Extensions of Tempesley-Lieb algebra & CFT

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- Continuum and lattice models are different.
- Connection is rather subtle
- Give examples where there is an algebraic structure on lattice.

Lattice Rep$^{\text{ns}}$	⟷	Continuum CFT rep$^{\text{ns}}$
• Null vectors • Generically irreducible		• Minimal models

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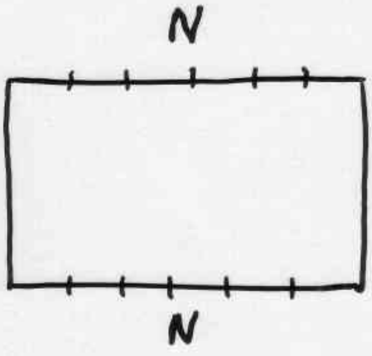
J. Stat. Mech (2005) P03003
" " (2006) P01003
" " (2006) L02004

math.RT/0703338.

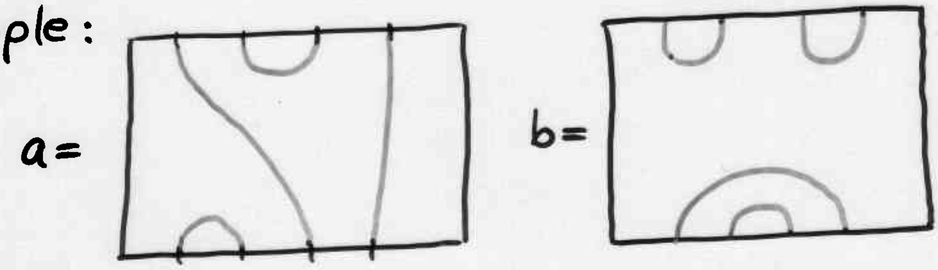
Temperley-Lieb diagram algebra:

(Temperley-Lieb '71)
Jones '80's

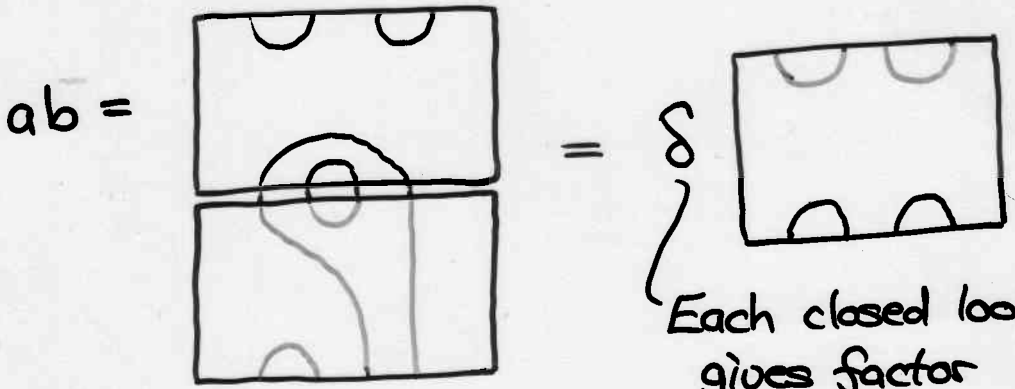


- Connect all points with non-intersecting arcs.
- Consider diagrams equivalent up to ISOTOPY.

Example:



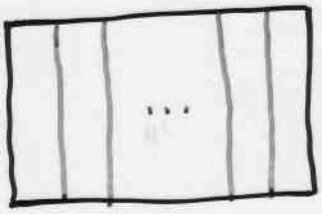
Multiplication:



Each closed loop gives factor δ.

Identity element:

$\mathbb{1} =$



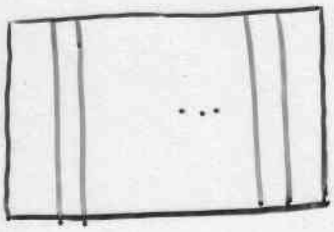
This is $TL_N(\delta)$ algebra.

- There are only $\underbrace{\frac{1}{N+1}\binom{2N}{N}}_{\text{Catalan number}}$ diagrams.

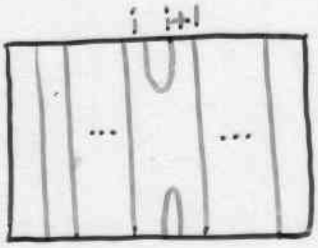
Algebraic description

Diagrams are generated by:

$\mathbb{1} =$



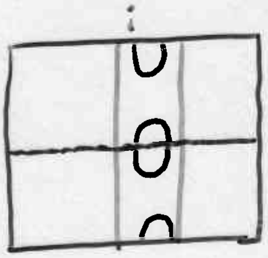
$e_i =$



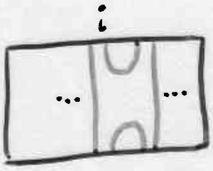
$1 \leq i \leq N-1$

Relations:

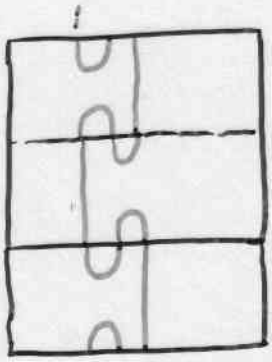
$e_i^2 = \delta e_i$



$= \delta$



$e_i e_{i\pm1} e_i = e_i$



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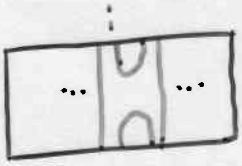
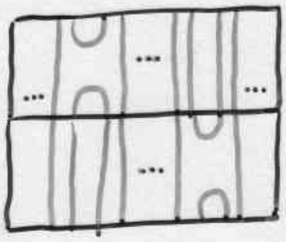


$e_i e_j = e_j e_i \quad |i-j|>1$



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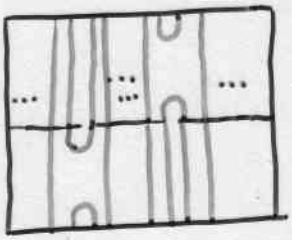


$TL_N(\delta)$.

Integrable models:

Yang-Baxter equation:-

$$R_i(u)R_{i+1}(u+v)R_i(v) = R_{i+1}(v)R_{i}(u+v)R_{i+1}(u)$$

$$R_i(u)\,R_j(v) = R_j(v)R_i(u) \qquad |i-j|>1$$

$$R_i(u)R_i(-u) \propto \mathbb{1}$$

Factorisable Scattering

A generic solution (i.e rep^n independent) is:

$$R_i(u) = e_i - \frac{[u+1]}{[u]} \qquad \text{where } \delta = q+q^{-1}$$

$$\& \quad [n] = \frac{q^n-q^{-n}}{q-q^{-1}}.$$

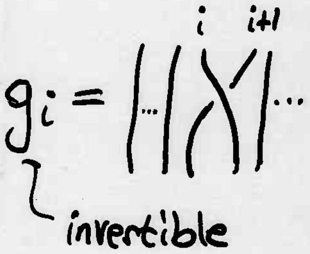
Standard proceduce

$$\text{Transfer matrix} \xrightarrow[\text{expansion in powers } u]{} \text{Conserved quantities}$$

Simplest: $\mathcal{H}^{TL} = -\sum_{i=1}^{N-1} e_i$

Braid group and Hecke algebra:

