+_{3,1}$ $(\frac{1}{3})$, $\mathcal{X}^-_{3,1}$ $(\frac{10}{3})$

$\mathcal{X}^+_{1,2}$ $(\frac{5}{8})$, $\mathcal{X}^-_{1,2}$ $(\frac{33}{8})$ $\mathcal{X}^+_{2,2}$ $(\frac{1}{8})$, $\mathcal{X}^-_{2,2}$ $(\frac{21}{8})$ $\mathcal{X}^+_{3,2}$ $(-\frac{1}{24})$, $\mathcal{X}^-_{3,2}$ $(\frac{35}{24})$

$\mathcal{X}^+_{1,2}\mathcal{X}^+_{1,2} = 2\mathcal{X}^-_{1,1} + 2\mathcal{X}^+_{1,1}$, $\mathcal{X}^+_{1,2}\mathcal{X}^+_{2,1} = \mathcal{X}^+_{2,2}$, $\mathcal{X}^+_{1,2}\mathcal{X}^+_{2,2} = 2\mathcal{X}^-_{2,1} + 2\mathcal{X}^+_{2,1}$,

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$\mathcal{X}^+_{2,2}\mathcal{X}^+_{2,2} = 2\mathcal{X}^-_{1,1} + 2\mathcal{X}^-_{3,1} + 2\mathcal{X}^+_{1,1} + 2\mathcal{X}^+_{3,1}$, $\mathcal{X}^+_{2,2}\mathcal{X}^+_{3,1} = 2\mathcal{X}^-_{1,2} + 2\mathcal{X}^+_{2,2}$,

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$\mathcal{X}^-_{1,1}$ acts as a simple current, $\mathcal{X}^-_{1,1}\mathcal{X}^{\pm}{}_{r,r'} = \mathcal{X}^{\mp}{}_{r,r'}$.

Beyond the Grothendieck ring

- The *TRUE* tensor algebra of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$-representations [K Erdmann et al]:

$r + s - p \leqslant 1$, then

$$\mathcal{X}_r^{\alpha} \otimes \mathcal{X}_s^{\beta} = \bigoplus_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \mathcal{X}_t^{\alpha\beta} \quad (\min(r,s) \text{ terms in the sum}).$$

$r + s - p \geqslant 2$, even: $r + s - p = 2n$ with $n \geqslant 1$, then

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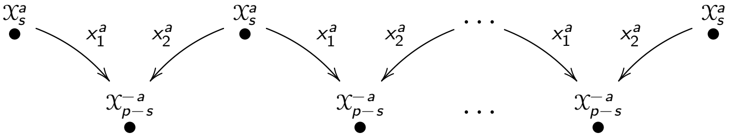
Indecomposable modules: $\mathcal{W}_s^a(n)$ and $\mathcal{M}_s^a(n)$

$1 \leqslant s \leqslant p-1$, $a = \pm$, and $n \geqslant 2$, module $\mathcal{W}_s^a(n)$:



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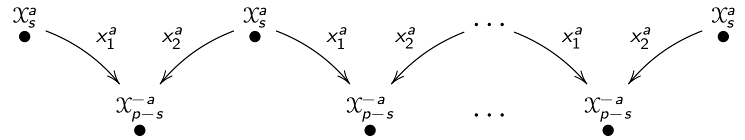

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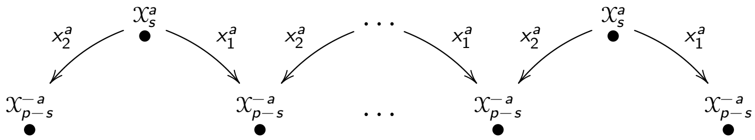
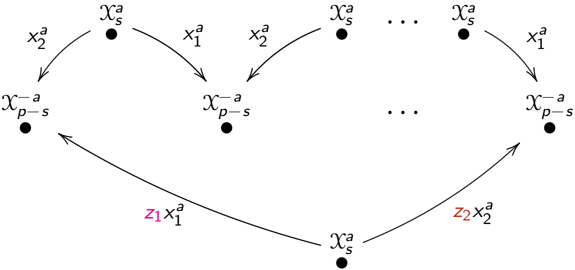


