Lectures on “Soft-Collinear Effective Theory”

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Helmholtz International Summer School on “Heavy Quark Physics”, Dubna, June 9/10, 2005

Soft-Collinear Effective Theory

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↪ provide field theory formalism for QCD effects relevant to rare decay and CP violation phenomenology
[← lectures by Ali, Fleischer, Höcker, Parkhomenko, ...]

↪ focus on basics of SCET + few illustrations
[→ also lectures by Lunghi]

School on Heavy Quark Physics, Dubna, June 9/10, 2005

SCET in Heavy Quark Physics

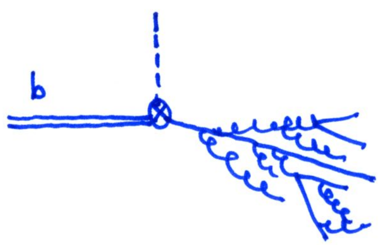
- HQET

$B \to D \ell \nu$, $B \to \pi \ell \nu$, inclusive B decays
[small π energy]

Heavy quark interacting with small-momentum ("soft") stuff

- SCET

Particles or jets with energy $\mathcal{O}(m_B)$



jet or single meson

$B \to \pi \ell \nu$ [large π energy]

$B \to \gamma \ell \nu$

$B \to M \gamma^{(*)}$, $M_1 M_2$

$B \to X_u(\text{jet})\, \ell \nu$, $X_s(\text{jet})\, \gamma$

Exclusive or semi-inclusive processes

↳ isolate strong coupling physics from calculable weak coupling physics
[$\alpha_s(m_b)$, $\alpha_s(\sqrt{m_b \Lambda}) \ll 1$]

↳ sum logs with renormalization group [not discussed in this lecture]

Plan of the lectures

I Learn how to reproduce Feynman integrals with two scales by contributions from relevant modes in the example $b \to u\,(\ell\nu)$.

Need of collinear modes.

Understand how to go from momentum regions to diagrammatic factorization and to fields with scaling rules.

II Derivation of the SCET Lagrangian and the effective $b \to u$ current including power corrections. Renormalization.

First steps towards $B \to X_u(\text{jet})\,\ell\nu$

III Learn how to prove factorization in SCET by factorizing hard from hard-collinear / soft and hard-collinear from soft.

Example $B \to X_u(\text{jet})\,\ell\nu$

Extending SCET to spectator interactions. The problem of endpoint singularities. $B \to \pi$ form factor as an example.

References

Expansion of Feynman integrals by momentum regions

[1] Beneke, Smirnov : NPB 522 (1998) 321 ; [2] Smirnov : Applied Asymptotic Expansions in Momenta and Masses, Springer, 2002

SCET

[3] Bauer, Fleming, Pirjol, Stewart : PRD 63 (2001) 114020 ; [4] Bauer, Pirjol, Stewart : PRD 65 (2002) 054022 (→ lecture 3)

[5] Beneke, Chapovsky, Diehl, Feldmann : NPB 643 (2002) 431 (→ lecture 2) ;
[6] Beneke, Feldmann : PLB 553 (2003) 267

$B \to X_u(\text{jet})\ell\nu$, $B \to X_s(\text{jet})\gamma$

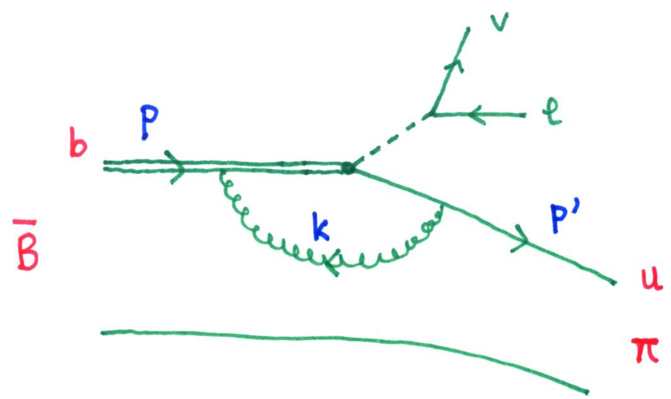
[7] Korchemsky, Sterman : PLB 340 (1994) 96 ; [8] Bosch, Lange, Neubert, Paz : NPB 699 (2004) 335 ;
power corrections : [9] Lee, Stewart : hep-ph/0409045 ; [10] Bosch, Neubert, Paz : JHEP 0411 (2004) 073 ; [11] Beneke, Campanario, Mannel, Pecjak : hep-ph/0411395 (to appear in JHEP)

B → light meson form factors at large recoil energy

[12] Charles et al. : PRD 60 (1999) 014001 ; [13] Beneke, Feldmann : NPB 592 (2001) 3 ; [14] Bauer, Pirjol, Stewart : PRD 67 (2003) 071502 ;
[15] Beneke, Feldmann : NPB 685 (2004) 249 (→ lecture 3) ;
[16] Lange, Neubert : NPB 690 (2004) 249

Factorization of Feynman diagrams ; momentum regions and effective fields

Example : weak decay $\bar{B} \to \pi l \nu$ [$b \to u l \nu$]



For illustration take b- and u-quark on-shell and massive gluon

$p^2 = m^2$ ($p = mv = m\,(1,0,0,0)$)

$p'^2 = 0$ ($p' = E n_- = E\,(1,0,0,-1)$)

$p \cdot p' = mE$

$\lambda^2 \ll m^2$ provides the small scale, analogous to Λ^2

$$I \equiv m^2 \frac{(4\pi)^2}{i} \int [dk] \, \frac{1}{[k^2-\lambda^2][k^2+2p\cdot k][k^2+2p'\cdot k]}$$

for simplification no numerator (all scalar propagators)

$$\equiv I\left(\frac{\lambda}{m}, E/m\right)$$

↑ small

UV + IR finite integral

The effective theory description of the process depends on the external momenta. Consider two different cases.

Case 1 ($\to$ HQET) : soft pion (u quark) $E = \mathcal{O}(\lambda)$

Even simpler: $E = 0$. Set $m = 1$, $\hat{\lambda} \equiv \lambda/m = \lambda$

$$I(\hat{\lambda}, 0) = \text{logs of square roots} = -\frac{\pi}{\lambda} + \left(-\frac{1}{2}\ln\lambda^2 + 1\right) + \mathcal{O}(\lambda)$$

To factorize the diagram, need to know the relevant integration regions.

Aim: construct expansion in λ ($\simeq \Lambda/m_b$!) from sum of all relevant momentum regions (modes)

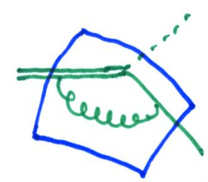
$$\boxed{\begin{array}{l}\text{HARD region}\\ \text{loop momentum}\\ k \sim m\end{array}}$$

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2][k^2+2p\cdot k][k^2]} + \begin{array}{l}\text{higher order}\\ \text{in } \lambda^2\end{array}$$

drop λ^2 because $k^2 \sim m^2 >> \lambda^2$

$p' = 0$

I_h is now IR-divergent $\longrightarrow$ need auxiliary regularisation; take dim. reg.



off-shell $\sim m^2$
short-distance

$$I_h = -\frac{1}{2\varepsilon} - \frac{1}{2}\ln\mu^2 + 1 + \mathcal{O}(\lambda^2)$$

μ scale of dim. reg.

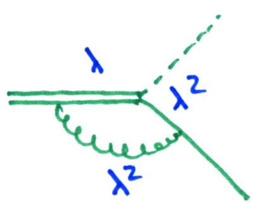
$\boxed{\text{SOFT region} \quad \text{loop momentum} \quad k \sim \lambda}$

$$I_s = m^2 \frac{(4\pi)^2}{i} \int \underbrace{[dk]}_{\lambda^4} \underbrace{\frac{1}{[k^2-\lambda^2][k^2][2p\cdot k]}}_{1/\lambda^5} \Big(\underbrace{1}_{1} - \underbrace{\frac{k^2}{2p\cdot k}}_{\lambda} + \dots \Big)$$

now $k^2 \sim \lambda^2$

I_s is UV-divergent from $\mathcal{O}(1)$ on

$\longrightarrow$ use SAME auxiliary regularization



internal lines with small virtuality

long-distance

$$= -\frac{\pi}{\lambda} + \left[\frac{1}{2\varepsilon} - \frac{1}{2}\ln\frac{\lambda^2}{\mu^2} \right] + \mathcal{O}(\lambda)$$

Sum of hard and soft reproduces the expansion of the exact result.

First lessons:

- Expansion in ratio of scales ($\rightarrow$ heavy quark expansion) can be obtained by adding contributions from relevant momentum regions

- Direct computation of the expansion is much simpler than the computation of the exact result, because all integrals involve only one scale, not several.

- Each term scales with a definite power of λ, which is determined before the calculation.

Q: Why did we integrate over all $\int d^dk$ even though k was assumed hard or soft?

A: Correct in dim. reg., because each integral contained only one scale, so in dim. reg. the result can only come from this scale (scaleless integrals vanish in dim. reg.

Factorization is particularly simple in dim. reg. !

