Applications of QCD Sum Rules to Heavy Quark Physics

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Preface

- QCD sum rule, the three key elements

Correlation function of quark-antiquark currents

$\Downarrow$ $\Downarrow$

Operator Product Expansion in terms of quark-gluon diagrams and universal QCD parameters	=	Dispersion Relation, a sum over hadronic amplitudes

- sum rules are analytical relations in continuum QCD

  as opposed to numerical simulation of the correlation functions in the lattice QCD

- the origin of the method:

  QCD and Resonance Physics. Sum Rules
  Mikhail A. Shifman, A.I. Vainshtein, Valentin I. Zakharov
  Published in **Nucl.Phys. B147 (1979) 385-447**
  ITEP-73-1978, ITEP-80-1978
  DOI: 10.1016/0550-3213(79)90022-1
  References | BibTeX | LaTeX(US) | LaTeX(EU)
  Detailed record - Cited by 4087 records 1000+

- major development of the QCD SR method in 1980's:

  "standard" 2-point sum rules; modifications: 3-,4-point SR's;

  Finite Energy Sum Rules; method of external fields

- QCD Light-cone sum rules (LCSR)
  [V. Braun, I. Balitsky et al; V. Chernyak, I.Zhitnisky, 1989,...]

  a more advanced technique to calculate hadronic transition form factors

- These lectures concentrate on the QCD sum rule applications to Heavy Quark Physics

- from ABC of the method to recent applications

Outline of the lectures

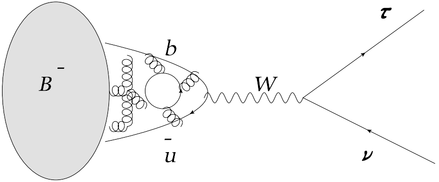
- Lecture 1: **Introducing the method of QCD Sum Rules**
  - calculation of B-meson decay constant
    $B \to \tau\nu_\tau$, $B_s \to \mu^+\mu^-$
- Lecture 2: **Light-cone Sum Rules for Heavy-Light Form Factors**
  - calculation of $B \to \pi$ form factor
    $|V_{ub}|$ from $B \to \pi\ell\nu_\ell$
- Lecture 3: **Hadronic effects in $B \to K^{(*)}\ell^+\ell^-$, $B \to K^*\gamma$ from LCSR's**

Lecture 1:

Introducing the method of QCD Sum Rules

Our study object: $B \to \tau\nu_\tau$ decay

- the decay amplitude in the Standard Model



$$A(B^- \to \tau^- \bar{\nu}_\tau)_{SM} = \frac{G_F}{\sqrt{2}} V_{ub}\, \bar{\tau}\gamma^\mu(1-\gamma_5)\nu_\tau \langle 0|\bar{u}\gamma_\mu\gamma_5 b|B\rangle$$

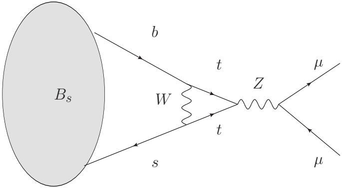
{$b \to u$ flavour-changing transition } $\otimes$ {QCD colour forces}

- hadronic matrix element $\Rightarrow$ decay constant:
  $\langle 0|\bar{u}\gamma^\mu\gamma_5 b|B(p_B)\rangle = ip_B^\mu f_B, \quad p_B^2 = m_B^2$

- partial width: (suppressed for $\ell = \mu, e$)

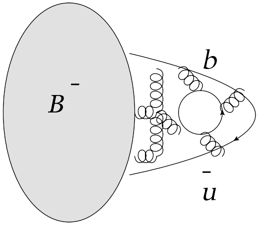
$$BR(B^- \to \tau^- \bar{\nu}_\tau)_{SM} = \frac{G_F^2 |V_{ub}|^2}{8\pi} m_\tau^2 m_B \left(1 - \frac{m_\tau^2}{m_B^2}\right)^2 f_B^2 \tau_{B^-} ,$$

Rare leptonic decays: $B_{s,d} \to \ell^+\ell^-$



- recently detected by LHCb
- in SM t, W, Z-loops, sensitive to $V_{ts}V_{tb}^*$,
- realistic chances to find/constrain new physics
- after integrating out the heavy particle loops:
  the hadronic matrix element in decay amplitude reduced to

  $\langle 0|\bar{s}\gamma^\mu\gamma_5 b|B_s(p_B)\rangle = ip_B^\mu f_{B_s}$, or f_{B_d} for $B_d \to \mu^+\mu^-$
- $f_{B_d} \simeq f_{B_u} \equiv f_B$ (isospin symmetry),
  but $f_{B_s} \neq f_B$, $(SU(3)_{fl}$ violation)