In $u, v \longrightarrow \infty$ limit Yang-Baxter eq^n becomes braid group:-



$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

$$g_i g_j = g_j g_i \qquad |i-j| > 1$$

Given $TL_N(\delta)$ with $\delta = q + q^{-1}$ then:-

$g_i = e_i - q^{-1}$ satisfy braid relations and:

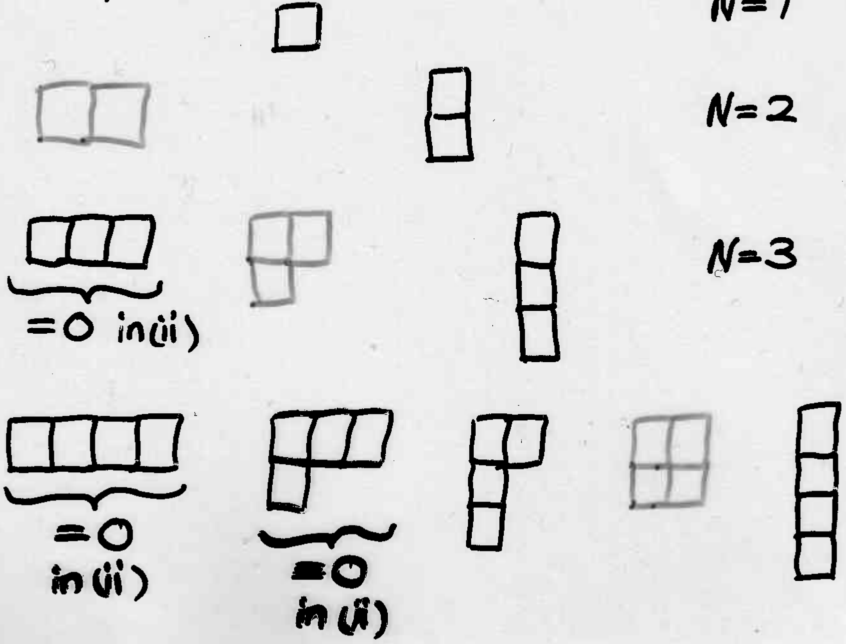
i) $(g_i - q)(g_i + q^{-1}) = 0$ Hecke algebra

ii) $g_i g_{i+1} g_i + q^{-1} g_i g_{i+1} + q^{-1} g_{i+1} g_i + q^{-2} g_i + q^{-2} g_{i+1} + q^{-3} = 0.$

The Hecke algebra is a deformation of Symmetric group $(q=1)$.

Irreps of Symmetric Group ~ Irreps of Hecke for "generic" q.

Young diagrams:



N=4.

ii) $\Rightarrow$ $\boxed{}\!\boxed{}\!\boxed{} = 0$.

Example: XXZ model.

$$\underbrace{\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\otimes\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}}_{N\text{ sites}}$$

$$e_i = -\tfrac{1}{2}\left[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cos\gamma\,\sigma_i^z\sigma_{i+1}^z - \cos\gamma + i\sin\gamma(\sigma_i^z - \sigma_{i+1}^z)\right]$$

- For $1 \leqslant i \leqslant N-1$ these obey $TL_N(\delta)$ with $\delta = 2\cos\gamma$.

$$H^{qg} = -\sum_{i=1}^{N-1} e_i = \tfrac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cos\gamma\,\sigma_i^z\sigma_{i+1}^z - \cos\gamma\right)$$
$$+\tfrac{1}{2}i\sin\gamma\,(\sigma_1^z - \sigma_N^z)$$

Quantum group symmetry: $q = e^{i\gamma}$

$$q^{\pm S^z} = q^{\pm\frac{1}{2}\sigma^z}\otimes\cdots\otimes q^{-\frac{1}{2}\sigma^z}$$

$$S^\pm = \sum_i q^{\frac{1}{2}\sigma^z}\otimes\cdots\otimes q^{\frac{1}{2}\sigma^z}\otimes\sigma_i^\pm\otimes q^{-\frac{1}{2}\sigma^z}\otimes\cdots\otimes q^{-\frac{1}{2}\sigma^z}$$

These generate $SU_q(2)$:-

$$q^{S^z}S^\pm q^{-S^z} = q^{\pm1}qS^\pm$$

$$[S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}$$

and:

$$[SU_q(2), e_i] = 0.$$

- Spectrum of H^{qg} can be found using Bethe Ansatz $|\uparrow\cdots\uparrow\rangle$ & $|\downarrow\cdots\downarrow\rangle$ - good ref. states.

3-state Potts model

$$\underbrace{\begin{pmatrix}A\\B\\C\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}A\\B\\C\end{pmatrix}}_{L}$$

There is a rep^{n} of $TL_{2L}(\sqrt{3})$ given by:

$$e_{2i}=\frac{1}{\sqrt{3}}\left(1+R_i R_{i+1}^{+}+R_i^{+}R_{i+1}\right)$$

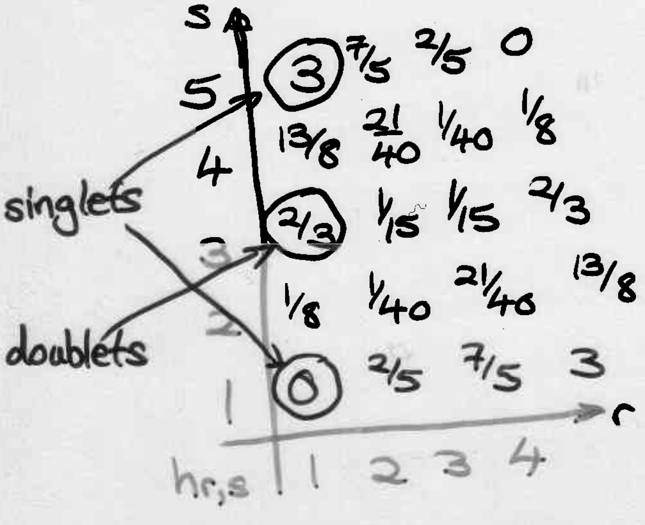
$$e_{2i-1}=\frac{1}{\sqrt{3}}\left(1+M_i+M_i^{+}\right).$$

$$R=\begin{pmatrix}1&0&0\\0&e^{2\pi i/3}&0\\0&0&e^{4\pi i/3}\end{pmatrix}\qquad M=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\mathcal{H}^{TL}=-\sum_{i=1}^{2L-1}e_i$$ gives critical Potts model. (free boundary)

Symmetry is S_3 → singlets & doublets in spectrum.