Indecomposable modules: $\mathcal{O}^{a}_{s}(n, z)$

$1 \leqslant s \leqslant p-1$, $a = \pm$, $n \geqslant 1$, and $z \in \mathbb{CP}^1$, module $\mathcal{O}^{a}_{s}(n, z)$:



$\mathbb{CP}^1 \ni z = (z_1 : z_2)$

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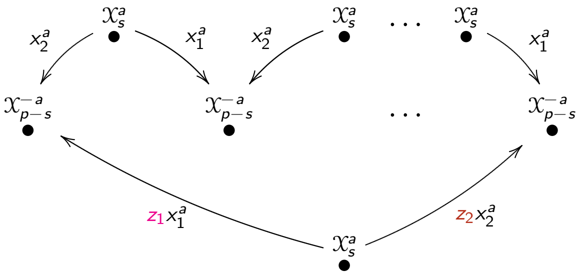

$$\mathbb{CP}^1 \ni z = (z_1 : z_2)$$

Summary

We know

- extended symmetry of the model: W-algebra
- W-algebra irreps
- Quantum-group projective modules
- Quantum-group Grothendieck ring/"fusion"
- $\mathfrak{Z}_{\mathrm{CFT}}$: Characters and generalized characters, modular transformations on $\mathfrak{Z}_{\mathrm{CFT}}$
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EVEN THOUGH I WAS CHEATING!!

Thank You :)

4 More

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From Free Fields to the Quantum Group

Consider the $(p,1)$ case. Screening: $E = \oint e^{-\sqrt{\frac{2}{p}}\varphi}$.

The $\mathcal{W}_p$ algebra:

$$W^-(z) = e^{-\sqrt{2p}\varphi(z)},\ W^0(z) = [S_+, W^-(z)],\ W^+(z) = [S_+, W^0(z)],$$

where $S_+ = \oint e^{\sqrt{2p}\varphi}$. The $W^{\pm,0}(z)$ are primary fields of dimension $2p-1$ with respect to the energy-momentum tensor

$$T(z) = \frac{1}{2}\partial\varphi\,\partial\varphi(z) + \left(\sqrt{2p} - \sqrt{\frac{2}{p}}\right)\partial^2\varphi(z).$$

On a suitably defined free-field space $\mathcal{F}$,

$$\mathrm{Ker}\, E\,\Big|_{\mathcal{F}} = \bigoplus_{r=1}^{p} \mathfrak{X}_r^+ \oplus \mathfrak{X}_r^-,$$

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Consider the $(p,1)$ case. Screening: $E = \oint e^{-\sqrt{\frac{2}{p}}\varphi}$.

The $\mathcal{W}_p$ algebra:

$$W^-(z) = e^{-\sqrt{2p}\varphi(z)},\ W^0(z) = [S_+, W^-(z)],\ W^+(z) = [S_+, W^0(z)],$$

where $S_+ = \oint e^{\sqrt{2p}\varphi}$. The $W^{\pm,0}(z)$ are primary fields of dimension $2p-1$ with respect to the energy-momentum tensor

$$T(z) = \frac{1}{2}\,\partial\varphi\,\partial\varphi(z) + \left(\sqrt{2p} - \sqrt{\frac{2}{p}}\right)\partial^2\varphi(z).$$

On a suitably defined free-field space $\mathcal{F}$,

$$\mathrm{Ker}\, E\,\Big|_{\mathcal{F}} = \bigoplus_{r=1}^{p} \mathfrak{X}_r^+ \oplus \mathfrak{X}_r^-,$$

a sum of $2p$ $\mathcal{W}_p$-representations.