Q: How did we know the relevant regions ?

A: They can be derived systematically from the structure of the denominator.

$k \sim \lambda^n$

$n=0$: hard (Taylor expansion of integrand)

$n=1$: soft

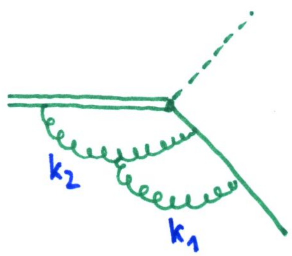
$n \geq 2$: $\int [dk] \frac{1}{[-\lambda^2] k^2 p\cdot k} = 0$

$k = v\cdot k \, v$ + smaller

$\rightarrow \int [dk] \frac{1}{[v\cdot k^2 - \lambda^2] (v\cdot k)^3} = 0$

scaleless in $k_\perp$
all poles in $v\cdot k$ on the same side in the complex plane

More loops



k_1	k_2
h	h
h	s
s	s

- just as in standard renormalization theory 1-loop additive factorization $\longrightarrow$ multiplicative factorization to all orders

- hard subgraphs polynomial in their soft external momenta $\longrightarrow$ local vertices like counterterms

$\Longrightarrow$ diagrammatic factorization

$$\langle u(p'=0)\,|\,\bar{u}\,\Gamma\, b\,|\,b(p)\rangle = \sum_i C^i_{hard}(m/\mu)\cdot M^i_{soft}(\mu/\lambda)$$

independent on external momenta

$\longrightarrow$ same when $b \to \bar{B}$

$u \to \pi, \rho, \ldots$

depends on Dirac matrix Γ

weak coupling in QCD

depends on soft external momenta (here $p'=0$) and IR scale λ

$\longrightarrow$ strong coupling in QCD

Effective fields and Lagrangians

Introduce fields for soft modes and construct Lagrangian that corresponds to the diagrammatic expansion rules $\longrightarrow$ reproduce M^i_{soft}

Heavy quark :

$p = mv + k$

$\sim \lambda$ changes by soft interactions

$h_v(x) \equiv \frac{1+\not{v}}{2} e^{imv\cdot x} \psi(x)$

fixed label v never changes

dual to k $x \sim 1/\lambda$

i.e. typical variations of h_v occur only over large distances

Diagrammatically, in the soft region

$\frac{\not{p}+m}{p^2-m^2} \longrightarrow \frac{1}{v\cdot k} + \dots$ corresponds to $\bar{\psi}(i\not{D}-m)\psi \longrightarrow \bar{h}_v iv\cdot D h_v + \dots$

(HQET)

Interactions of heavy quarks with soft modes are described by heavy quark effective theory.

Scaling rules

$$\mathcal{L}_{eff} = \bar{h}_v \, iv\cdot D h_v + \dots + \sum_{\text{light quarks}} \bar{q} i \not{D} q - \frac{1}{4} G^2$$

$$\bar{u}\Gamma b = \bar{u}\Gamma h_v + \dots$$

$$\underbrace{h_v(x)\bar{h}_v(y)}\ \sim \int \underbrace{d^4k}_{\lambda^4}\, e^{ik(x-y)} \underbrace{\frac{i}{v\cdot k}}_{1/\lambda} \sim \lambda^3 \quad \Rightarrow \quad \text{assign scaling} \quad h_v \sim \lambda^{3/2}$$

Object	Scaling
h_v, q	$\lambda^{3/2}$
A^μ	λ
$iD^\mu = i\partial^\mu + gA^\mu$	λ

$x \sim 1/\lambda \quad \partial \sim \lambda$

$$S_{eff} = \int \underbrace{d^4x}_{1/\lambda^4}\, \mathcal{L}_{eff} = \mathcal{O}(\lambda^0) + \text{corrections}$$

In HQET power counting is simple, because there is only one mode (soft).
λ can only occur as λ/m so λ-expansion $\simeq$ $1/m$ expansion

$\simeq$ dimensional analysis

Case 2 ($\to$ SCET) : energetic pion (u-quark) $E = \mathcal{O}(m)$

Even simpler $E = E_{max} = \frac{m}{2}$

$$I(\hat{\lambda}, \tfrac{1}{2}) = \text{logs and di-logs} = -\frac{1}{4}\left(\ln^2\lambda^2 + \pi^2\right) + \mathcal{O}(\lambda)$$

Hard contribution

$k \sim m$

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2]\,[k^2 + 2p\cdot k]\,[k^2 + 2p'\cdot k]} + \ldots$$

cannot be dropped now

$$= -\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon}\ln\mu^2 - \frac{1}{4}\ln^2\mu - \frac{\pi^2}{24} + \mathcal{O}(\lambda^2)$$

Soft contribution

$k \sim \lambda$

$$I_s = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2 - \lambda^2]\,[2p\cdot k]\,[2p'\cdot k]} + \ldots$$

$$= -\Gamma(\varepsilon) \int_0^\infty \frac{dx}{x} \left(x^2 + \lambda^2\right)^{-\varepsilon} + \ldots$$

ill-defined in dim. reg., so $I \neq I_h + I_s$

A relevant momentum region is missing!

External kinematics : additional vector $p' = En_- = E(1,0,0,-1)$ $\quad n_-^2 = 0$

$\longrightarrow$ introduce $n_+ = (1,0,0,1)$ such that $n_+^2 = 0$, $n_+ \cdot n_- = 2$

and write

$$q = n_+ \cdot q \, \frac{n_-}{2} + q_\perp + n_- \cdot q \, \frac{n_+}{2}$$ (light-cone decomposition)

$\longrightarrow$ $n_+ \cdot p' \sim m$ large

$p'^2 = n_+ \cdot p' n_- \cdot p' + p_\perp'^2 = 0$ small

COLLINEAR region

loop momentum

$n_+ \cdot k \sim m$

$k^2 \sim \lambda^2$

in general : $k^2 \ll n_+ \cdot k$

$$I_c = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2 - \lambda^2][m n_+ \cdot k][k^2 + n_+ \cdot p' n_- \cdot k]} + \ldots$$

- also ill-defined in dim. reg.

  need additional intermediate regularization (analytic - so that scaleless integrals still vanish)

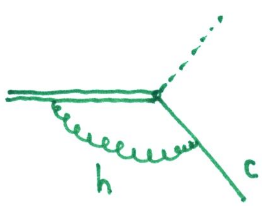
- $I_s + I_c$ is well-defined in dim. reg.

$$I_c + I_s = \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \ln \mu^2 + \frac{1}{4} \ln^2 \mu^2 - \frac{1}{4} \ln^2 \lambda^2 - \frac{5\pi^2}{24} + \mathcal{O}(\lambda)$$

$I_h + I_c + I_s$ reproduces the expanded exact result. This works to all orders in λ.

When there are energetic, nearly on-shell, nearly massless external lines, one must introduce collinear loop momentum configurations. The infrared structure is different from the case $E \sim \lambda$.

An important difference is non-locality



$n_+ \cdot p_c \sim m$ collinear momentum

$\Rightarrow$ hard subgraphs cannot be expanded in $n_+ \cdot p_c$ of external collinear momenta

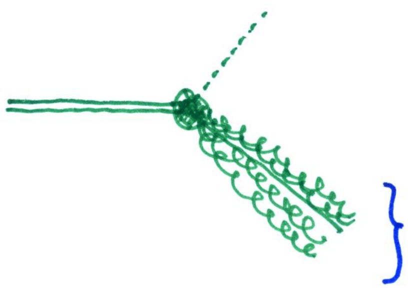
$\Rightarrow$ non-polynomial in $n_+ \cdot p_c$ $\cong$ non-local in position space

expect $\dfrac{1}{i n_+ \partial}$

Next lessons:

- For a given process (external kinematics) find the relevant momentum regions to construct the expansion.
- Introduce effective fields and vertices which reproduce this $\rightarrow$ effective Lagrangian

In the following consider



assume:
jet with
$E \sim m_b$
invariant mass
$p^2 \sim m_b^2 \lambda$

Relevant modes (set $m_b = 1$)

hard-collinear $\quad n_+ p \sim 1 \quad n_- p \sim \lambda \quad p_\perp \sim \lambda^{1/2}$

soft $\quad p \sim \lambda$

later add
collinear $\quad n_+ p \sim 1 \quad n_- p \sim \lambda^2 \quad p_\perp \sim \lambda$

Note: $\quad p_{hc}^2 \sim \lambda \gg p_s^2 \sim p_c^2 \sim \lambda^2$

Can add a soft line to a jet $(p_{hc}+p_s)^2 \sim \lambda$, but not a line with $p \sim \lambda^{1/2}$, hence $p_{hc}^2 \gg p_s^2$