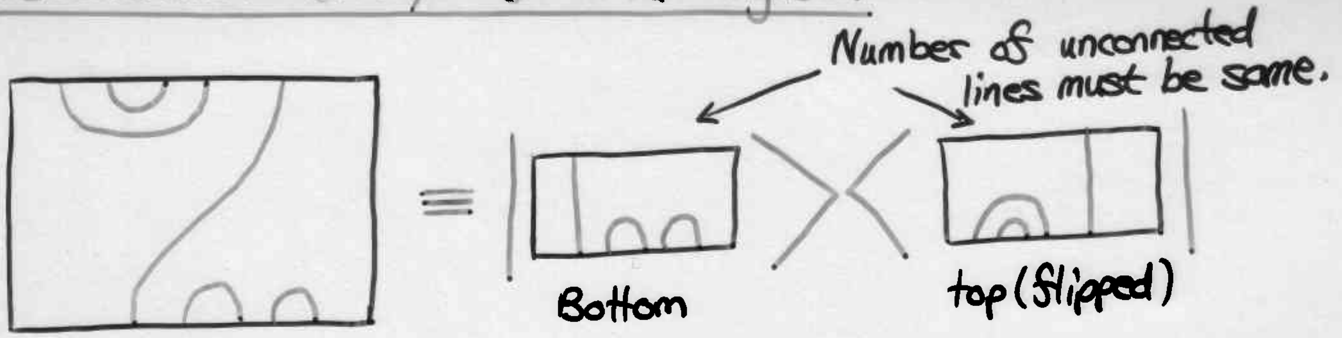
Continuum: $c_{6,5}=4/5$ (Cardy NPB 324, 591 '89)



5	3	7/5	2/5	0
4	13/8	21/40	1/40	1/8
3	2/3	1/15	1/15	2/3
2	1/8	1/40	21/40	13/8
1	0	2/5	7/5	3
	1	2	3	4

$$h_{r,s}=\frac{(6r-5s)^2-1}{120}$$

Representation theory of TL algebra



Full diagram

This is UNIQUE!

Consider only half-diagrams :-

- Action of generators cannot increase number of unconnected lines :-

Ex. N=4 ||||↺

↓

∩|| ; |∩| ; ||∩ ↺

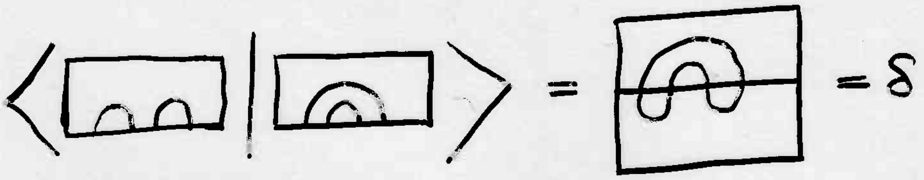
↓

(∩) ; ∩∩ ↺

→ : Action of algebra.

Irreducible Representations :-	Dimension
$H_0 = \{$ (∩) ; ∩∩ $\}$	2
$H_1 = \{$ ∩|| ; |∩| ; ||∩ $\}/H_0$	3
$H_2 = \{$ |||| $\}/H_1$	1

Inner product :-



$X = |x_1\rangle\langle x_2|$ $\quad$ $Y = |y_1\rangle\langle y_2|$

$$XY = |x_1\rangle\langle x_2|y_1\rangle\langle y_2| = \underbrace{\langle x_2|y_1\rangle}_{\text{indpt of } x_1, y_2} |x_1\rangle\langle y_2|$$

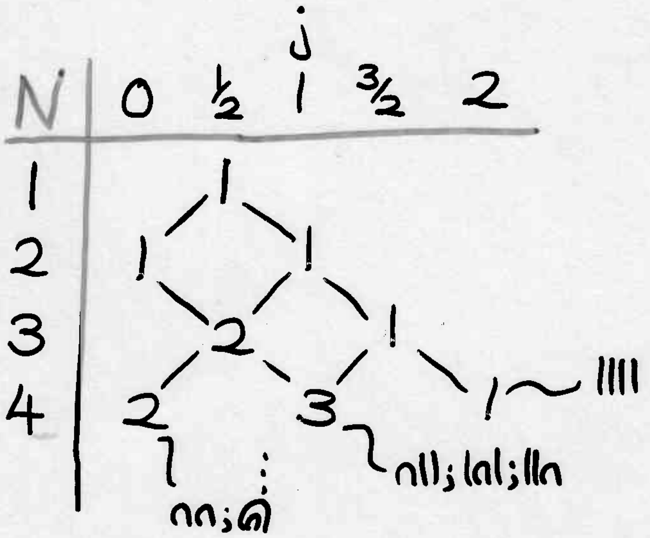
If $\{|v_i\rangle\}$ are diagrams then:-

$$G_{ij} = \langle v_i | v_j \rangle \qquad ; \quad G_{ij} = G_{ji}$$

For irreducibility $\det G \neq 0$.

$$\delta = q + q^{-1}$$

- q GENERIC $\quad (q^n \neq 1 \quad n \in \mathbb{Z})$ $\quad$ Semisimple.



V_j has $2j$ unconnected lines

$j = \frac{N}{2}, \frac{N}{2}-1, \frac{N}{2}-2, \ldots$

$$\dim V_j^{(N)} = \binom{N}{\frac{N}{2}-j} - \binom{N}{\frac{N}{2}+j+1}$$

Dimensions of irreducible representations

For XXZ model:

$$\sum_{j=0}^{N/2} (2j+1) \underbrace{\dim V_j^{(N)}}_{V_j^{(N)} \text{ is spin-}j \text{ representation of } SU_q(2).} = 2^N.$$

The XXZ repN is FAITHFUL - It contains all repNs of T-L algebra.

If $q^p = \pm 1$ for $p \in \mathbb{N}$ then we have null states.
(Non-Semisimple)

- There are now less irreducible represention[s] $0 \leqslant j \leqslant \frac{p-2}{2}$.

E.g. $p=6$ $(0 \leqslant j \leqslant 2)$

doub Potts sing.

N	V_0	$V_{1/2}$	V_1	$V_{3/2}$	V_2			$V_0 \oplus 2V_1 \oplus V_2$
1		1						–
2	1		1					3
3		2		1				–
4	2		3		1			9
5		5		4		*		–
6	5		9		5		*	27

$$\dim V_j^{(N)} = \Gamma_j^{(N)} - \Gamma_{j+p}^{(N)} + \Gamma_{j+2p}^{(N)} - \Gamma_{j+3p}^{(N)} + \cdots$$

where $\Gamma_j^{(N)} = \binom{N}{\frac{N}{2}-j} - \binom{N}{\frac{N}{2}+j+1}$ "generic" dimension of $V_j^{(N)}$.

LATTICE THEORY. CONTINUUM THEORY.

Temperley-Lieb algebra

$q = e^{i\pi/(m+1)}$

Irreducible Rep^N: V_j

$(0 \leqslant j \leqslant \frac{m-1}{2})$

$\rightsquigarrow$ F.S.S.

Virasoro algebra

$c = 1 - \frac{6}{m(m+1)}$

Irreducible Rep^N $h_{1,2j+1}$

$(1 \leqslant 2j+1 \leqslant m)$

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

- Agrees with Potts model results
- DO NOT GET WHOLE CFT!

Finite size scaling:-

Consider $H = -\sum_{i=1}^{N-1} e_i$ in sector V_j

- Step 1 : Rescaling H

$$H' = -\frac{\gamma}{\sin\gamma} \sum_{i=1}^{N-1} e_i$$

Ensures conf. inv^{t} in the limit (sound velocity = 1)

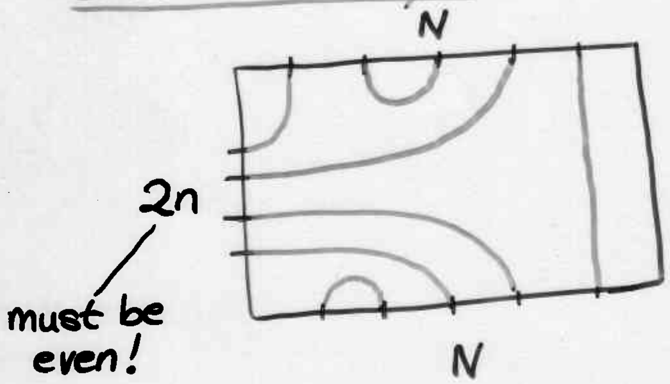
- Step 2 : Consider

$$\underbrace{\overline{F}_i}_{\text{rescaled differences}}(N;\gamma) = \frac{N}{\pi}\left(\underbrace{E_i(N;\gamma)}_{\text{Energies in sector } V_j} - \underbrace{E_0(N;\gamma)}_{\text{Ground state energy}}\right)$$

Now : $\lim_{N\to\infty} \sum_i z^{\overline{F}_i(N;\gamma)} \longrightarrow$ "Characters" of C.F.T.