The task is to calculate the hadronic matrix element
$\langle 0|\bar{u}\gamma^\mu\gamma_5 b|B\rangle \sim f_B$ in QCD

Quantum Chromodynamics (QCD)

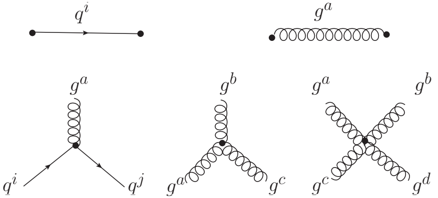
- Quantum field theory of quarks, gluons and their interactions

$$L_{QCD}(x) = -\frac{1}{4} G^{a}_{\mu\nu}(x) G^{a\,\mu\nu}(x)$$
$$+ \sum_{q=u,d,s,c,b,t} \bar{q}^{\,i}(x)(iD_\mu \gamma^\mu - m_q) q^{\,i}(x)$$

$D_\mu = \partial_\mu - i g_s \frac{\lambda^a}{2} A^a{}_\mu$, $G^a_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g_s f^{abc} A^b{}_\mu A^c{}_\nu$,

colour charges $i = 1, 2, 3$, $a = 1, .., 8$, $\alpha_s = g_s^2/4\pi$ flavour neutrality

- basic elements of Feynman graphs:

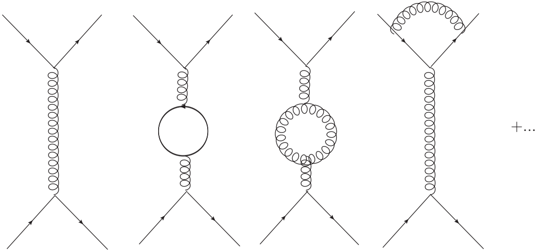


Propagators

Vertices

Asymptotic Freedom of QCD

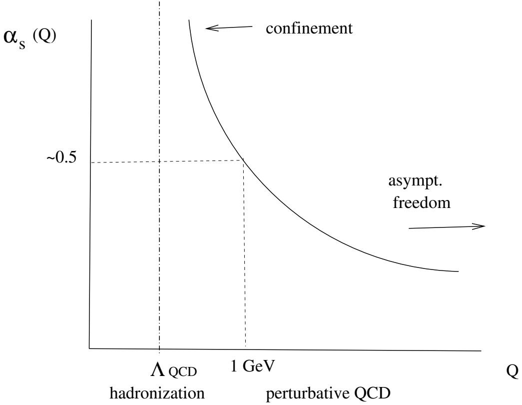
- Quark-quark scattering, energy/momentum transfer Q



- the quantum loop corrections play a crucial role:
  $\alpha_s \to \alpha_s(Q)$, effective, scale-dependent coupling
- $\alpha_s(Q)$ small for processes with $Q \geq 1$ GeV
  $\Rightarrow$ perturbative expansion in powers of α_s applicable

QCD at long distances

- $\alpha_s(Q) \to \infty$ at small momenta $\sim$ long distances



- an intrinsic scale emerges: $\Lambda_{QCD} \sim 200 - 300$ MeV
- at $Q \sim \Lambda_{QCD}$ quarks/gluons strongly interact, hadronization, confinement

QCD Vacuum

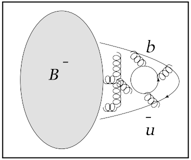
- the lowest energy state, no hadrons
  contains fluctuating quark-antiquark and gluon fields:
  vacuum condensates
- e.g., $\langle 0|\overline{q}q|0\rangle \neq 0$, $\quad q = u, d, s$
  -spontaneous breaking of chiral symmetry
- $\langle 0|G_{\mu\nu}G^{\mu\nu}|0\rangle \neq 0$, $\langle 0|\overline{q}\sigma_{\mu\nu}G^{\mu\nu}q|0\rangle \neq 0$,...
- universal set of vacuum condensate densities with dimension $d = 3, 4, 5, ..$