Soft-collinear effective theory - fields and power counting

- light quarks

collinear quark $\quad \psi_c = \xi + \eta \qquad \xi \equiv \frac{\not{n}_-\not{n}_+}{4}\psi_c$ large $\qquad \eta \equiv \frac{\not{n}_+\not{n}_-}{4}\psi$ small

since $\not{p}' u_c(p') = 0$

$\longrightarrow$ η is integrated out (see below)

$$\underbrace{\xi(x)\bar{\xi}(y)} \sim \int \underbrace{d^4p}_{\lambda^2}\, e^{-ip(x-y)}\, \underbrace{\frac{in_+ p}{p^2}}_{\frac{1}{\lambda}} \sim \lambda \qquad \Rightarrow \qquad \text{Field } \xi \text{ with } \not{n}_-\xi = 0 \text{ and } \xi \sim \lambda^{1/2}$$

soft quark

$$\underbrace{q(x)\bar{q}(y)} \sim \int \underbrace{d^4k}_{\lambda^4}\, e^{-ik(x-y)}\, \underbrace{\frac{i\not{k}}{k^2}}_{1/\lambda} \sim \lambda^3 \qquad q \text{ with } q \sim \lambda^{3/2}$$

- heavy quarks

can never be collinear since always $n_+p \sim 1$ and $n_-p \sim 1$

soft heavy quark
(as in HQET)

$$Q = e^{-imv\cdot x}(h_v + \underbrace{H_v}_{\text{integrate out}}) \qquad \Rightarrow \qquad h_v \sim \lambda^{3/2} \quad (\text{as before})$$

$\not{v} h_v = h_v$

- gluons

collinear gluon $\qquad \underbrace{A_c^\mu(x) A_c^\nu(y)} \sim \int d^4p \, e^{-ip(x-y)} \, \frac{i}{p^2} \left(-g^{\mu\nu} + (1-\xi) \frac{p^\mu p^\nu}{p^2} \right)$

$\Rightarrow \qquad n_+ \cdot A_c \sim 1 \,, \quad A_{\perp c} \sim \lambda^{1/2} \,, \quad n_- \cdot A_c \sim \lambda$ (same as collinear momentum)

soft gluon $\qquad A_s \sim \lambda$

- derivatives on fields

soft fields vary over distances $x \sim 1/\lambda \quad \Rightarrow \quad \partial \phi_s \sim \lambda \phi_s$

collinear fields vary differently in different directions $\qquad n_+ \cdot x \sim 1/n_- \cdot p_c \sim 1/\lambda \qquad x_\perp \sim 1/p_{c\perp} \sim 1/\lambda^{1/2}$

$n_- \cdot x \sim 1/n_+ \cdot p_c \sim 1$

$$\Rightarrow \quad \begin{aligned} n_+ \cdot \partial \phi_c &\sim \phi_c \\ \partial_\perp \phi_c &\sim \lambda^{1/2} \phi_c \\ n_- \cdot \partial \phi_c &\sim \lambda \phi_c \end{aligned}$$

(same as momentum)

$\Longrightarrow$ Power counting for operators (field products)

Derivation of the soft-collinear Lagrangian

- until otherwise mentioned "collinear" will now mean "hard-collinear". Recall that these collinear modes have <u>larger</u> virtuality than soft modes.
- first set q (soft quark field) to 0.

Step 1: Integrate out small components of the collinear quark field

$$\mathcal{L}_c = \bar{\Psi}_c \, i\not{D} \, \Psi_c \underset{\Psi_c = \xi + \eta}{=} \bar{\xi} \frac{\not{n}_+}{2} in_- \cdot D \, \xi + \bar{\eta} \frac{\not{n}_-}{2} in_+ \cdot D \, \eta + \bar{\xi} \, i\not{D}_\perp \eta + \bar{\eta} \, i\not{D}_\perp \xi$$

Gaussian path integral over η can be done exactly and sets

$$\eta = -\frac{\not{n}_+}{2} \frac{1}{in_+ \cdot D + i\varepsilon} i\not{D}_\perp \xi$$

definition!

Functional determinant $\det[in_+ \cdot D] = \det[in_+ \cdot \partial]$ is a field-independent (irrelevant constant)

$$\Rightarrow \quad \mathcal{L}_c = \bar{\xi} \left(in_- \cdot D + i\not{D}_\perp \frac{1}{in_+ \cdot D} i\not{D}_\perp \right) \frac{\not{n}_+}{2} \xi$$

non-local

This is exact!

Remarks:

(1) No degrees of freedom have been integrated out!

$\mathcal{L}_c$ is simply the QCD Lagrangian in a frame where all particles are boosted to large momentum

(2) Non-locality can be made explicit in terms of Wilson lines.

Define W by $\quad [\, i n_+ \cdot D \, W \,] = 0$:

$$W(x) = P \exp\left(ig \int_{-\infty}^{0} ds \; n_+ \cdot A(x + s n_+) \right)$$

path ordering of fields

$A = A_c + A_s$

$$\Rightarrow \quad \frac{1}{i n_+ \cdot D} = W \frac{1}{i n_+ \cdot \partial} W^\dagger \, , \quad W W^\dagger = 1$$

$$\mathcal{L}_c = \bar{\xi} \left(i n_- \cdot D + i \not{D}_\perp W \frac{1}{i n_+ \cdot \partial} W^\dagger i \not{D}_\perp \right) \frac{\not{n}_+}{2} \xi$$

$$= \bar{\xi}(x) \, i n_- \cdot D \frac{\not{n}_+}{2} \xi(x) + i \int_{-\infty}^{0} ds \, [\bar{\xi} \, i \overleftarrow{\not{D}}_\perp W](x) \, [W^\dagger i \not{D}_\perp \frac{\not{n}_+}{2} \xi](x + s n_+)$$

$$\left[\frac{1}{i n_+ \cdot \partial + i\varepsilon} f(x) = -i \int_{-\infty}^{0} ds \, f(x + s n_+) \right]$$

one $n_- \cdot A$, up to two $A_\perp$, infinitely many $n_+ \cdot A$ (vanish in light-cone gauge)

Step 2: Expansion of the exact Lagrangian in $\lambda^{1/2}$; Light-front multipole expansion

$$i n_-\cdot D = \underset{\lambda}{i n_-\cdot\partial} + \underset{\lambda}{g n_-\cdot A_c} + \underset{\lambda}{g n_-\cdot A_s}$$

$$i D_\perp = \underset{\lambda^{1/2}}{i\partial_\perp} + \underset{\lambda^{1/2}}{g A_{\perp c}} + \underset{\lambda}{g A_{\perp s}} \equiv i D_{\perp c} + g A_{\perp s}$$

$$\frac{1}{i n_+\cdot D} = \underset{\lambda^0}{\frac{1}{i n_+\cdot D_c}} - \frac{1}{i n_+\cdot D_c}\, \underset{\lambda}{g n_+\cdot A_s}\, \frac{1}{i n_+\cdot D_c} + \dots$$

Multipole expansion

$$\int d^4x \;\bar\xi(x)\, n_-\cdot A_s(x)\, \frac{\not{n}_+}{2}\, \xi(x)$$

dominated by rapid variations of collinear fields

$n_-\cdot x \sim 1$, $x_\perp \sim 1/\lambda^{1/2}$

$n_+\cdot x \sim 1/\lambda$

slowly varying $x \sim 1/\lambda$ in all components

$\Rightarrow$ Taylor expansion in $n_-\cdot x$, $x_\perp$ around

$$x_-^\mu \equiv n_+\cdot x\, \frac{n_-^\mu}{2}$$

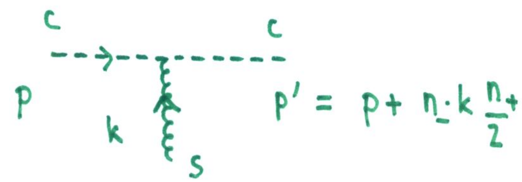
Hence, in products of collinear and soft fields, expand the soft fields:

$$\phi_s(x) = \underset{\lambda^0}{\phi_s(x_-)} + \underset{\lambda^{1/2}}{[x_\perp\cdot\partial\phi_s](x_-)}$$

$$+ \frac{n_-\cdot x}{2}\underset{\lambda}{[n_+\partial\phi_s](x_-)} + \frac{1}{2} x_\perp^\mu x_\perp^\nu \underset{\lambda}{[\partial_\mu\partial_\nu\phi_s](x_-)}$$

$$+ \dots\dots$$

Interpretation:



$k_\perp, n_+ k$ are expanded, because $k_\perp \ll p_\perp$, $n_+ k \ll n_+ p$ (diagrammatic interpretation)

cf. atomic (non-relativistic) physics: $\underbrace{e^{i\vec{k}\cdot\vec{x}}}_{\text{phase of light wave}} \approx 1 + \underbrace{i\vec{k}\cdot\vec{x}}_{\sim \frac{\text{size of atom}}{\text{wavelength of light}} \ll 1} + \dots$

"dipole approximation"

Here x_- plays the role of time t, which is not expanded

$$\mathcal{L}_c = \mathcal{L}_\xi^{(0)} + \mathcal{L}_\xi^{(1)} + \dots = \bar{\xi}\left(in_- D + i\not{D}_{\perp c}\frac{1}{in_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi$$

$$+ \bar{\xi}\left(x_\perp \cdot \partial\, g n_- A_s + i\not{D}_{\perp c}\frac{1}{in_+ D_c} g\not{A}_{\perp s} + g\not{A}_{\perp s}\frac{1}{in_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi + \dots$$