Boundary extensions of T-L algebra :-

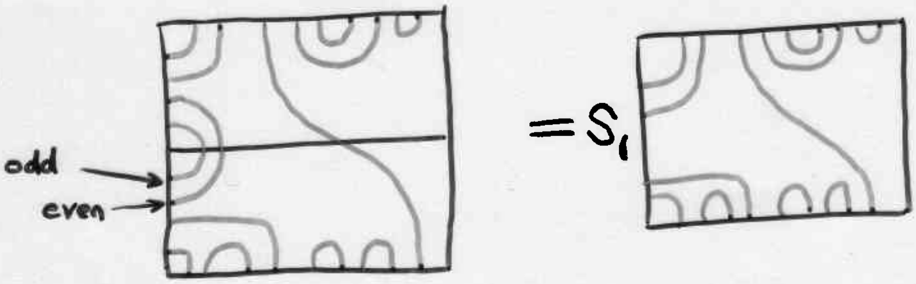
- One-boundary :-



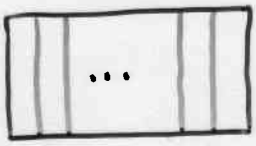
- Join points as before.
- No left-left connections.

Multiplication as before

- Remove closed loops factor δ.
- Mark left-left arcs odd/even (# points below lowest point on arc.)
  - Remove even arcs factor 1
  - Remove odd arcs factor s_1.



- Identity as before:

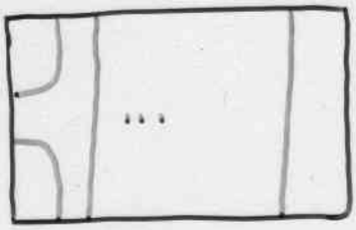


- This is $1BTL_N(\delta, s_1)$ diagram algebra.

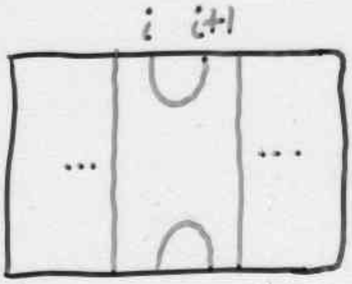
– There are now $\binom{2N}{N}$ possible diagrams.

Fundamental generators:-

$e_0 =$



$e_i =$



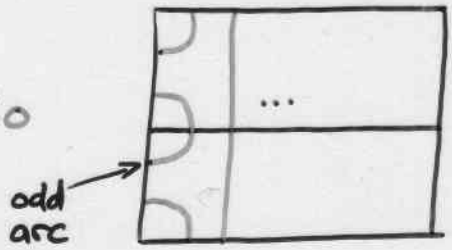
$1 \leqslant i \leqslant N-1$

T-L relations:-

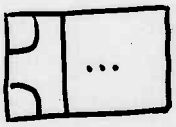
$e_i^2 = \delta e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \quad |i-j|>1$

Also:

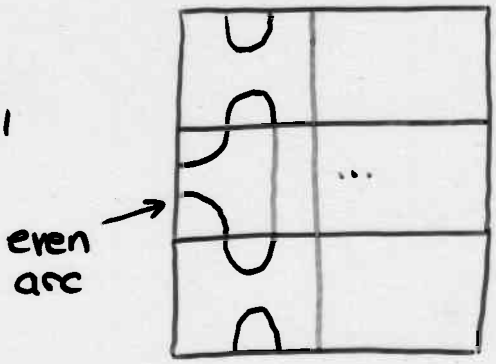
$e_0^2 = s_1 e_0$



$= s_1$



$e_1 e_0 e_1 = e_1$



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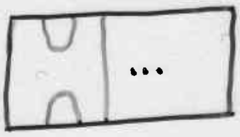
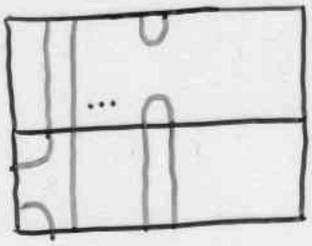


$e_0 e_i = e_i e_0$

$i > 1$



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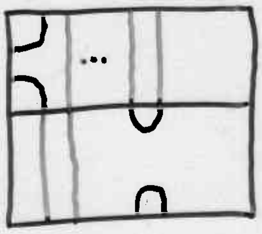
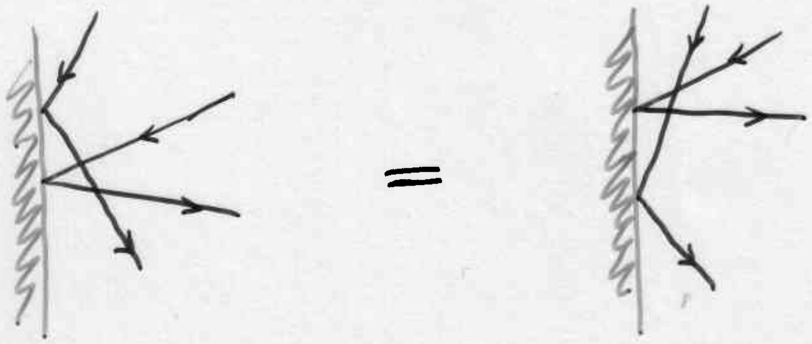


These are algebraic relations for $1BTL_N(\delta, s_1)$.

Reflection Equation :-



$$K_0(2v)R_1(u+v)K_0(2u)R_1(u-v) = R_1(u-v)K_0(2u)R_1(u+v)K_0(2v)$$

$$R_i(u)\,K_0(v) = K_0(v)\,R_i(u) \qquad i>1$$

$$K_0(u)\,K_0(-u) \propto \mathbb{1}$$

T-L algebra $\longrightarrow$ $R_i(u) = e_i - \dfrac{[u+1]}{[u]}$ $\qquad \delta = q + q^{-1}.$

$$s_1 = \frac{[\omega_1]}{[\omega_1+1]}$$

$$K_0(u) = e_0 - \frac{[(u-\omega_1+\xi)/2][(u-\omega_1\mp\xi)/2]}{[u][\omega_1+1]}$$

ξ is arbitrary

Integrable Hamiltonian :-

$$H = -a\,e_0 - \sum_{i=1}^{N-1} e_i$$

arbitrary parameter (related to ξ)

Adding a boundary to braid group
(braid grp type B)

As before:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

$$g_i g_j = g_j g_i \qquad |i-j|>1.$$

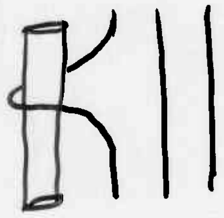
Add g_0: $$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0$$

$$g_i g_0 = g_0 g_i \qquad i>1.$$

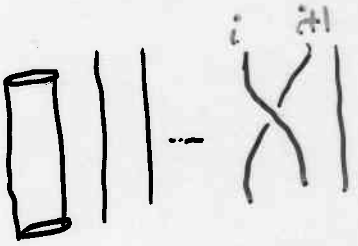
- All generators are invertible.