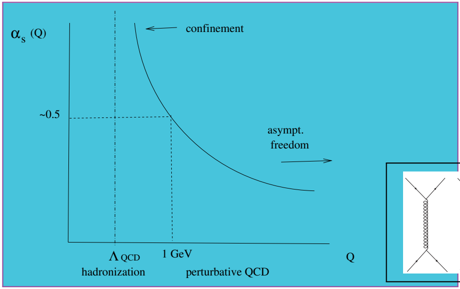
B-meson annihilation in QCD

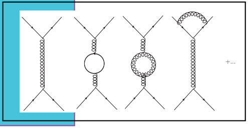
- energy scale of quark-gluon interactions binding b and $\bar{u}$ inside B:

  $$\bar{\Lambda} \sim m_B - m_b \sim 500\text{-}700 \text{ MeV},$$

  ($m_B \simeq 5.3$ GeV, $m_b = 4.6 - 4.8$ GeV ("pole" quark mass))







- no perturbative expansion in $\alpha_s(\bar{\Lambda})$ can be used:

  $\Rightarrow$ nonperturbative QCD

B-meson annihilation in QCD

- m_B , f_B and other B-meson observables are described by long-distance (soft) quark-gluon interactions in QCD
- in addition to "valence" quarks, partonic components with soft gluons and $\bar{q}q$ -pairs in the B-meson state

$$|B^-\rangle = |b\bar{u}\rangle \oplus |b\bar{u}G\rangle \oplus |b\bar{u}\bar{q}q\rangle \oplus \dots$$

- the QCD vacuum state $\langle 0|$, (the lowest energy state, no hadrons) is populated by fluctuating quark-antiquark and gluon fields

  $\Rightarrow$ even for the simlpest hadronic matrix element $\langle 0|\bar{u}..b|B\rangle \sim f_B$ there is no exact solution in QCD

- alternatives
  - numerical simulation on the lattice
  - use of QCD sum rules (SVZ method):

  the main idea: construct an object calculable in QCD and simultaneously related to f_B

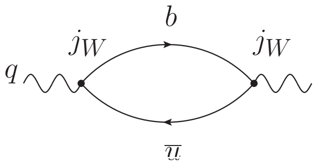
Correlation function of $\overline{u}b$ currents

- formal definition of the vacuum correlation function:

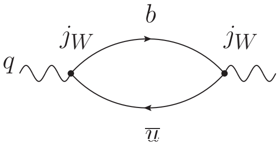
$$\Pi_{\mu\nu}(q^2) = \int d^4x\, e^{iqx} \langle 0 | T\{ j^W_\mu(x) j^{W\dagger}_\nu(0) \} | 0 \rangle \,,$$

a quantum amplitude of emission and absorbtion of $\overline{u}b$ pair in vacuum by the external current:

$$j^W_\mu = \bar{u}\gamma_\mu\gamma_5 b,$$
$$j^{W\dagger}_\mu = \bar{b}\gamma_\mu\gamma_5 u$$

$\Pi_{\mu\nu}(q^2) =$ 

Correlation function far below the B hreshold



- 4-momentum of the $b\bar{u}$ pair: $q = (q_0, \vec{q})$, $q^2 = q_0^2 - \vec{q}^2$, rest frame: $\vec{q} = 0$, $q^2 = q_0^2$, fix the energy $q_0 \ll m_b, m_B$
- the $b\bar{u}$-pair is virtual: $\Delta E \Delta t \sim 1$, the energy deficit $\Delta E \sim m_b$, $\Delta t \sim 1/m_b$

  $m_b \gg \Lambda_{QCD}$: $\Delta t \ll 1/\Lambda_{QCD}$
- virtual quarks propagate during short times, are asymptotically free,
- at $q^2 \ll m_b^2, m_B^2$,

  $\Pi_{\mu\nu}(q^2) \simeq$ simple loop diagram

  $\oplus$\{ calculable QCD corrections\}

Transforming to pseudoscalar currents

- simplifying trick: multiply the correlation function,

  $q^\mu q^\nu \Pi_{\mu\nu}(q^2) \equiv \Pi_5(q^2)$, scalar function of q^2

  note that

  $q^\mu \bar{u}\gamma_\mu\gamma_5 b = (p^b_\mu + p^u_\mu)\bar{u}\gamma^\mu\gamma_5 b = (m_b + m_u)\bar{u}i\gamma_5 b \equiv j_5$

  (apply Dirac equation for both quark fields))

  $\Rightarrow \Pi_5(q^2) = \int d^4x\, e^{iqx}\langle 0|T\{j_5(x)j_5^\dagger(0)\}|0\rangle$,