- all soft fields are taken at x_-
- translation invariance not manifest (no momentum conservation at vertices)
- $\mathcal{L}_\xi^{(0)}$ contains only the $n_- A_s$ component (at x_-) – key property for factorization proofs
- gauge-invariance not manifest

Can transform $\mathcal{L}_\xi^{(1)}$ into

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}(x)\, x_\perp^\mu n_-^\nu W_c\, g F^s_{\mu\nu}(x_-) W_c^\dagger \frac{\not{n}_+}{2} \xi$$

analogue of the $\vec{x}\cdot\vec{E}$ dipole interaction

$$W_c = W|_{A\to A_c} \quad \text{collinear Wilson line}$$

Pedestrian derivation for abelian gauge fields :

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}\left\{ i\left[g x_\perp\cdot A_s,\, i\not{D}_{\perp c}\frac{1}{in_+\cdot D_c} i\not{D}_{\perp c}\right] + (x_\perp\cdot\partial\, g n_-\cdot A_s)\right\}\frac{\not{n}_+}{2}\xi$$

$\vdots$ use

$$i\left[x_\perp^\mu,\, i\not{D}_{\perp c}\frac{1}{in_+\cdot D_c} i\not{D}_{\perp c}\right] = \gamma_\perp^\mu \frac{1}{in_+\cdot D_c} i\not{D}_{\perp c} + i\not{D}_{\perp c}\frac{1}{in_+\cdot D_c}\gamma_\perp^\mu$$

$$= \bar{\xi}\left\{ - x_\perp^\mu\, g n_-\cdot\partial A_{s\mu} + x_\perp\cdot\partial\, g n_-\cdot A_s\right\}\frac{\not{n}_+}{2}\xi$$

$\vdots$ use equation of motion

$$i\not{D}_{\perp c}\frac{1}{in_+\cdot D_c} i\not{D}_{\perp c}\frac{\not{n}_+}{2}\xi = -\, i n_-\cdot D\frac{\not{n}_+}{2}\xi + O(\lambda^{1/2})$$

$$= \bar{\xi}\, x_\perp^\mu n_-^\nu\, g F^s_{\mu\nu}\frac{\not{n}_+}{2}\xi$$

This becomes very complicated in higher orders. Impractible for non-abelian gauge fields, where one must use the collinear gluon eq. of. motion

Note on gauge symmetry

- separate gauge symmetry for c and s fields
- must not mix powers of $\lambda^{1/2}$
  must transform c fields into c fields and s fields into s fields

collinear gauge symmetry

$$\xi \to U_c \xi \qquad n_+ \cdot A_c \to U_c n_+ \cdot A_c U_c^\dagger + \frac{i}{g} U_c [n_+ \cdot \partial, U_c^\dagger]$$

$$q \to q \qquad A_{\perp c} \to U_c A_{\perp c} U_c^\dagger + i/g\, U_c [\partial_\perp, U_c^\dagger]$$

$$A_s \to A_s \qquad n_- \cdot A_c \to U_c n_- \cdot A_c U_c^\dagger + i/g\, U_c [n_- \cdot D_s(x_-), U_c^\dagger]$$

$$W_c \to U_c W_c \qquad W_c \to U_s W_c U_s^\dagger$$

soft gauge symmetry

$$\xi \to U_s \xi$$

$$q \to U_s q$$

$$A_c \to U_s A_c U_s^\dagger$$

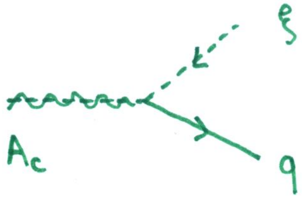
$$A_s \to U_s A_s U_s^\dagger + \frac{i}{g} U_s [\partial, U_s^\dagger]$$

$$U_s = U_s(x_-)$$

$\Longrightarrow$ collinear invariant building blocks $W_c^\dagger \xi$, $\underbrace{[W_c^\dagger i D_{\perp c}^\mu W_c]}_{A_{\perp c}^\mu \text{ in light-cone gauge } n_+ \cdot A_c = 0}$

$\Rightarrow$ systematic procedure to construct the Lagrangian to any order - see ref. [6]

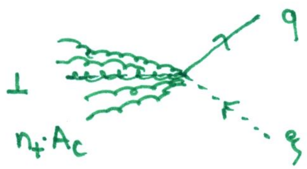
Step 3 Add soft quarks



$$\bar\Psi \not{A} \Psi \ni \bar q \underbrace{\not{A}_c}_{\substack{n_+\cdot A_c \frac{\not n_-}{2} \;+\; \not A_{\perp c} \;+\; n_-\cdot A_c \frac{\not n_+}{2} \\ \lambda^0 \qquad\quad \lambda^{1/2} \qquad \lambda}} \xi + \ldots \approx \bar q \not{A}_{\perp c}\, \xi + \ldots$$

but $\not n_- \xi = 0$

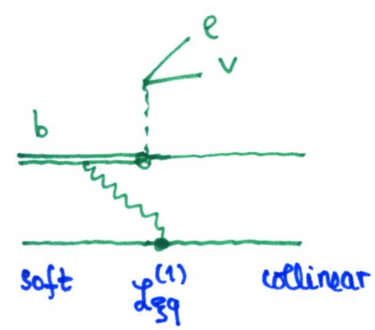
but this is not all:



$$\mathcal{L}^{(1)}_{\xi q} = \underbrace{\underset{\lambda^{3/2}}{\bar q}\; W_c^\dagger\; \underset{\lambda^{1/2}}{i\not D_{\perp c}}\; \underset{\lambda^{1/2}}{\xi}}_{\lambda^{5/2}} + h.c.$$

gauge-invariant

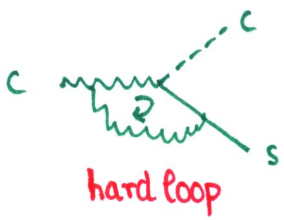
- soft quark interactions are power-suppressed, $\mathcal{O}(\lambda^{1/2})$
  higher-order terms are known
- $\mathcal{L}^{(1)}_{\xi q}$ is very important for exclusive B decays
  Soft spectator quark must be converted into collinear.



(No) Renormalization of the SCET Lagrangian

Up to now interactions of soft and collinear modes at tree-level.

Expect that hard loops (short-distance fluctuations) modify the effective vertices



$$\mathcal{L}_{SCET,tree} = \sum_i O_i \longrightarrow \mathcal{L}_{SCET} = \sum_{i'} C_{i'} O_{i'}$$

expansion in $g^2(m) \ll 1$

includes new operators

However, if factorization is done in dim. reg.:

$C_i \equiv 1$ to all orders

no new operators

<u>The tree Lagrangian is exact!</u>

Reason:

- Hard loops must depend on invariants of order m_b^2.
  The hard momenta are $\tilde{p}_i \equiv n_+ p_i \frac{n_-}{2}$ but $\tilde{p}_i \cdot \tilde{p}_j \propto n_-^2 = 0$

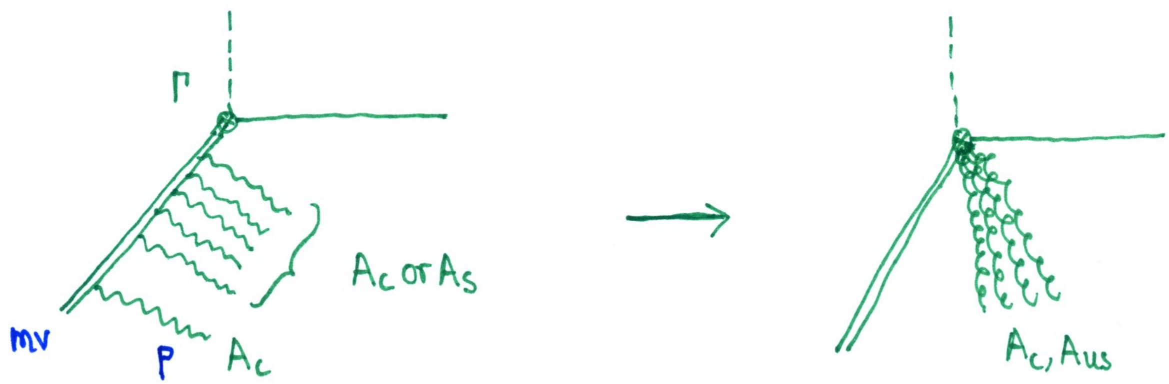
  There are no invariants, so all loops vanish.

- The notion "collinear" has a Lorentz-invariant meaning only in the presence of external sources; nothing has been integrated out, really.