Pictures:

$g_0 =$



$g_i =$



$$g_0 = e^{i\omega_1} - 2i\sin(\omega_1+\gamma)e_0 \qquad s_1 = \frac{\sin\omega_1}{\sin(\omega_1+\gamma)}.$$

$$g_i = e_i - \cancel{2}1 e^{-i\gamma}$$

$$(g_0 - e^{i\omega_1})(g_0 - e^{-i\omega_1}) = 0.$$

Type B
Hecke algebra.

XXZ representation

$$e_i = -\frac{1}{2}\left[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cos\gamma\,\sigma_i^z\sigma_{i+1}^z - \cos\gamma + i\sin\gamma(\sigma_i^z - \sigma_{i+1}^z)\right]$$

$$e_0 = -\frac{1}{2\sin(\omega_1+\gamma)}\left(-i\cos\omega_1\,\sigma_1^z - \sigma_1^x - \sin\omega_1\right)$$ } most general one-site boundary term.

$$\delta = 2\cos\gamma \qquad s_1 = \frac{\sin\omega_1}{\sin(\omega_1+\gamma)}.$$

$$H^{nd} = -a e_0 - \sum_{i=1}^{N-1} e_i$$

Is there any analogue to $SU_q(2)$? $\quad q = e^{i\gamma}$

$$X = \frac{1}{2\sin(\gamma+\omega)}\left(e^{-\frac{i}{2}\gamma}S^+ q^{-S^z} + e^{\frac{i}{2}\gamma}S^- q^{-S^z} - \frac{\cos\omega}{\sin\gamma}\left(q^{-2S^z}-1\right)\right)$$

$$\Delta(X) = 1\otimes X + X\otimes q^{-2S^z}$$ (Not ~~Hopf~~ quantum group as $[q^{-2S^z}, e_0] \neq 0$)

$$[X, e_0] = 0 \qquad [X, e_i] = 0 \quad 1 \leq i \leq N-1$$

Naively $SU_q(2) \longmapsto U(1)$.

- Generically X can be fully diagonalised
  Abelian symmetry $[X,H]=0$.

- For $\omega = k\gamma \qquad k = -(N-1), -(N-2), \ldots, N-1$

  X becomes of Jordan form $\begin{pmatrix} \lambda_1 & & \\ & \boxed{\begin{matrix}\lambda & 1 \\ 0 & \lambda\end{matrix}} & \\ & & \lambda_m \end{pmatrix}$

Now $[X,H]=0 \Rightarrow H$ has degeneracy!

Potts representation: $\delta = \sqrt{3}$

Two possibilities:

$$A{:})\quad e_0 = \begin{pmatrix} \sqrt{3} & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$S_1 = \sqrt{3}$$

$$\underbrace{\begin{pmatrix} A \\ B \\ C \end{pmatrix} \otimes \begin{pmatrix} A \\ B \\ C \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} A \\ B \\ C \end{pmatrix}}_{L}$$

$$AB)\quad e_0 = \begin{pmatrix} \sqrt{3}/2 & & \\ & \sqrt{3}/2 & \\ & & 0 \end{pmatrix}$$

$$S_1 = \frac{\sqrt{3}}{2}.$$

- Both break $S_3 \to \mathbb{Z}_2$

Consider $H = -a e_0 - \sum_{i=1}^{N-1} e_i$ for $a > 0$.

A leads to lower energy for states $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

AB leads to lower energy for states $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

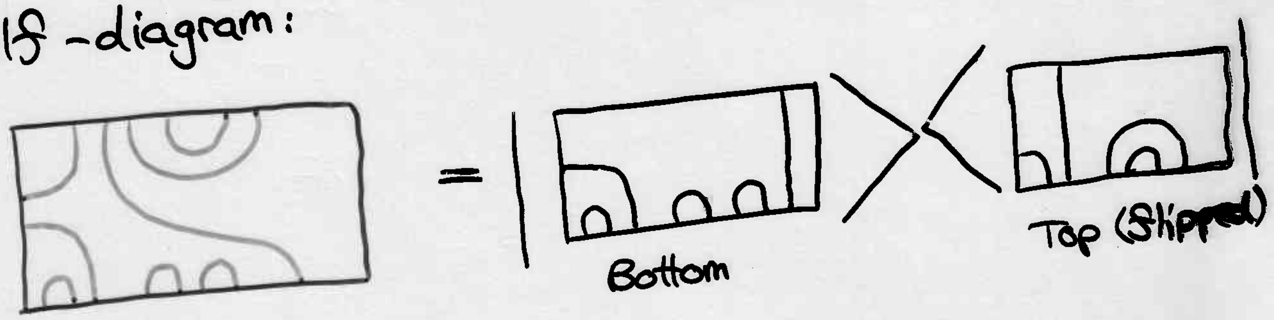
For $a \to \infty$ we obtain FIXED boundary at one end

$$\left.\begin{array}{l} A: \; h_{4,4} = 1/8,\; h_{4,2} = 13/8. \\ AB: \; h_{2,2} = 1/40,\; h_{2,4} = 21/40 \end{array}\right\} \text{Known results from Cardy.}$$

1BTL algebra.

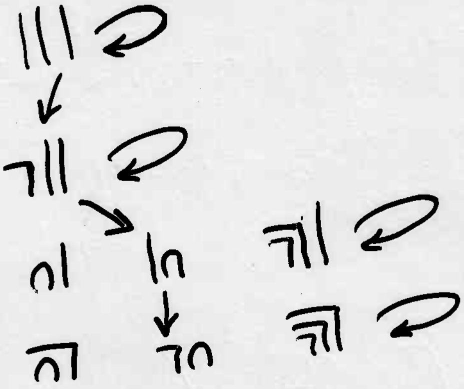
- Same techniques as for Temperley-Lieb.

Half-diagram:

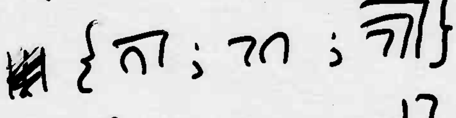
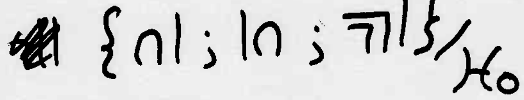
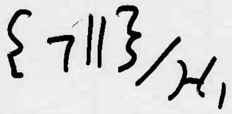
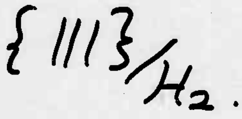


- Again UNIQUE.