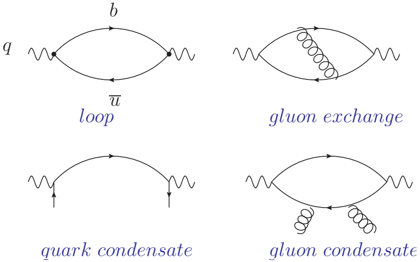
- equivalent definition of the decay constant:

  at $q = p_B$ $(q^2 = m_B^2)$,

  $q_\mu\langle 0|\bar{u}\gamma^\mu\gamma_5 b|B(p_B)\rangle = \langle 0|j_5|B(p_B)\rangle = m_B^2 f_B$

Calculating the correlation function at $q^2 \ll m_B^2$

- adding perturbative gluon exchanges to the simple loop , $\alpha_s(m_B) \ll 1$
- including nonperturbative effects due to condensates
- typical diagams



- technically, using Feynman rules of QCD and considering the vacuum quark-antiquarks and gluons as external static fields.
- The result: analytical expression for $\Pi_5(q^2)$ in terms of m_b, m_u and universal QCD parameters α_s, $\langle \bar{q}q \rangle$,...

- interpreting the calculation as an operator-product expansion:

$$T\{j_5(x)j_5^\dagger(0)\} = \sum_{d=0,3,4,..} C_d(x^2, m_b, m_u, \alpha_s) O_d(0)$$

in local operators with the quantum numbers of vacuum

(Lorentz-scalar, C-,P-,T-invariant, colorless) and growing dimensions:

$O_0 = 1, O_3 = \bar{q}q, O_4 = G^{\mu\nu}G_{\mu\nu}, ...$ (no operator of dimension 2 in QCD !)

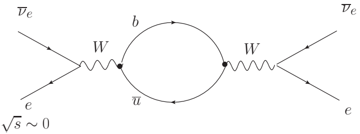
- vacuum average, integrating over x

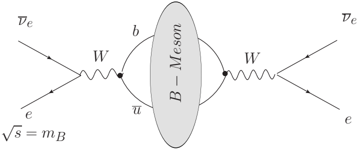
$$\Pi_5(q^2) = \int d^4x\, e^{iqx} \langle 0|T\{j_5(x)j_5^\dagger(0)\}|0\rangle$$
$$= \sum_{d=0,3,4,..} \overline{C}_d(q^2, m_b, m_u, \alpha_s)\langle 0|O_d|0\rangle$$

- perturbative loops $\rightarrow$ Wilson coefficeints $\overline{C}_d$ as series in α_s, $d \neq 0$, $\langle 0|O_d|0\rangle \sim (\Lambda_{QCD})^d$ - vacuum condensate densities,
- at $q^2 \ll m_b^2$, high-d terms suppressed by $O[(\Lambda_{QCD}/m_b)^d]$ the OPE can safely be truncated

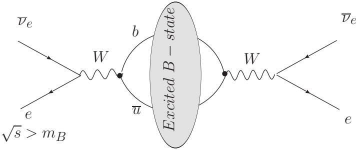
Correlation function above B threshold

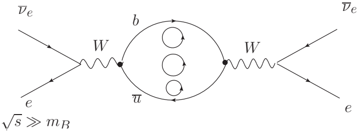
- Hypothetical neutrino-electron scattering, varying c.m. energy $\sqrt{s} = \sqrt{q^2}$,
- $\Pi_5(q^2)$ is the part of the scattering amplitude





highly virtual quark pair, B-meson, resonance

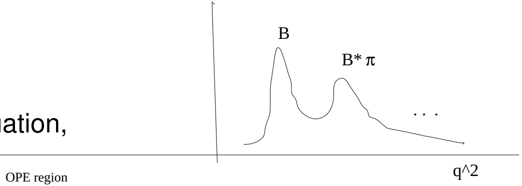




excited B mesons multiple hadrons (continuum)

Hadronic representation of the correlation function

- at $q^2 \ll m_b^2$:
  a short-distance,
  short-lived $b\bar{u}$ -fluctuation,
  $\Pi_5(q^2) \simeq \Pi_5^{(OPE)}(q^2)$



- $\Pi_5(q^2)$ at $q^2 \geq m_B^2$, describes propagation of B meson and excited and multiparticle B states
- the hadronic representation (dispersion relation):

$$\Pi_5(q^2) = \frac{\langle 0|j_5|B\rangle\langle B|j_5^\dagger|0\rangle}{m_B^2 - q^2} + \sum_{B_{exc}} \frac{\langle 0|j_5|B_{exc}\rangle\langle B_{exc}|j_5^\dagger|0\rangle}{m_{B_{exc}}^2 - q^2}$$

rigorous derivation: analyticity of $\Pi_5(q^2) \oplus$ Cauchy theorem $\oplus$ unitarity