The effective $b \to u$ transition current

Aim: represent the operator $\underset{\bar{u}}{\bar{\Psi}}\,\Gamma\, \underset{b}{Q}$ in SCET. Ψ carries momentum of order m_b

Treelevel matching is non-trivial



After the first emission, the heavy quark is off-shell

$(mv+p)^2 \approx m\, n_+ \cdot p = \mathcal{O}(m^2)$

$\Rightarrow$ must be integrated out

$\Rightarrow$ effective vertex

leading power:

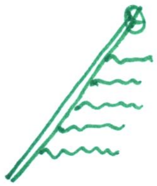
any number of $n_+ \cdot A_c$

$\lambda^{1/2}$: one $A_{\perp c}$

λ: two $A_{\perp c}$, one $n_- \cdot A_c$

or one A_{us}

etc.



$$= \overset{\lambda^0}{W_c h_v} - \frac{\not n_-}{2m_b}\overset{\lambda^{1/2}}{[iD_{\perp c} W_c]}\, h_v + \mathcal{O}(\lambda h_v)$$

$$= \overset{\lambda^0}{\bar\xi} - \overset{\lambda^{1/2}}{\bar\xi\, i\overleftarrow{\not D}_{\perp c} \frac{1}{i n_+ \overleftarrow{D}_c}\frac{\not n_+}{2}} + \mathcal{O}(\lambda^{1/2}\bar\xi)$$

infinite number of attachments sum to a Wilson line

$$\Rightarrow \qquad [\bar\Psi \Gamma Q](x) = e^{-imv\cdot x}\left(\sum_i C_i^{(0)} * J_i^{(0)} + \sum_i C_i^{(1)} * J_i^{(1)} + \ldots \right)$$

short-distance correction (hard loops), see below.

$$J_j^{(A0)} = (\bar\xi W_c)(x + s n_+)\, \Gamma_j'\, h_v(x_-) \equiv (\bar\xi W_c)_s\, \Gamma_j'\, h_v$$

leading power

$$J_j^{(A1)} = (\bar\xi W_c)_s\, i\overleftarrow{\partial}_\perp^\mu \frac{1}{i n_+ \overleftarrow{\partial}} \frac{\not n_+}{2} \Gamma_j'\, h_v$$

$$J_j^{(B1)} = \frac{1}{m_b} (\bar\xi W_c)_{s_1} (W_c^\dagger i D_{\perp c}^\mu W_c)_{s_2}\, \Gamma_j'\, h_v$$

$\mathcal{O}(\lambda^{1/2})$ suppressed "two-body" and "three-body"

Quark bilinears and symmetries

QCD

$\langle u(p') | \bar\psi \Gamma Q | b(mv) \rangle$

$\Gamma = 1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}$

$\Rightarrow$ 10 independent form factors
(16−6 e.o.m.)

SCET

$$\bar\xi\, \Gamma_j' h_v = \bar\xi \underbrace{\frac{\not{n}_+ \not{n}_-}{4}}_{} \Gamma_j' \not{v} h_v$$

⋮ occurs in $J^{(A0,A1,B1)}$

projections reduce the number of independent structures

between $\frac{\not{n}_+ \not{n}_-}{4}$ and $\not{v}$:

$1 \longrightarrow 1$

$\gamma_5 \longrightarrow \gamma_5$

$$\gamma^\mu = \not{n}_+ \frac{n_-^\mu}{2} + \cancel{\not{n}_- \frac{n_+^\mu}{2}} + \gamma_\perp^\mu$$

$\not{n}_+ = 2\not{v} - \cancel{\not{n}_-}$

$$\longrightarrow n_-^\mu \cdot 1 + \gamma_\perp^\mu$$

$$\gamma^\mu\gamma_5 \longrightarrow -n_-^\mu \gamma_5 + \underbrace{\gamma_\perp^\mu \gamma_5}$$

$$= -\frac{i}{6} \varepsilon^{\mu_\perp \nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \longrightarrow \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_{\nu\perp} n_{-\rho} n_{+\sigma}$$

nothing new

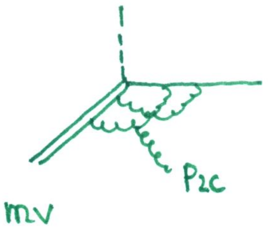
$$i\sigma^{\mu\nu} \longrightarrow \frac{n_-^\mu n_+^\nu - n_+^\mu n_-^\nu}{2} \cdot 1 + \gamma_\perp^\mu n_-^\nu - \gamma_\perp^\nu n_-^\mu + \underbrace{\frac{1}{2}[\gamma_\perp^\mu, \gamma_\perp^\nu]}$$

$$= -\frac{i}{2} \varepsilon^{\mu_\perp \nu_\perp \rho\sigma} \frac{1}{2}[\gamma_\rho, \gamma_\sigma]\gamma_5 \longrightarrow -\frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} n_{-\rho} n_{+\sigma} \gamma_5$$

nothing new

$\Rightarrow$ only $\Gamma_j' = \{1, \gamma_5, \gamma_\perp^\mu\}$ $\Rightarrow$ only 3 rather than 10 heavy-to-light form factors at leading power

Short-distance corrections



hard loops

Now can built $O(m^2)$ invariants : m^2 , $m\, n_+ \cdot p_{ic}$

$\Rightarrow$ modification of tree-level coefficients by functions $C_i\left(m^2/\mu^2, \frac{n_+ \cdot p_{ic}}{m}\right)$

dependence on collinear external momentum $\rightarrow$ non-locality in position space

$$[\bar\Psi \Gamma Q](x) = e^{-imv\cdot x} \sum_i \int_{-\infty}^{0} ds\, \tilde C_i^{\Gamma}(m/\mu, s\mu)\, (\bar\xi W_c)(x+sn_+)\, \Gamma_i'\, h_v(x_-) + \ldots\ldots$$

then

$$\langle \bar u(p') | \bar\Psi \Gamma Q | b(p) \rangle = \sum_i C_i^{\Gamma}\left(m/\mu, \frac{n_+ p'}{m}\right) \langle u(p') | (\bar\xi W_c) \Gamma_i' h_v | b(p) \rangle + \ldots$$

with $C_i^{\Gamma} = \int_{-\infty}^{0} ds\, e^{isn_+ \cdot p'} \tilde C_i^{\Gamma}$

In general $C_i = C_i\left(\frac{m}{\mu}, \frac{2E}{m}, \tau_1, \ldots, \tau_{n-2}\right)$

$\frac{2E}{m} = \frac{\sum n_+ \cdot p_{ci}}{m}$ $\quad \tau_j = \frac{n_+ \cdot p_{ci}}{\sum_j n_+ \cdot p_{cj}}$ momentum fractions

E total energy

[see Beneke, Kiyo, Yang ; Hill, Becher, Lee, Neubert ; Becher, Hill ; Beneke, Yang for $J^{(B1)}$]

Examples of factorization with SCET

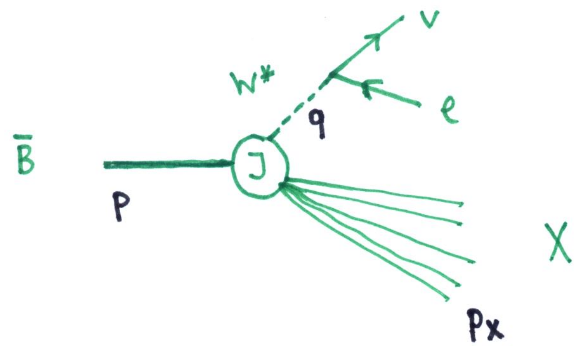
1) $\bar{B} \to X_u(\text{jet})\, \ell\nu$

(inclusive $b \to u\ell\nu$ in the "shape function" region)

2) $\bar{B} \to \pi\ell\nu$, energetic π

(heavy-to-light form factors at large recoil energy)

Kinematics of $\bar{B} \to X_u e \nu$



hadronic final state

Sum over all hadronic final states

+ optical theorem give

$$\frac{d\Gamma}{dq^2 dE_e} = G_F^2 |V_{ub}|^2 * \text{kinematical factors} * \operatorname{Im} T$$

where

$$T \equiv i \int d^4x \, e^{-iq\cdot x} \langle \bar{B}(p) | T(J^{+}_{\mu}(x) J^{\mu}(0)) | \bar{B}(p) \rangle$$