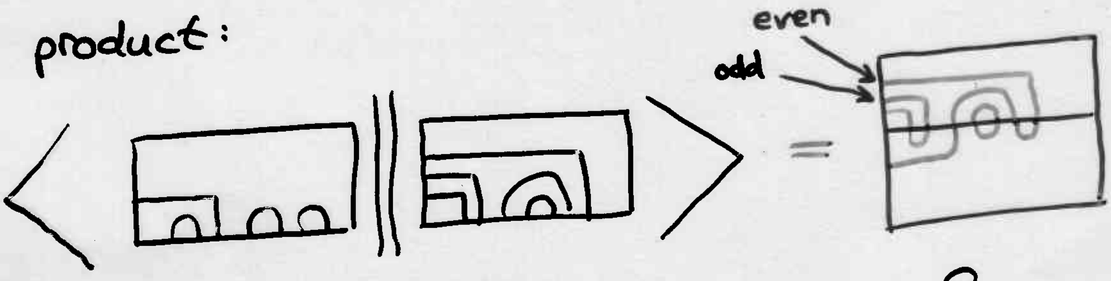
Ex. $L=3$



Irreducible Rep^{N}s:-

		Dimension
$\mathcal{H}_0 =$		3
$\mathcal{H}_1 =$		3
$\mathcal{H}_2 =$		1
$\mathcal{H}_3 =$		1

Inner product:



$= s_1 \delta.$

Generic ω_1:

$\omega_1 \neq k\gamma \qquad k = -(N-1), -(N-2), \ldots, N-1$

irreps $W_Q^{(N)}$

N		W_{-2}	$W_{-3/2}$	W_{-1}	$W_{-1/2}$	W_0	$W_{1/2}$	W_1	$W_{3/2}$	W_2	
1					1		1				
2				1		2		1			
3			1		3		3		1		
4		1		4		6		4		1	
5	1		5		10		10		5		1

$$\dim W_Q^{(N)} = \binom{N}{\frac{N}{2}-Q}$$

$$\sum_{Q=-N/2}^{N/2} \dim W_Q^{(N)} = 2^N$$

XXZ chain contains a single copy of each irrep.

Exceptional ω_1 (and γ)

Ex. $\delta = \sqrt{3}$, $s_1 = \sqrt{3}$

N	$W_{-3/2}$	W_{-1}	$W_{-1/2}$	W_0	$W_{1/2}$	Potts $W_{-1} \oplus W_0$
1			1		1	–
2		1		2		$3 \sim_{L=1}$
3	1		3		2	–
4		4		5		$9 \sim_{L=2}$
5	~~5~~4		9		5	–
6		13		14		$27 \sim_{L=3}$

A Surprise!

$$H^{nd} = -a e_0 - \sum_{i=1}^{N-1} e_i$$

with $a = \dfrac{2\sin\gamma\sin(\omega_1+\gamma)}{\cos\omega_1+\cos\delta_1}$

$$H^{nd} = \frac{\sin\gamma}{\cos\omega_1+\cos\delta_1}\left(i\cos\omega_1+\sigma_1^x-\sin\omega_1\right)$$
$$-\frac{1}{2}\left\{\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\,\sigma_i^z\sigma_{i+1}^z+\cos\gamma\right)+i\sin\gamma\left(\sigma_1^z-\sigma^z_{\underset{N}{\cancel{k}}}\right)\right\}$$

has <u>same</u> spectrum as:-

$$H^{diag} = -\frac{1}{2}\left\{\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y-\cos\gamma\,\sigma_i^z\sigma_{i+1}^z+\cos\gamma\right)+\cancel{i\sin\gamma}\right.$$
$$\left.+\sin\gamma\left[\tan\left(\frac{\omega+\delta}{2}\right)\sigma_1^z+\tan\left(\frac{\omega-\delta}{2}\right)\sigma^z_{\underset{N}{\cancel{k}}}+\frac{2\sin\omega}{\cos\omega+\cos\delta}\right]\right\}$$

- Checked using exact diagonalisation ($\overset{N}{\cancel{k}}\leq 4$)
- Bethe Ansatz

Generically $\quad U H^{nd} U^{-1} = H^{diag}$

<u>BUT</u> for $\omega = k\gamma \quad k = -(N-1), -(N-2), \ldots, N-1$

U does not exist (singular)

Reason: At these points get extra degeneracy

$$H^{nd} \simeq \begin{pmatrix}\ddots & & \\ & \boxed{\begin{matrix}\lambda & 1\\ 0 & \lambda\end{matrix}} & \\ & & \ddots\end{pmatrix} \qquad \underset{\substack{?\\ \text{Hermitian}}}{H^{diag}} \simeq \begin{pmatrix}\ddots & & \\ & \boxed{\begin{matrix}\lambda & 0\\ 0 & \lambda\end{matrix}} & \\ & & \ddots\end{pmatrix}.$$

Finite size scaling.

Consider $H = -a e_0 - \sum_{i=1}^{N-1} e_i$ in sector W_Q.

Trick: Use known results from diagonal chain.
(δ - independence)

$$\Downarrow$$

FSS scaling is indpt of a $(a>0)$

$$H = -a e_0 - \sum_{i=1}^{N-1} e_i \text{ in } W_Q$$

Continuum CFT

$$C_{m+1,m} = 1 - \frac{6}{m(m+1)}$$

$$s_1 = \frac{\sin \omega_1}{\sin(\omega_1 + \gamma)} \qquad \delta = 2\cos\gamma \quad \xrightarrow[\text{F.S.S.}]{} \quad \text{Field } h_{r,s}$$

$$\omega_1 = r\gamma \qquad \gamma = \frac{\pi}{m+1}$$

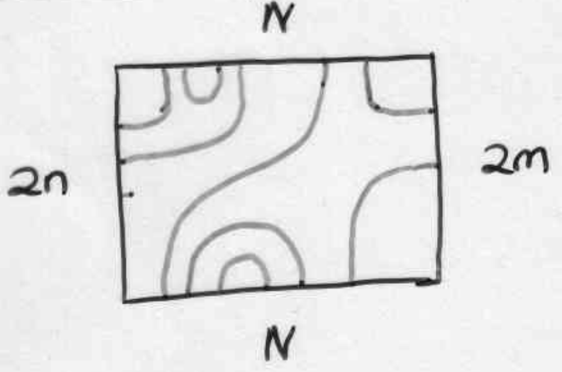
$$Q = \frac{s-r}{2} \qquad 1 \leqslant r \leqslant m-1$$

$$1 \leqslant s \leqslant m$$

BY ADDING BOUNDARY TERMS WE SEE ALL FIELDS FROM CFT.

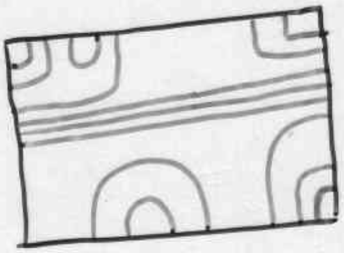
- For Potts model this <u>exactly</u> reproduces results of Cardy.

Two boundary Temperley-Lieb.