- As a result: at $q^2 \ll m_b^2$ there is a relation between $\Pi_5^{(OPE)}(q^2)$ and a hadronic sum containing f_B

Deriving the sum rule for f_B^2

- isolating the ground-state B-state and introducing the spectral density of excited hadronic states

$$\Pi_5(q^2) = \frac{f_B^2 m_B^4}{m_B^2 - q^2} + \int\limits_{s_h}^{\infty} ds \frac{\rho^h(s)}{s - q^2}$$

- expressing the OPE result as a dispersion relation

$$\Pi(q^2)^{(OPE)} = \frac{1}{\pi}\int\limits_{m_b^2}^{s_0} ds \frac{\mathrm{Im}\Pi_5^{(OPE)}(s)}{s - q^2} + \frac{1}{\pi}\int\limits_{s_0}^{\infty} ds \frac{\mathrm{Im}\Pi_5^{(OPE)}(s)}{s - q^2}$$

equating the two representations at $q^2 \ll m_b^2$
- global quark-hadron duality

- at sufficiently large s the local duality is also valid:

$$\rho^h(s) \simeq \frac{1}{\pi}\mathrm{Im}\Pi_5^{(OPE)}(s),$$

Deriving the sum rule for f_B^2

- semilocal quark-hadron duality is used, the effective threshold s_0

$$\int\limits_{s_h}^{\infty} ds \frac{\rho^h(s)}{s-q^2} \simeq \frac{1}{\pi}\int\limits_{s_0}^{\infty} ds \frac{\mathrm{Im}\Pi_5^{(OPE)}(s)}{s-q^2}$$

- this yields approximate analytical relation for decay constant:

$$\frac{f_B^2 m_B^4}{m_B^2-q^2} = \frac{1}{\pi}\int\limits_{m_b^2}^{s_0} ds \frac{\mathrm{Im}\Pi_5^{(OPE)}(s)}{s-q^2}$$

- Borel transformation

$$\Pi_5(M^2) \equiv \mathcal{B}_{M^2}\Pi(q^2) = \lim_{\substack{-q^2,\, n\to\infty \\ -q^2/n=M^2}} \frac{(-q^2)^{(n+1)}}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_5(q^2)\,.$$

suppresses the higher-state contributions to the hadronic sum, the sum rule less sensitive to the duality approximation

$$\mathcal{B}_{M^2}\left(\frac{1}{m^2-q^2}\right) = \exp(-m^2/M^2)$$

The resulting QCD sum rule

$$f_B^2 m_B^4 e^{-m_B^2/M^2} = \int\limits_{m_b^2}^{s_0} ds e^{-s/M^2} \, \mathrm{Im}\Pi_5^{(OPE)}(s, m_b, m_u, \alpha_s, \langle 0|\bar{q}q|0\rangle, ...)$$

- current accuracy of $\Pi_5^{(OPE)}(q^2)$ at $q^2 \ll m_b^2$:
  vacuum condensates with $d \leq 6$
  loop $\oplus$ $O(\alpha_s) \oplus O(\alpha_s^2)$
  [K.Chetyrkin, M.Steinhauser (2001)]

- standard way to fix s_0:
  calculate the mass of B-meson from the same sum rule:

$$m_B^2 = -\frac{\frac{d}{d(1/M^2)}[SR]}{SR}$$

Input parameters

- sensitivity to the b-quark mass (in $\overline{MS}$ scheme)
- independent determination of m_b - quarkonium sum rules:
  the correlation function of two $\bar{Q}\gamma_\mu Q$ currents ($Q = b, c$)
  calculated in QCD up to $O(\alpha_s^3)$ and matched to the sum over
  $J^{PC} = 1^{--}$ heavy quarkonia levels
  measured in $e^+e^- \to \Upsilon, \Upsilon(2S), ...$ or
  $e^+e^- \to J/\psi, \psi(2S), ...$
- heavy quark masses in $\overline{MS}$: $\bar{m}_b(\bar{m}_b) = (4.18 \pm 0.03)\,\text{GeV}$,
  $\bar{m}_c(\bar{m}_c) = (1.275 \pm 0.025)\,\text{GeV}$,
  PDG values including QCD SR for quarkonia [K.Chetyrkin et al. (2012)]
- light quark masses and quark condensate density:
  (QCD SR for strange meson channels)
  [A.K., K. Chetyrkin (2005); M.Jamin, Oller, A.Pich (2006)]
  $m_s(\mu = 2\,\text{GeV}) = (98 \pm 16)\,\text{MeV}$,
  $m_s \oplus \text{ChPT} \to m_{u,d} \to \langle\bar{q}q\rangle(2GeV) = -(277^{+12}_{-10}\,\text{MeV})^3$