"hadronic tensor"

$$J_\mu = \bar{u} \gamma_\mu (1-\gamma_5) b$$

Scales

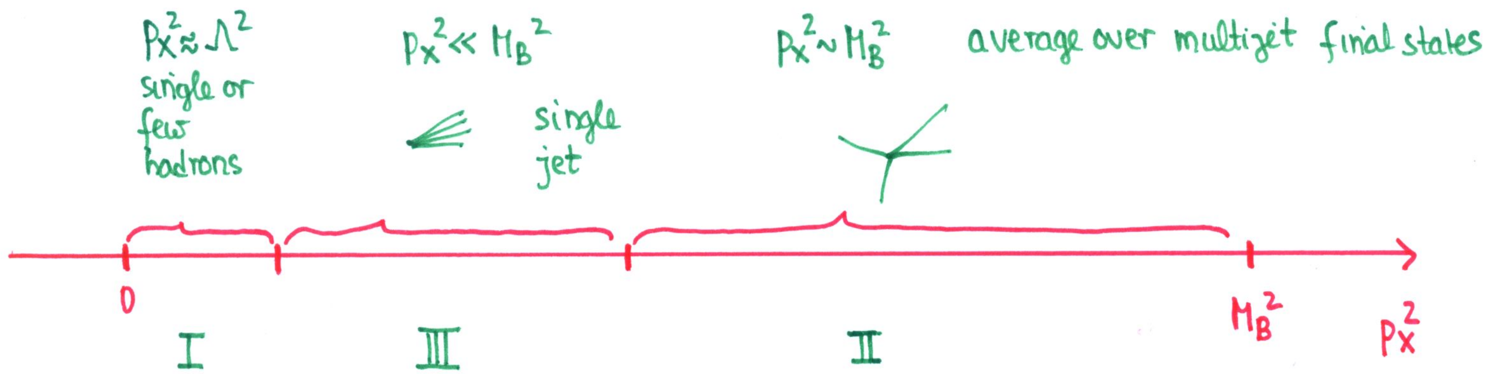
1) $p^2 = M_B^2$ heavy quark mass

2) q^2 lepton-invariant mass

3) $2p\cdot q$ or $p_X^2 = (p-q)^2 = M_B^2 + q^2 - 2M_B(E_e + E_\nu)$ in the $\bar{B}$ rest frame

invariant mass of the hadronic final state

Assume $v \cdot p_X = E_X$ = hadronic energy $\sim \mathcal{O}(M_B)$



I exclusive or resonance region ($B \to \pi \ell \nu$, $B \to \rho \ell \nu$), see later example

III semi-inclusive region, collinear modes are relevant, but perturbative since

$$p_X^2 \sim M_B^2 \lambda \gg \Lambda^2$$

i.e. $\lambda \sim \frac{p_X^2}{M_B} \gg \frac{\Lambda^2}{M_B^2}$

e.g. $\lambda \sim \frac{\Lambda}{M_B}$

This is the region where SCET as discussed up to now applies.

II inclusive region; no direction singled out $\to$ collinear modes not relevant, only hard and soft

$\to$ OPE of hadronic tensor + HQET

Aim: Show that [in region II]

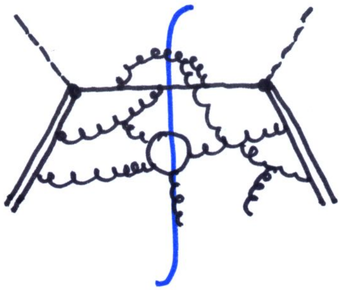
$$\mathrm{Im}\,T = \underset{\substack{\text{scale } m_b^2 \\ \text{(hard)}}}{H} \cdot \underset{\substack{\text{scale } m_b^2\lambda \\ \text{(collinear modes only)}}}{J} * \underset{\substack{\text{scale } m_b^2\lambda^2 \text{ (and smaller)} \\ \text{(soft)}}}{S} + \mathcal{O}(\lambda) \text{ corrections}$$

Factorization at leading power (Ref.[7] diagrammatically)

$$\mathrm{QCD} \xrightarrow[\text{hard modes}]{\text{integrate out}} \mathrm{SCET} \xrightarrow[\text{modes perturbatively}]{\text{integrate out collinear}} \mathrm{HQET}$$

since $p_X^2 \sim m_b^2\lambda \gg \Lambda^2$

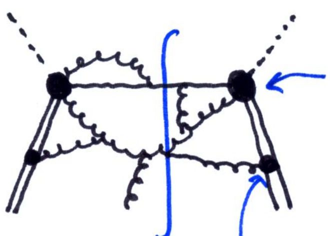
Step 1 Factorization of hard modes



hard, collinear, soft

cut for $\mathrm{Im}\, T$

$\downarrow$



effective current

effective vertex from $\mathcal{L}_{HQET}$

$$\mathcal{L}_{QCD} \longrightarrow \mathcal{L}_{HQET} + \mathcal{L}_{SCET}$$

$$J_\mu(x) \longrightarrow e^{-imv\cdot x} \sum_i \tilde{C}_i * O_i$$

convolution

$[\bar{u}\gamma_\mu(1-\gamma_5)b](x)$

$\bar{\xi}(x)\Gamma_j' W_c(x) h_v(x_-)$ in leading power

$$T \longrightarrow i\int d^4x\, e^{i(m_b v - q)\cdot x} \sum_{i,j} C_i(m_b, E_x)\, C_j(m_b, E_x)$$

$$\times \langle \bar{B}_v | T(O_i^\dagger(x) O_j(0)) | \bar{B}_v \rangle$$

- note: hard modes cannot be cut when $p_X^2 \ll M_B^2$, therefore drop local term  in T

$$\Rightarrow \qquad \mathrm{Im}\, T = \sum_k \underbrace{H_k(m, E_x, \mu)}_{\text{product } C_i \cdot C_j} \; \underbrace{*}_{\text{convolution in general}} \; \underbrace{\mathrm{Im}\, T_k^{SCET}(E_x, \mu)}_{\text{SCET current product}}$$

This can be done including power corrections in λ. Just keep power-suppressed effective currents.

In leading power T_k^{eff} is :

$$i \int d^4x \, e^{i(m_b v - q)\cdot x} \langle \bar{B}_v | T\left([\bar{h}_v \Gamma_1^{\dagger} W_c^{\dagger} \xi](x) \, [\bar{\xi} \Gamma_2 W_c h_v](0) \right) | \bar{B}_v \rangle$$

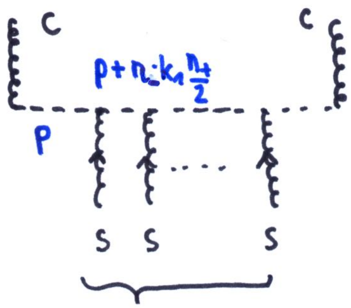
Step 2 Decoupling of soft modes from collinear modes

Still have two scales :

$M_B^2 \lambda \gg \Lambda_{QCD}^2$ virtuality of collinear modes, i.e. the hadronic jet

$M_B^2 \lambda^2$ virtuality of soft modes

AIM : decouple collinear from soft and integrate them out



n attachments of soft gluons to a collinear quark

$$\frac{i n_+ p}{\left(p+n_-(k_1+\dots+k_n)\frac{n_+}{2}\right)^2}\frac{\not n_-}{2}\; i g\, n_- A_s \frac{\not n_+}{2} \cdots\cdots i g\, n_- A_s \frac{\not n_+}{2}\frac{i n_+ p}{\left(p+n_- k_1 \frac{n_+}{2}\right)^2}\frac{\not n_-}{2}\; i g\, n_- A_s \frac{\not n_+}{2}\frac{i n_+ p}{p^2}\frac{\not n_-}{2}$$

$$= \underbrace{(-g)^n \frac{n_- A_s}{n_+(k_1+\dots+k_n)} \cdots\cdots \frac{n_- A_s}{n_+ k_1}}\; \frac{i n_+ p}{p^2}\frac{\not n_-}{2}$$

This is the Feynman rule for the nth term in the expansion of the soft Wilson line

$$Y(x) \equiv P e^{ig\int_{-\infty}^{0} ds\, n_- A_s(x_- + s n_-)}$$

This suggests the field redefinition

$$\xi(x) \equiv Y(x)\, \xi^{(0)}(x)$$

$$A_c(x) \equiv Y(x)\, A_c^{(0)\mu}(x)\, Y^\dagger(x)$$

$$W_c^{(0)} \equiv P e^{ig\int_{-\infty}^{0} ds\, n_+ A_c^{(0)}(x+s n_+)}$$

Then $Y \to U_s Y$ under a soft gauge transformation, so $\xi^{(0)}$ and $A_c^{(0)}$ are invariant under soft gauge transformations and hence may not couple to soft gluons

Plug this into the Lagrangian

$$\bar{\xi}^{(0)} Y^\dagger \left(i n_- D_s + Y g n_- A_c^{(0)} Y^\dagger + [i\not{\partial}_\perp + Y g \not{A}_{c\perp}^{(0)} Y^\dagger] \, Y W_c^{(0)} Y^\dagger \frac{1}{i n_+ \partial} Y W_c^{(0)\dagger} Y^\dagger \, [i\not{\partial}_\perp + Y g \not{A}_{c\perp}^{(0)} Y^\dagger] \right) \frac{\not{n}_+}{2} Y \xi^{(0)}$$

use :

$$Y^\dagger Y = 1 \,, \quad i n_- \cdot D_s Y = Y \, i n_- \cdot \partial \,, \quad i \partial_\perp Y = Y i \partial_\perp \,, \quad \frac{1}{i n_+ \partial} Y = Y \frac{1}{i n_+ \partial}$$

$$= \bar{\xi}^{(0)} \left(i n_- D_c^{(0)} + i \not{D}_{\perp c}^{(0)} \frac{1}{i n_+ D_c^{(0)}} i \not{D}_{\perp c}^{(0)} \right) \frac{\not{n}_+}{2} \xi^{(0)}$$

i.e. $\mathcal{L}_\xi^{(0)}$ is independent of soft fields when expressed through $^{(0)}$- fields. Same for Yang-Mills Lagrangian at leading power. Crucial was that $\mathcal{L}_\xi^{(0)}$ depended only on $n_- A_s$

$$\Rightarrow \quad \mathcal{L}_{SCET} = \underbrace{\mathcal{L}_\xi^{(0)} + \mathcal{L}_{YM}^{(0)}}_{\text{depends ONLY on collinear fields } \xi^{(0)},\, A_c^{(0)}} + \underbrace{\bar{q}\, i\not{D}_s q + \bar{h}_v \, i v \cdot D_s h_v}_{\text{depends ONLY on soft fields } q,\, A_s,\, h_v} + \text{power corrections}$$