- No left-left arcs
- No right-right arcs

Now also have:

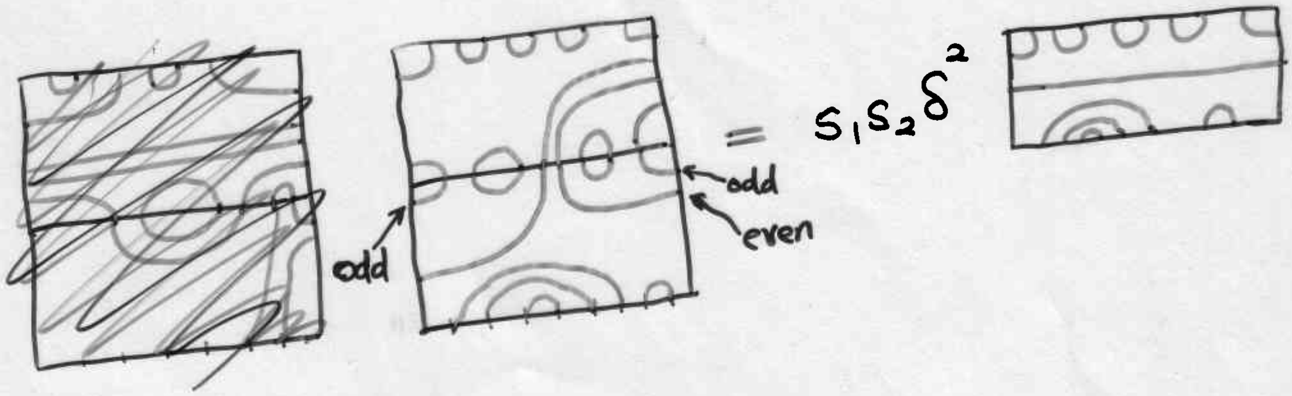


Infinite number of diagrams!

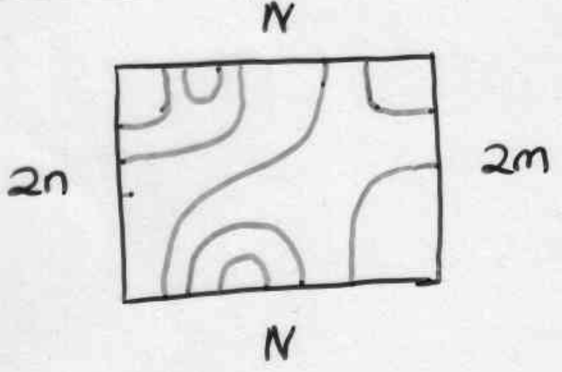
Composition:

Multiply as before:

- Remove closed loops factor δ
- Mark left-left & right-right arcs odd/even
  - Remove even arcs factor 1
  - Remove left add arcs factor s_1
    right odd arcs factor S_2.



Two boundary Temperley-Lieb.



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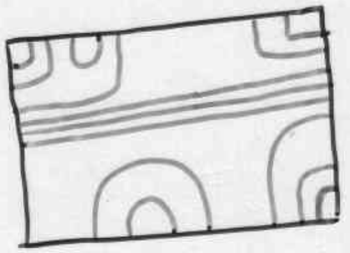

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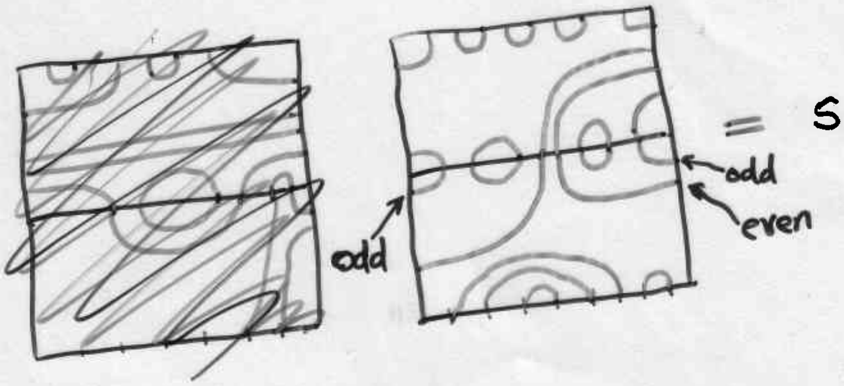
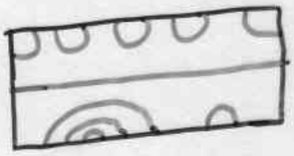
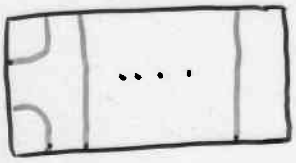
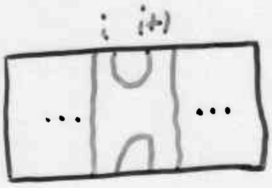
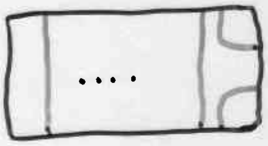


$= s_1 s_2 \delta^2$



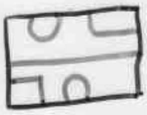
Algebraic generators:

$e_0 =$  $e_i =$  $e_N =$ 

Relations: $e_i e_{i\pm 1} e_i = e_i$ $\quad e_i^2 = \delta e_i$

$e_1 e_0 e_1 = e_1$ $\quad e_0^2 = s_1 e_0$

$e_{N-1} e_N e_{N-1} = e_{N-1}$ $\quad e_N^2 = s_2 e_N$

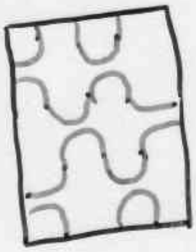
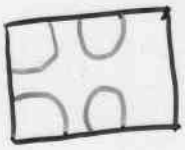
e.g. $N=$ ~~4~~ 3 $\quad e_0 e_2 e_1 e_3 =$  $\simeq$ 

Integrable Hamiltonian :-

$$\mathcal{H}^{2BTL} = -a e_0 - a' e_N - \sum_{i=1}^{N-1} e_i$$

All irreducible rep^Ns of $2BTL_N(\delta, s_1, s_2)$ lie in quotient:

- We remove pairs of horizontal lines (left-right) with some factor, b.

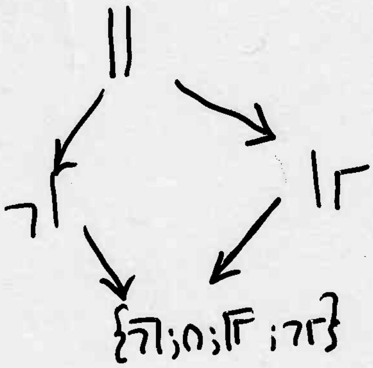
e.g. $N=3$
$e_0 e_2 e_1 e_3 e_0 e_2 =$  $= b$ 

$= b e_0 e_2.$

There are now a finite number of diagrams.

- Study half-diagrams as before:-

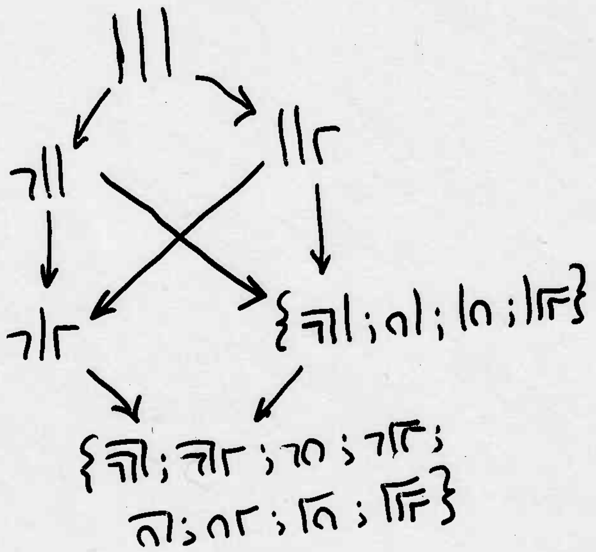
- N=2



b indpt. action.

Action of gens depends on b.

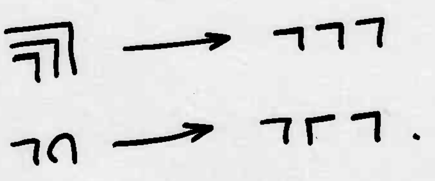
- N=3



b indpt action

Action of gens depends on b.