Universality of the method

- $\bar{q}Q$ currents with various flavour and J^P in the correlation functions $\Rightarrow$ sum rules for decay constants of B_s, D, D_s, π ρ, K, K^*, also baryonic, gluonic currents

  (any Lorentz-covariant and colour-invariant local operator)

- the coefficients in the OPE depend on the currents, inputs are universal (quark masses, α_s , condensates)

- QCD (SVZ) sum rules address the question: why are the hadrons not alike ?

- $SU(3)_{flavour}$ and heavy-quark symmetry violations can be estimated (finite quark masses, strange/nonstrange condensates)

$B_{(s)}$ and $D_{(s)}$ decay constants, recent update

P.Gelhausen, AK, A.A.Pivovarov, D.Rosenthal, 1305.5432 hep/ph]

Decay constant	Lattice QCD [ref.]	this work
f_B[MeV]	196.9 ± 9.1 [1] 186 ± 4 [2]	207^{+17}_{-9}
f_{B_s}[MeV]	242.0 ± 10.0 [1] 224 ± 5 [2]	242^{+17}_{-12}
f_{B_s}/f_B	1.229 ± 0.026 [1] 1.205 ± 0.007 [2]	$1.17^{+0.04}_{-0.03}$
f_D[MeV]	218.9 ± 11.3 [1] 213 ± 4 [2]	201^{+12}_{-13}
f_{D_s}[MeV]	260.1 ± 10.8 [1] 248.0 ± 2.5 [2]	238^{+13}_{-23}
f_{D_s}/f_D	1.188 ± 0.025 [1] 1.164 ± 0.018 [2]	$1.15^{+0.04}_{-0.05}$

[1]-Fermilab/MILC, [2]-HPQCD

Can we trust QCD sum rule estimates?

The story of $\Gamma(J/\psi \to \eta_c\gamma)$

- QCD sum rules from three-point correlator with $\bar{c}c$ currents

$\Gamma(J/\psi \to \eta_c\gamma) = 2.2 \pm 0.8$ KeV $(m_c = 1.25 GeV)$

[A.K. (1980); including gluon condensate correction (1984)]

- use of disp. relation for $\eta_c \to 2\gamma$ amplitude:

$\Gamma(J/\psi \to \eta_c\gamma) = 2.6 \pm 1.1$KeV *[M. Shifman, 1979]*

- three-point sum rules including $O(\alpha_s)$ corrections:

$\Gamma(J/\psi \to \eta_c\gamma) = 2.6 \pm 0.5 KeV$ *[V.Beilin, A.Radyushkin (1985)]*

- the only measurement in 1977 [Crystal Ball] $\to$ PDG:

$\Gamma(J/\psi \to \eta_c\gamma) = 1.3 \pm 0.4$ keV

- for many years considered a serious problem

for QCD sum rule approach

"QCD will not survive if $m_{\eta_c} = 2.977$ GeV and $J/\psi \to \eta_c\gamma$ rate is lower than 2 keV...
I expect that the experimental result will turn out close to 2 keV"

[M. Shifman, Z. Phys. (1980)]

- after thirty (!) years the CLEO data (2008):
$BR(J/\psi \to \eta_c \gamma) = 1.98 \pm 0.09 \pm 0.3\%$
$\Gamma(J/\psi \to \eta_c \gamma) = 1.84 \pm 0.29$ keV
(later confirmed by Novosibirsk experiment)

- the agreement of QCD sum rule prediction with experiment is restored.
The accuracy of the latter can be improved but demands two-loop, three-point, multi-scale perturbative diagrams , still a technical challenge...

Concluding:

- QCD sum rules: a method based on the twofold treatment of correlation functions:
  1) OPE
  2) hadronic dispersion relations

- B, D decay constants calculated with an accuracy, comparable to the one in lattice QCD

- further reading: reviews with more details

  A. Khodjamirian and R. Rückl, [hep-ph/9801443].

  "Snapshots of hadrons or the story of how the vacuum medium determines the properties of the classical mesons which are produced, live and die in the QCD vacuum" Mikhail A. Shifman [hep-ph/9802214]

  P. Colangelo and A. Khodjamirian, [hep-ph/0010175].

- several interesting applications ahead