$\Rightarrow$ decoupling of soft and collinear (after redefinition) fields

Power corrections

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}^{(0)} W_c^{(0)} x_\perp^\mu n_-^\nu \, Y^\dagger F^s_{\mu\nu} Y \, \frac{\not{n}_+}{2} W_c^{(0)\dagger} \xi$$

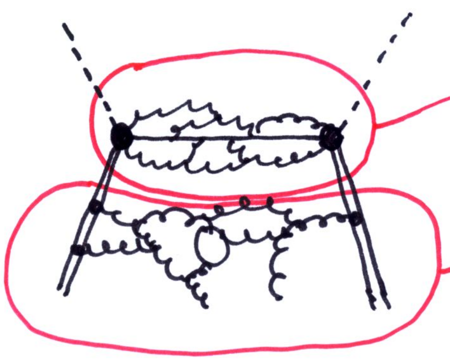
$$\mathcal{L}_{\xi q}^{(1)} = \bar{q} Y \, W_c^{(0)} \, i\not{D}_{\perp c}^{(0)} \xi^{(0)} + \text{h.c.}$$

do contain soft fields

no decoupling at $\mathcal{O}(\lambda^{1/2})$

Step 3 Soft-collinear factorization of the hadronic tensor

$$T^{SCET} = i \int d^4x \, e^{i(m_b v - q)\cdot x} \langle \bar{B}_v | T([\bar{h}_v Y \Gamma_1^\dagger W_c^{(0)\dagger} \xi^{(0)}](x) \, [\bar{\xi}^{(0)} W_c^{(0)} \Gamma_2 Y^\dagger h_v](0) | \bar{B}_v \rangle$$



only collinear

only soft

no coupling at leading power

The state $|\bar{B}_v\rangle$ cannot contain collinear modes since $(m_b v + p_c)^2 \sim \mathcal{O}(m_b^2)$

but not near m_b^2 (on-shell)

$$|\bar{B}_v\rangle \simeq |\bar{B}_v\rangle_{soft} \otimes |\Omega\rangle_{collinear}$$

collinear vacuum

$$\Rightarrow \quad T^{SCET} = i\int d^4x\, e^{i(m_b v-q)x} \langle \bar{B}_v | [\bar{h}_v Y \Gamma_1^{+}]_{\alpha(x)} [\Gamma_2 Y^{+} h_v]_{\beta(0)} | \bar{B}_v\rangle \langle \Omega | T([W_c^{(0)+}\xi^{(0)}]_{\alpha(x)} [\bar{\xi}^{(0)} W_c^{(0)}]_{\beta(0)}) |\Omega\rangle$$

soft matrix element depends only on x_-

collinear matrix element can only be $\propto \left(\frac{\not{n}_-}{2}\right)_{\alpha\beta}$

B meson distribution function (soft matrix element)

$$\langle \bar{B}_v | (\bar{h}_v Y)_{\alpha(x_-)} (Y^{+} h_v)_{\beta(0)} | \bar{B}_v\rangle = \langle \bar{B}_v | \bar{h}_{v(x_-)\alpha}\, P e^{ig\int_0^1 ds\, n_-\cdot A_s(sn_-)} h_{v(0)\beta} | \bar{B}_v\rangle$$

$$\equiv \frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha} \int d\ell_+\, e^{i\ell_+/2\, n_+\cdot x}\, S(\ell_+)$$

probability to find b quark in $\bar{B}_v$ with residual momentum ℓ with component ℓ_+ in the direction $n_+/2$

Jet function (collinear matrix element)

$$\langle \Omega | T([W_c^{(0)+}\xi^{(0)}]_{\alpha(x)} [\bar{\xi}^{(0)} W_c^{(0)}]_{\beta(0)}) |\Omega\rangle \equiv \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \hat{J}(p)\, i\, \frac{\not{n}_-}{2}{}_{\alpha\beta}$$

This is just the $\xi^{(0)}$ propagator in light-cone gauge $n_+\cdot A_c^{(0)} = 0$ where $W_c^{(0)} \equiv 1$.

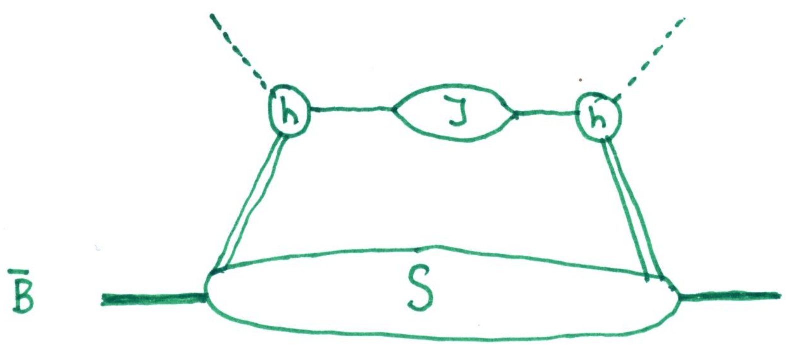
$$\Rightarrow \quad T^{SCET} = \frac{1}{2}\,\mathrm{tr}\left(\frac{1+\not v}{2}\Gamma_1^{+}\frac{\not n_-}{2}\Gamma_2\right)\int d^4x\, e^{i(m_b v-q)x}\int d\ell_+\, e^{\frac{i\ell_+}{2}n_-\cdot x}\, S(\ell_+)\int\frac{d^4p}{(2\pi)^4}\, e^{-ipx}\,\hat{J}(p)$$

$$= \# \int d\ell_+\, S(\ell_+)\,\hat{J}\left(m_b v - q + \frac{\ell_+}{2}n_+\right)$$

$\hat{J}$ can depend only on $p^2 = \left(p_X + \frac{\ell_+}{2}n_+\right)^2 = \underbrace{n_+\cdot p_X}_{2E_X}\,(n_-\cdot p_X + \ell_+)$ $\qquad p_X = m_b v - q$

$J(n_- p_X + \ell_+) \equiv \mathrm{Im}\,\hat{J}\left(m_b v - q + \frac{\ell_+}{2}n_+\right)$

$$\Rightarrow \quad \mathrm{Im}\,T = \underset{\substack{\uparrow\\ \text{hard}}}{H(m_b, E_X)}\cdot\int d\ell_+\, \underset{\substack{\uparrow\\ \text{soft}\\ \text{(non-perturbative)}}}{S(\ell_+)}\, \underset{\substack{\uparrow\\ \text{collinear}}}{J(E_X, n_-\cdot p_X + \ell_+)} + \underset{\text{corrections}}{\mathcal{O}(\lambda)}$$

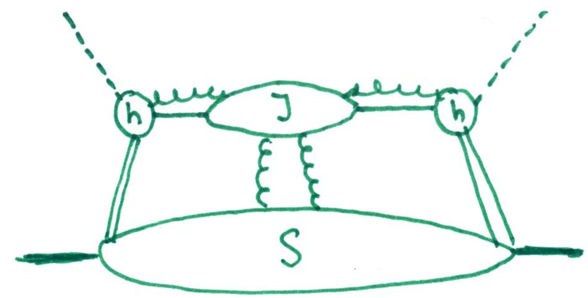


Factorization formula for inclusive decay in the shape-function region

With SCET the factorization formula can be extended to power corrections. Technically more involved (Refs. [9-11]), but follows essentially the same steps

- keep power-suppressed current-products $J^{(1)\dagger} J^{(1)}$, $J^{(2)\dagger} J^{(0)}$

- add up to two insertions of $\mathcal{L}_\xi^{(1)}$ and $\mathcal{L}_{\xi q}^{(1)}$ and one insertion of $\mathcal{L}_\xi^{(2)}$



- Define jet functions (many, but perturbative) and soft matrix elements, for instance

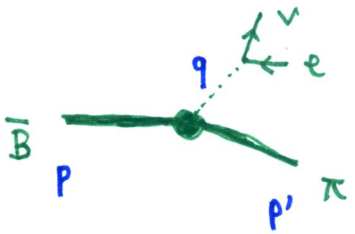
$$\langle \bar{B}_v | (\bar{h}_v Y)(x_-)\,(Y^\dagger i D_\perp Y)(z_{1-})\,(Y^\dagger i D_\perp Y)(z_{2-})\,(Y^\dagger h_v)(0) | \bar{B}_v \rangle$$

multi-local functions

$$\mathrm{Im}\, T = \sum H(u_1, \ldots, u_i) * J(u_1, \ldots u_i; \ell_{+1}, \ldots, \ell_{+n}) * S(\ell_{+1}, \ldots, \ell_{+n})$$