There is always a repⁿ of dimension 2^N!



Use symbols ┐, ┌.

- What is this representation?

- This is isomorphic to the spin chain rep$^{\text{n}}$:

Space: $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}$

$$e_0 = -\frac{1}{2\sin(\omega_1+\gamma)}\left(-i\cos\omega_1\sigma_1^z - \sigma_1^x - \sin\omega_1\right)$$

$$e_i = -\frac{1}{2}\left[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \cos\gamma\,\sigma_i^z\sigma_{i+1}^z - \cos\gamma + i\sin\gamma\,(\sigma_i^z - \sigma_{i+1}^z)\right]$$

$$e_N = -\frac{1}{2\sin(\omega_2+\gamma)}\left(i\cos\omega_2\sigma_N^z + \cos\phi\,\sigma_N^x - \sin\phi\,\sigma_N^y - \sin\omega_2\right)$$

$$\delta = 2\cos\gamma \qquad s_1 = \frac{\sin\omega_1}{\sin(\omega_1+\gamma)} \qquad s_2 = \frac{\sin\omega_2}{\sin(\omega_2+\gamma)}$$

For N odd: $$b = -\frac{\sin\left(\frac{\omega_1-\omega_2-\phi}{2}\right)\sin\left(\frac{\omega_1-\omega_2+\phi}{2}\right)}{\sin(\omega_1+\gamma)\sin(\omega_2+\gamma)}$$

For N even: $$b = \frac{\sin\left(\frac{\gamma+\omega_1+\omega_2-\phi}{2}\right)\sin\left(\frac{\gamma+\omega_1+\omega_2+\phi}{2}\right)}{\sin(\omega_1+\gamma)\sin(\omega_2+\gamma)}$$

N.B. $H = -a\,e_0 - a'e_N - \sum_{i=1}^{N-1} e_i$ is most general XXZ model with NON-DIAGONAL boundary terms (5 parameters : $\underbrace{a, a'}_{\text{Hamiltonian}}, \underbrace{\omega_1, \omega_2, \phi}_{\text{2BTL algebra}}$)

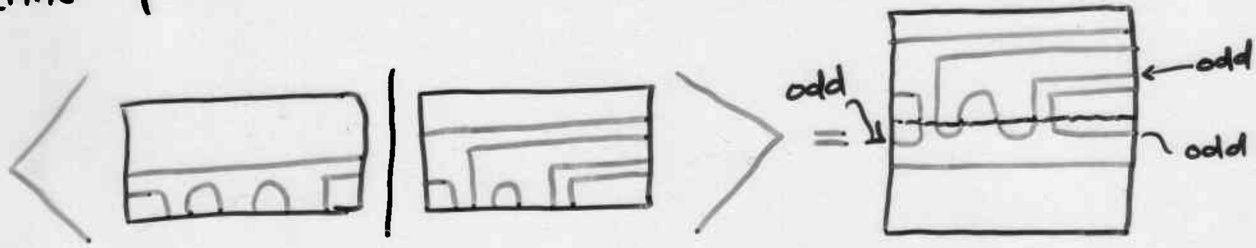
- When is this rep$^{\text{n}}$ irreducible?

i.e. $$H \cong \left(\begin{array}{c|c} /// & /// \\ \hline 0 & /// \end{array}\right)$$ ← NOT IRRED.

← This does NOT imply Jordan cells e.g. $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$.

- This is easy using diagram rep^N

Inner product:



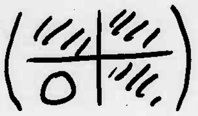
$= s_1 s_2^2 b.$

Gram matrix $G_{ij} = \langle v_i | v_j \rangle$. det $G \neq 0 \Leftrightarrow$ IRRED.

RESULT:

- For generic ϕ the rep^N is irreducible.
- For $\phi = (N-1-2n)\gamma + \varepsilon_1 \omega_1 + \varepsilon_2 \omega_2$ $\varepsilon_{1,2} = \pm 1$.

the ~~space~~ rep^N is indecomposable.



$n = 0, 1, 2, \dots, \frac{N-2}{2}$ N even

$n = 0, 1, 2, \dots, \frac{N-1}{2}$ N odd

Surprisingly these are precisely the points where the Bethe Ansatz for XXZ has been achieved!

(Nepomechie, Cao et al.)

There are two sets of Bethe equations and the splitting of the 2^N eigenvalues agrees with splitting from 2BTL algebra!

Irreducible rep^{N}s of $2BTL_N(\delta, s_1, s_2)$

generic

1

1 1 2 1 1 N=1

1 3 3 1 4 1 3 3 1 N=2

1 4 7 7 4 1 8 1 4 7 7 4 1 N=3

1 5 11 15 15 11 5 1 16 1 5 11 15 15 11 5 1 N=4

||

1 6 16 26 31 31 26 16 6 1 32 1 6 16 26 31 31 26 16 6 1

⋮ ⋮

$\}$ 2^N dimensional rep^N

All these numbers can be found by studying Bethe Ansatz eq^{N}s numerically and counting number of solutions.

- Connections with $c=-2$.

Take $\gamma = \pi/2$ $\quad \omega_1=\omega_2=\pi/2$ $\qquad$ EXCEPTIONAL PTS.

$\delta = 0$; $s_1 = s_2 = 0$

N even

- $\theta = \pi/2$ $\quad \chi_{\mathcal{V}_0} + \chi_{\mathcal{V}_1}$
  (b=0)

N odd

- $\theta = 0$ $\quad \chi_{\mathcal{V}_{-1/8}}$
  (b=1)
- $\theta = \pi$ $\quad \chi_{\mathcal{V}_{3/8}}$
  (b=0)

} We know where to look (rep.^{N} theory)

where:

$$\chi_{\mathcal{V}_0} = \frac{1}{2} z^{-1/8}\left(\Theta_{1,2} + \partial\Theta_{1,2}\right) P(z)$$

$$\chi_{\mathcal{V}_1} = \frac{1}{2} z^{-1/8}\left(\Theta_{1,2} - \partial\Theta_{1,2}\right) P(z)$$

$$P(z) = \prod_{n\geq 1} (1-z^n)^{-1}$$

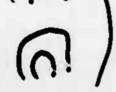
$$\chi_{\mathcal{V}_{-1/8}} = z^{-1/8}\,\Theta_{0,2} P(z)$$

$$\chi_{\mathcal{V}_{3/8}} = z^{-1/8}\,\Theta_{2,2} P(z)$$

$\chi_R \equiv \chi_{R_0} = \chi_{R_1} =$ ~~$2\Theta_{1,2}$~~ $2z^{-1/8}\,\Theta_{1,2} P(z)$.

$$\Theta_{\lambda,k} = \sum_{n\in\mathbb{Z}} z^{(2kn+\lambda)^2/4k}$$

$$\partial\Theta_{\lambda,k} = \sum_{n\in\mathbb{Z}} (2kn+\lambda) z^{\frac{(2kn+\lambda)^2}{4k}}$$

For sure there are indecomposable rep^{N}s: e.g. $e_2\Big($  $\Big) = \begin{pmatrix} 0 & 0 \\ 1 & \underset{\delta=0}{0} \end{pmatrix}$

EXERCISE: Understand them.