- worked out at tree level up to now

Second example : Exclusive $B \to \pi \ell \nu$ at large recoil energy



$E_\pi \approx \frac{M_B}{2}$ for $q^2 \to 0$

$$\langle \pi(p') | \bar{q} \gamma^\mu b | \bar{B}(p) \rangle = f_+(q^2) \left[(p+p')^\mu - \frac{M_B^2 - m_\pi^2}{q^2} q^\mu \right] + f_0(q^2) \frac{M_B^2 - m_\pi^2}{q^2} q^\mu$$

+ tensor current $\Rightarrow$ 3 form factors $f_{+,0,T}$ for $B \to \pi$

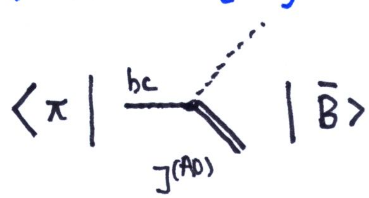
7 " for $B \to \rho$

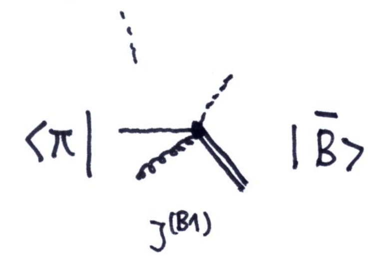
Step 1 : QCD $\to$ SCET$_I$

see below

$$J_{QCD} \to J^{(A0)}, J^{(A1)}, J^{(B1)}, \ldots\ldots$$

$$f_i(E) = C_i^{(A0)} \xi(E) + \int d\tau\, C^{(B1)}_{(\tau)} \Xi(\tau; E)$$





Ξ is the matrix element of the B1 current

$$(\bar{\xi} W_c)(W_c^\dagger i \not{D}_{\perp c} W_c)_s h_v$$

- There is only one A0 current $(\bar{\xi} W_c) h_v$ for $B \to \pi$ and one B1 current, so

$$f_{+,0,T}(q^2) \longrightarrow \quad \xi(E), \quad \Xi(\tau; E)$$

- But why does $J^{(B1)}$ contribute at all? Wasn't it $\mathcal{O}(\lambda^{1/2})$ suppressed?

Pions don't consist of collinear modes with virtuality $M_B \Lambda$!

Indeed, the analysis of regions gives:

HARD - COLLINEAR (hc)

$$p^2 \sim M_B^2 \lambda \qquad n_+ p \sim M_B, \quad p_\perp \sim M_B \lambda^{1/2}, \quad n_- p \sim M_B \lambda$$

COLLINEAR (c)

$$p^2 \sim M_B^2 \lambda^2 \qquad n_+ p \sim M_B, \quad p_\perp \sim M_B \lambda, \quad n_- p \sim M_B \lambda^2$$

(same virtuality as soft)

$$\begin{tikzcd}[column sep=small]
 & \text{QCD} \arrow[d, "\cancel{\text{hard}}"] & \\
 & \text{SCET}_{\text{I}} \ [\text{contains hc and c}] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\cancel{\text{hard-collinear}}"] & \\
\text{HQET} & & \text{HQET} + \text{SCET}_{\text{II}}
\end{tikzcd}$$

HQET [only soft - what we discussed before]

HQET + SCET$_{\text{II}}$ [soft + collinear]

$B \to \pi$ in $SCET_{II}$

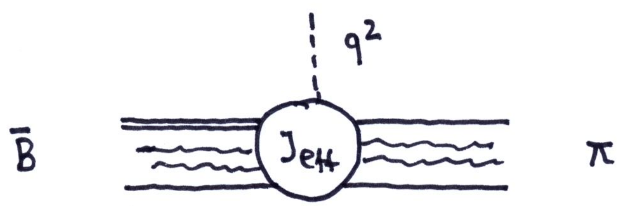
- No collinear-soft interactions

  $(p_c + p_s)^2 \sim M_B \lambda \;\longrightarrow$ integrated out

  $\Rightarrow$ trivial Lagrangian
  (see, however, below !)

$$\mathcal{L}_{SCET_{II}} = \mathcal{L}_{soft} + \mathcal{L}_{collinear}$$

$$\mathcal{L}_{soft}: \quad \bar{q}\, i\not{D}_s\, q \qquad \mathcal{L}_{collinear}: \quad \bar{\xi}\left(i n_- D_c + i\not{D}_{\perp c}\frac{1}{i n_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi$$

- 

  only soft fields with quantum numbers of the $\bar{B}$ meson

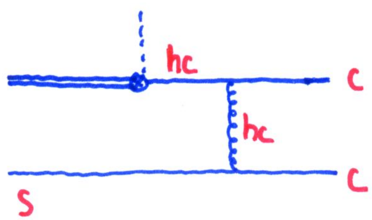
  only collinear fields with quantum numbers of the pion

After integrating out the hc modes, all the physics is in the effective $SCET_{II}$ current

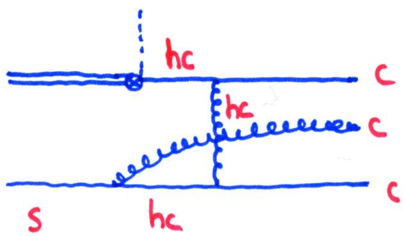
$$J_{eff} \sim \underbrace{[\text{product of c fields}]}_{\pi \text{ quantum numbers}} \cdot \underbrace{[\text{product of s fields}]}_{\bar{B} \text{ quantum numbers}}$$

Leading currents

From A0:



but also:

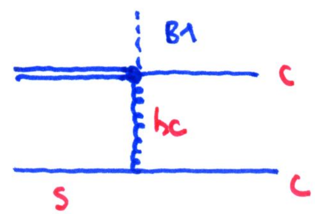


+ many more

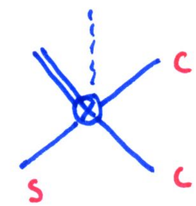
$J^{(A0)}$ i.e. $\xi(E)$ has a very complicated representation in $SCET_{II}$

$$\xi \sim \lambda^{3/2}$$

FROM B1:



in general



i.e. only four-quark operators appear

$J^{(B1)}$ i.e. $\Xi(\tau, E)$ has a simple representation in $SCET_{II}$

$$\Xi(\tau, E) \sim \lambda^{3/2}$$

same order as ξ in $SCET_{II}$

To prove this:

- dimensional analysis
- boost invariance
- λ power counting

Final step (sketch)

$$\Xi \sim \underset{\vdots}{J} * \langle \pi | [\bar{\xi}\xi][\bar{q} h_v] | \bar{B} \rangle |_{\mathrm{SCET}_{II}} \sim \int_0^\infty \frac{d\omega}{\omega} \int_0^1 du \; J(\tau; \ln\omega, u)\, \phi_B(\omega)\, \phi_\pi(u)$$

short-distance ($p^2 \sim M_B^2 \lambda$)
coefficient

where $\phi_\pi \sim$ Fourier transform of $\langle \pi | (\bar{\xi} W_c)_s (W_c^\dagger \xi) | 0 \rangle$ [Note: Dirac structure left out !]

$=$ light-cone distribution amplitude of the pion

$\phi_B \sim$ Fourier transform of $\langle 0 | (\bar{q} Y)_t (Y^\dagger h_v) | \bar{B} \rangle$

$=$ light-cone distribution amplitude of the B meson

$$\Longrightarrow \quad F_i(q^2) = \overset{h}{C_i^{(A0)}}\, \xi(E) + \phi_B \underset{\omega}{*} \left[\overset{h}{C^{(B1)}} \underset{\tau}{*} \overset{hc}{J} \right] \underset{u}{*} \phi_\pi$$

defined in SCET_I ; too complicated in SCET_{II}

so

π : 3 form-factors $\longrightarrow$ ξ, ϕ_B, ϕ_π

ρ : 7 form-factors $\longrightarrow$ $\xi_\parallel, \xi_\perp, \phi_B, \phi_\parallel, \phi_\perp$

WARNING: $\mathrm{SCET}_{\mathrm{II}}$ as currently understood is inconsistent. Use it at your own risk.

Problem: No regulator of $\mathrm{SCET}_{\mathrm{II}}$ is known that preserves the factorization $\mathcal{L} = \mathcal{L}_{\text{soft}} + \mathcal{L}_{\text{collinear}}$

"Factorization anomaly" (seen in analytic regularization, or as "messenger modes" in dimensional regularization)

$\Rightarrow$ $$\langle \pi \,|\, [\text{collinear fields}]\,[\text{soft fields}] \,|\, \bar{B} \rangle \big|_{\mathcal{L}_{\mathrm{SCET}_{\mathrm{II}}}} = \langle \pi |\, [\text{collinear fields}]\, |0\rangle \langle 0|\, [\text{soft fields}]\, |\bar{B}\rangle$$ is <u>wrong in general</u>!

$\Rightarrow$ related to divergences in convolution integrals in longitudinal momenta, which are not regulated ("endpoint divergences")

Can show that Ξ factorizes as above and is unaffected by this problem. But ξ does not factorize naively – another reason for leaving it as a $\mathrm{SCET}_{\mathrm{I}}$ matrix element in the final result.